Ordinary Differential Equations

Course Code: 21M03CC

UNIT - IV

Dr. A. Tamilselvan
Professor and Chair
School of Mathematical Sciences

BHARATHIDASAN UNIVERSITY
Tiruchirappalli- 620024, Tamil Nadu, India

Qualitative Properties of Solutions

x, $\sin x$

$f(0)=0$, $f'(0)=1$ | $f(0)=1$, $f'(0)=0$

Ex:

$f(x)=x$ $f'(x)=1$

$f(0)=0$

$f'(0)=1$

Ex: $\cos x$

diff^l eq^n ?

$$f''+f=0 \qquad \begin{matrix}\cos x\\ \sin x\end{matrix}$$

$\begin{cases} f(0)=0 \\ f'(0)=1 \end{cases}$ & $f(0)=1$, $f'(0)=0$

$f_1: \sin x$ $\qquad$ $f_2: \cos x$

$$f_1' = f_2 \quad , \quad f_2' = -f_1$$

$$f_1^2 + f_2^2 = 1$$

$f_1, f_2 \to$ linearly independent

$$W[f_1, f_2] = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix} = -1$$

In general

f is a periodic function with period T if

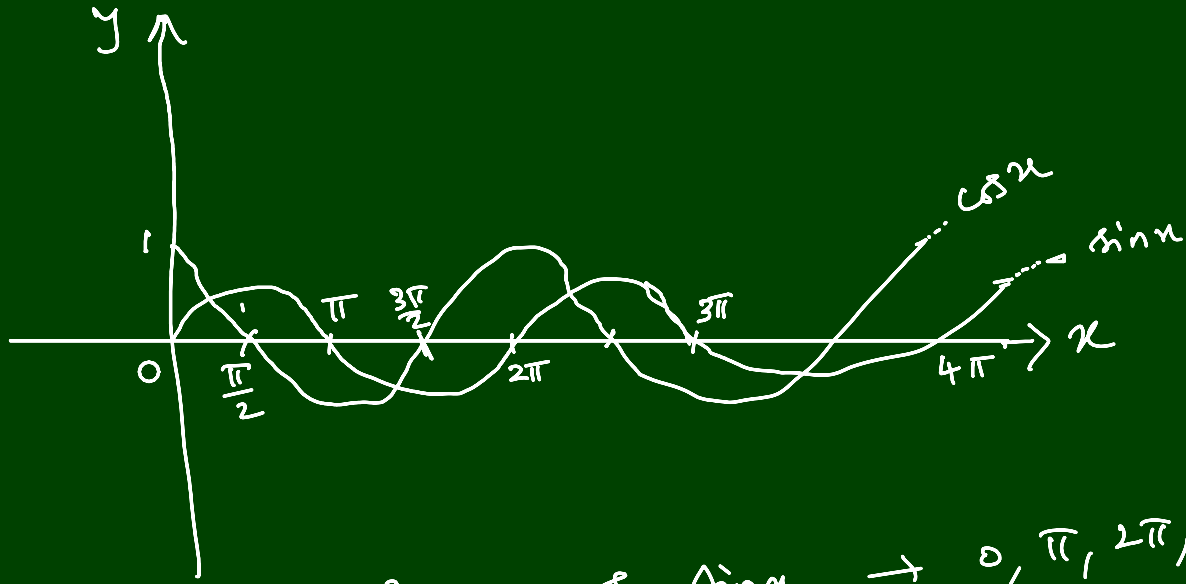
$$f(x+T) = f(x)$$

$\cos(2\pi + x) = \cos x$

$\sin(2\pi + x) = \sin x$

$\cos nx \to 2\pi/n$

$\cos 2x \to \pi$ $\qquad$ $\frac{2\pi}{2} = \pi$



0, 2π not successive zeros of $\sin x$

Zeros of $\sin x \to 0, \pi, 2\pi, 3\pi, \ldots$

of $\cos x \to \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$

0 & π are two successive zeros of $\sin x$

$\frac{\pi}{2}$ is a zero of $\cos x$

π & 2π are two successive zeros of $\sin x$

$\frac{3\pi}{2}$ is a zero of $\cos x$

$\frac{\pi}{2}$ & $\frac{3\pi}{2}$ two successive zeros of $\cos x$

π is a zero of $\sin x$

Zeros of these functions are distinct and occur alternatively

(ie, $\cos x$ vanishes exactly once between any two successive zeros of $\sin x$] $\Rightarrow$ (Sturm separation theorem

Tcm 11/1/21

Recall Sturm separation theorem

(Qualitative analysis)

Find $f(x)$ $\ni$ $\left.\begin{array}{l} f(0)=1 \\ f'(0)=0 \end{array}\right\}$ and $\left\{\begin{array}{l} f(0)=0 \\ f'(0)=1 \end{array}\right.$

f_1, f_2

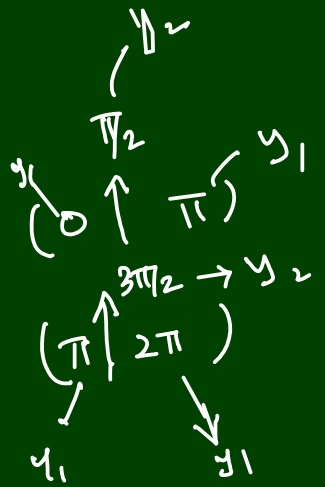
Ans: $f(x)$ must be the soln of $f''+f=0$.

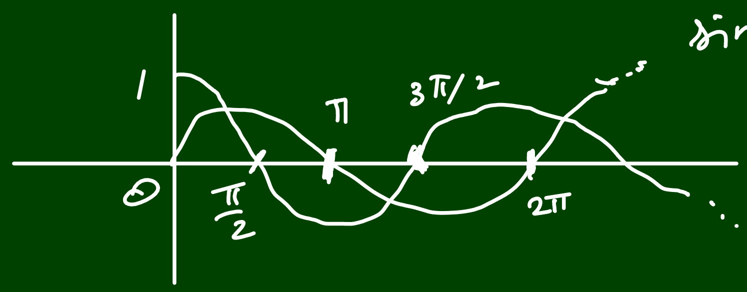
$f_1 = \cos x$, $\sin x = f_2$

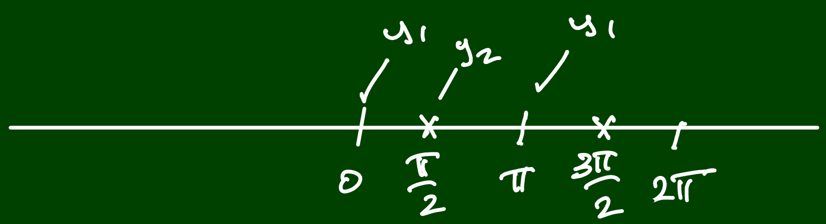
zeros of f_1 and f_2

0 $\quad \frac{\pi}{2}$ $\quad \pi$

x_1 $\quad$ x_2







$0, \pi \to$ consecutive

$0, 2\pi \to$ not consecutive

Theorem 1 (Sturm Separation Theorem)

Statement: If $y_1(x)$ and $y_2(x)$ are two linearly independent solutions of $y'' + P(x)y' + Q(x)y = 0$ then the zeros of these functions are distinct and occur alternatively. (in the sense that $y_1(x)$ vanishes exactly once between any two successive zeros of $y_2(x)$ and conversely)

—o—

Pf: Let x_1 and x_2 be the successive zeros of $y_1(x)$. That is $y_1(x_1) = y_1(x_2) = 0$.

Then $y_2(x_1) \neq 0$ and $y_2(x_2) \neq 0$.

for if $y_2(x_1) = y_2(x_2) = 0$, then the wronskian $W(y_1, y_2) = y_1 y_2' - y_1' y_2$ would be zero there, which is a contradiction to the fact that y_1 and y_2 are linearly independent.

Claim(i): $y_2(x)$ vanishes between x_1 and x_2

Suppose $y_2(x)$ does not vanish between x_1 and x_2.

Consider the function $\phi(x) = \dfrac{y_1(x)}{y_2(x)}$

Then $\phi(x)$ is

(i) continuous on $[x_1, x_2]$

(ii) differentiable on (x_1, x_2) and

(iii) vanishes at x_1 and x_2

f is cont^{s} on $[a,b]$, diff^{l} on (a,b) and $f(a) = f(b) = 0$ then $\exists\, c \in (a,b) \ni f'(c) = 0$

Hence by Rolle's theorem $\phi'(x) = 0$ at some point in (x_1, x_2).

But $$\phi'(x) = \frac{y_2(x)y_1'(x) - y_1(x)y_2'(x)}{y_2^2(x)} = \frac{-W(y_1, y_2)}{y_2^2(x)}$$

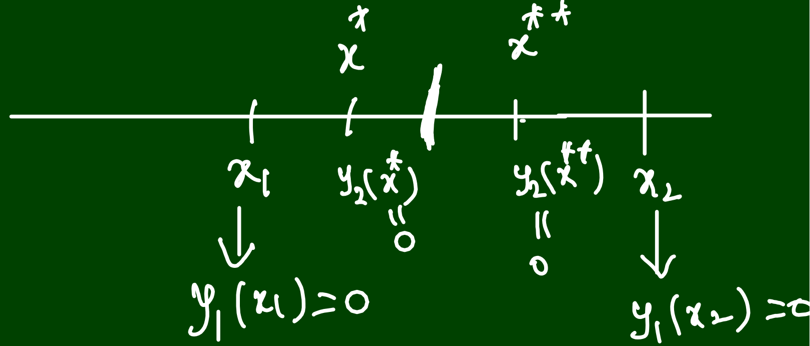
$\neq 0$ (as y_1 & y_2 are linearly independent)

$\Rightarrow \Leftarrow$ to Rolle's theorem.

$\therefore$ $y_2(x)$ vanishes at least once in (x_1, x_2).

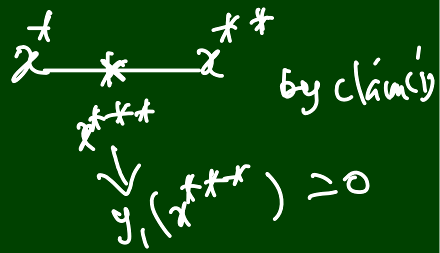
Claim (ii): $y_2(x)$ vanishes exactly once between x_1 and x_2

Suppose $y_2(x)$ vanishes twice in (x_1, x_2)



Then by claim (i), $y_1(x)$ would have a zero between them.

$\Rightarrow$ x_1 and x_2 are not consecutive zeros of $y_1(x)$



$\Rightarrow\Leftarrow$

Hence $y_2(x)$ has exactly one zero in (x_1, x_2).

Similarly $y_1(x)$ has exactly one zero between two successive zeros of $y_2(x)$

Observe.

$\sin 2(\pi/2) = \sin \pi = 0$

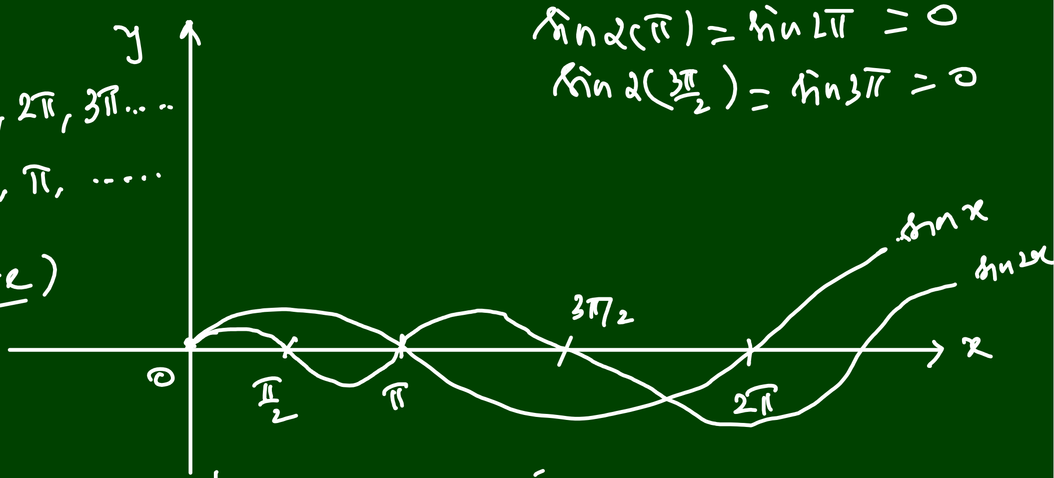
$\sin 2(\pi) = \sin 2\pi = 0$

$\sin 2(\frac{3\pi}{2}) = \sin 3\pi = 0$

$\psi(x) \leftarrow \sin x \rightarrow 0, \pi, 2\pi, 3\pi \ldots$

$\phi(x) \leftarrow \sin 2x \rightarrow 0, \frac{\pi}{2}, \pi, \ldots$

(oscillates more)



$\sin x \rightarrow y'' + y = 0$ coeff of $y \rightarrow 1$

$\sin 2x \rightarrow y'' + 4y = 0$ coeff of $y \rightarrow 4$

$4 > 1$

$\phi(x) > \psi(x)$

Theorem 2 (Sturm comparison Theorem)

Let $\phi(x)$ and $\psi(x)$ be nontrivial solutions of $y'' + p(x) y = 0$ and $y'' + q(x) y = 0$ respectively, where $p(x)$ and $q(x)$ are positive functions such that $p(x) > q(x)$. Then between any two zeros of $\psi(x)$, there is a zero of $\phi(x)$.

— o —

Proof: Let x_1 and x_2 be the consecutive zeros of $\psi(x)$.

$\boxed{\begin{array}{l}\psi(x_1)=0\\ \psi(x_2)=0\end{array}}$

claim: $\phi(x)$ vanishes in (x_1, x_2)

Suppose $\phi(x)$ does not vanish in (x_1, x_2).

Since the zeros of a function y are same as those of $-y$, we can assume that $\phi(x) > 0$ on (x_1, x_2). for if $\phi(x) < 0$, we can simply replace $\phi(x)$ by $-\phi(x)$.

$\therefore$ we have

$$W(\phi(x), \psi(x) : x_1) = \phi(x_1)\psi'(x_1) - \phi'(x_1)\psi(x_1)$$
$$= \phi(x_1)\psi'(x_1) \geq 0$$

$$W(\phi(x), \psi(x) : x_2) = \phi(x_2)\psi'(x_2) - \phi'(x_2)\psi(x_2)$$
$$= \phi(x_2)\psi'(x_2) \leq 0 \qquad (*)$$

$W(\phi, \psi) = \phi \psi' - \phi' \psi$

However
$$\frac{dW(\phi,\psi,x)}{dx} = \phi\psi'' + \phi'\psi' - \phi'\psi' - \phi''\psi$$
$$= \phi\psi'' - \phi''\psi$$
$$= \phi(-q(x)\psi) - (-p(x)\phi)\psi$$
$$= \phi\psi(p(x) - q(x))$$
$$\geq 0 \quad \text{on } x_1 < x < x_2$$

$$\phi'' + p(x)\phi = 0$$
$$\psi'' + q(x)\psi = 0$$
$$p(x) > q(x)$$

Hence W is nondecreasing on (x_1, x_2)

$\Rightarrow\Leftarrow$ to $(*)$

Thus we proved

D. Tamil
12/01/2021

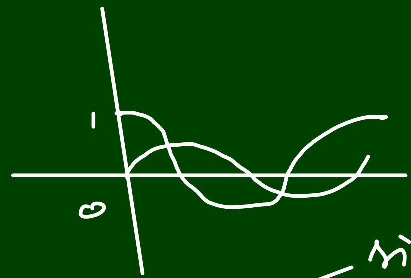
Recall Theorem 1 } Sturm separation Thm
Theorem 2 } Sturm comparision Thm

$\sin x, \cos x$ (y_1, y_2) $\quad y''+y=0$

$y''+p(x)y=0$

$y''+q(x)y=0$

$p(x) > q(x)$



$y''+y=0$ — $\sin x, \cos x$

$y''+4y=0$ — $\sin 2x, \cos 2x$

$4>1$

oscillations

$y''+p(x)y=0$ — ①

$p(x)>0 \rightarrow$ the solutions of ① oscillate (eg) $y''+y=0$ → $\sin x, \cos x$

$p(x)=1>0$

$p(x)<0 \rightarrow$?

The solutions of ① do not oscillate

eg. $y''-y=0 \rightarrow e^{x}, e^{-x}$

$p(x)=-1<0$

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① — $y''+p(x)y'+q(x)y=0 \rightarrow$ standard form

② — $u''+q(x)u=0 \rightarrow$ normal form

Q: Can ① be written in the form of ② ?

Ans: Yes, ① can be written in the form of ②

Put $y(x) = u(x)v(x)$ $\qquad y = uv$

Then $y' = uv' + u'v$

$y'' = uv'' + \underbrace{u'v' + u'}v' + u''v$

Then ① becomes

$uv'' + 2u'v' + \underline{\underline{u''v}} + P(uv' + u'v) + Q(uv) = 0$

$vu'' + [2v' + Pv]u' + [v'' + Pv' + Qv]u = 0$ — ③

Set the coefficient of u' as zero.

ie, $2v' + Pv = 0$

$\Rightarrow 2\frac{dv}{dx} + Pv = 0 \Rightarrow 2\frac{dv}{dx} = -Pv$

$\Rightarrow \frac{dv}{v} = -\frac{1}{2}P\,dx$

Integrating (w.r.t x), $\log v = \int -\frac{1}{2}P\,dx$

$$\Rightarrow \boxed{v = e^{-\frac{1}{2}\int P\,dx}}$$

Now, $v' = e^{-\frac{1}{2}\int P\,dx} \cdot \left(-\frac{1}{2}P\right) = -\frac{1}{2}P\,e^{-\frac{1}{2}\int P\,dx}$

$v'' = -\frac{1}{2}P\left[e^{-\frac{1}{2}\int P\,dx}\left(-\frac{1}{2}P\right)\right] + e^{-\frac{1}{2}\int P\,dx}\left(-\frac{1}{2}P'\right)$

$v'' = \frac{1}{4}P^2 \cdot e^{-\frac{1}{2}\int P\,dx} - \frac{1}{2}P'\,e^{-\frac{1}{2}\int P\,dx}$

$\therefore$ 3 becomes

$$e^{-\frac{1}{2}\int p\,dx} \cdot u'' + \left[\frac{1}{4}p^2 e^{-\frac{1}{2}\int p\,dx} - \frac{1}{2}p' e^{-\frac{1}{2}\int p\,dx} + p\left(-\frac{1}{2}p e^{-\frac{1}{2}\int p\,dx}\right) + Q e^{-\frac{1}{2}\int p\,dx}\right] u = 0$$

$$\Rightarrow u'' + \left[\frac{1}{4}p^2 - \frac{1}{2}p' - \frac{1}{2}p^2 + Q\right] u = 0$$

$$\Rightarrow u'' + \left[Q - \frac{1}{4}p^2 - \frac{1}{2}p'\right] u = 0$$

which can be written in the normal form

as $\quad u'' + q(x)\,u = 0, \quad$ where

$$q(x) = Q - \frac{1}{4}p^2 - \frac{1}{2}p'$$

Problem: Find the normal form of Bessel's equation $x^2y'' + xy' + (x^2 - p^2)y = 0$, p a const.

—o—

solution Given that

$$x^2y'' + xy' + (x^2 - p^2)\,y = 0$$

$$y'' + \frac{1}{x}y' + \frac{(x^2 - p^2)}{x^2}\,y = 0 \quad \text{— 1 (Std form)}$$

$$P(x) = \frac{1}{x}, \quad Q(x) = (x^2 - p^2)/x^2$$

The normal form is

$$u'' + q(x)\,u = 0, \quad \text{where } q(x) = Q(x) - \frac{1}{4}P^2(x) - \frac{1}{2}P'(x)$$

i.e., $$q(x) = \frac{x^2-p^2}{x^2} - \frac{1}{4}\left(\frac{1}{x}\right)^2 - \frac{1}{2}\left(-\frac{1}{x^2}\right)$$

$$= \frac{x^2-p^2}{x^2} - \frac{1}{4x^2} + \frac{1}{2x^2}$$

$$= \frac{x^2-p^2}{x^2} + \frac{1}{4x^2} = 1 - \frac{p^2}{x^2} + \frac{1}{4x^2}$$

$$= 1 + \frac{1-4p^2}{4x^2}$$

$\therefore$ The normal form is

$$u'' + \left[1 + \frac{1-4p^2}{4x^2}\right]u = 0$$

Theorem: 3 If $q(x) < 0$ and $\phi(x)$ is any nontrivial solution of $y'' + q(x)y = 0$, then $\phi(x)$ has atmost one zero.

Given $\phi(x) \not\equiv 0$

Proof: Suppose $\phi(x_0) = 0$.

x_0

$\phi(x_0) = 0$

Then $\phi'(x_0) \neq 0$.

for, if $\phi'(x_0) = 0$ then $\phi(x) = 0$, by uniqueness theorem.

Let $\phi'(x_0) > 0$.

Then for $x > x_0$, $\phi(x) > 0$.

$\phi''(x) + q(x)\phi(x) = 0$

Hence $\phi''(x) = -q(x)\phi(x)$
$\geqslant 0$

Thus $\phi(x)$ is a monotonic function and hence it has no zero for $x > x_0$.

lly $\phi(x)$ has no zero for $x < x_0$.

A similar argument holds for $\phi'(x_0) < 0$.

Hence $\phi(x)$ has atmost one zero.

D. Kumar 19/1/21

centre code: 040

course code		Title of the course
19SIM01CC	-	Linear Algebra
19SIM02CC	-	Real Analysis I
19SIM03CC	-	Ordinary Differential Equations
19SIM04CC	-	Theory of Numbers
19SIM05CC	-	Graph Theory

Recall $y'' + P(x)y' + Q(x)y = 0 \rightarrow$ Std form

$u'' + q(x)u = 0 \longrightarrow$ normal form

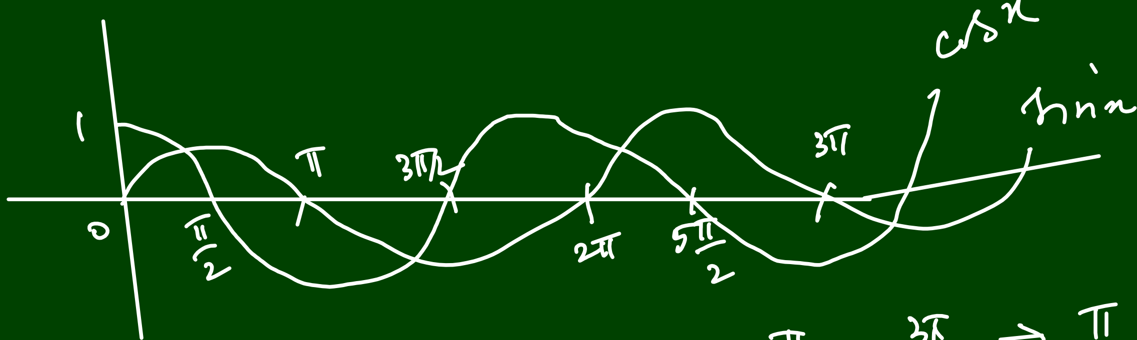
where $q(x) = Q(x) - \frac{1}{4}P^2(x) - \frac{1}{2}P'(x)$

Oscillation: $u'' + q(x)u = 0$, $q(x) > 0$

Ex: $u'' + u = 0$, $\sin x, \cos x$

don't oscillate: $u'' + q(x)u = 0$, $q(x) < 0$

Ex: $u'' - u = 0$, e^x, e^{-x}, $\sinh x, \cosh x$



$\cos x$: $\frac{\pi}{2}, \frac{3\pi}{2} \rightarrow \pi$

$\sin x$: $0, \pi \rightarrow \pi$

$\pi, 2\pi \rightarrow \pi$

$2\pi, 3\pi \rightarrow \pi$

Sturm separation theorem,

y_1, y_2

$y'' + p(x) y' + q(x) y = 0$

Zeros of y_1 and y_2 are distinct and occur alternatively

Sturm comparision theorem

$\phi(x) \to y'' + p(x) y = 0$

$\psi(x) \to y'' + q(x) y = 0$

$p(x) > 0,\ q(x) > 0$ with $p(x) > q(x)$

Then between any two zeros of $\psi(x)$ there is a zero of $\phi(x)$

i.e. $\phi(x)$ oscillates more than $\psi(x)$

Ex: $y'' + 4y = 0$
$y'' + y = 0$ $\quad 4 > 1$

Bessel's eqn of order p

$$x^2 y'' + x y' + (x^2 - p^2) y = 0$$

normal form: $y(x) = u(x) v(x) \to v = e^{-\frac{1}{2}\int p\, dx}$ $\quad$ +ve x-axis $(0, \infty)$

$$u'' + \left[1 + \frac{1 - 4p^2}{4x^2}\right] u = 0$$

$u'' + u = 0$

Theorem Let $y_p(x)$ be a nontrivial solution of Bessel's equation on the positive x-axis.

(i) If $0 \le p < \frac{1}{2}$, then every interval of length π contains at least one zero of $y_p(x)$.

(ii) If $p = \frac{1}{2}$, then the distance between successive zeros of $y_p(x)$ is exactly π

(iii) If $p > \frac{1}{2}$, then every interval of length π contains at most one zero of $y_p(x)$.

Proof: HW //

Theorem: Let $\phi(x)$ be any nontrivial solution of $y'' + q(x)y = 0$, where $q(x) > 0$, $\forall\ x > 0$. If $\int_1^\infty q(x)\,dx = \infty$, then $\phi(x)$ has infinitely many zeros on the positive x-axis.

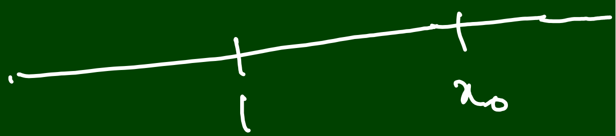
—o—

Proof: Claim: $\phi(x)$ has infinitely many zeros on $(0,\infty)$.

Suppose, $\phi(x)$ vanishes atmost a finite number of times for $0 < x < \infty$.

Therefore a point $x_0 > 1$ exists with the property that $\phi(x) \neq 0$, $\forall\ x \geq x_0$.

W.L.G, we assume that



$$\phi(x) > 0 \quad \forall\ x \geq x_0$$

Put $$v(x) = -\frac{\phi'(x)}{\phi(x)} \quad \text{for } x > x_0$$

Then
$$v'(x) = -\left[\frac{\phi\phi'' - \phi'\cdot\phi'}{\phi^2}\right]$$

$$= -\frac{\phi''}{\phi} + \frac{\phi'^2}{\phi^2}$$

$$\phi'' + q(x)\phi = 0$$
$$q(x) = -\frac{\phi''}{\phi}$$

$$= q(x) + (v(x))^2$$

Integrating this from x_0 to x $(x > x_0)$ we have

$$\int_{x_0}^{x} v'(x)\,dx = \int_{x_0}^{x} q(x)\,dx + \int_{x_0}^{x} (v(x))^2\,dx$$

$$\Rightarrow \int_{x_0}^{x} d(v(x)) = \int_{x_0}^{x} q(x)\,dx + \int_{x_0}^{x} (v(x))^2\,dx$$

$$\left[v(x)\right]_{x_0}^{x}$$

$$v(x) - v(x_0)$$

$$\Rightarrow v(x) = v(x_0) + \int_{x_0}^{x} q(x)\,dx + \int_{x_0}^{x} (v(x))^2\,dx$$

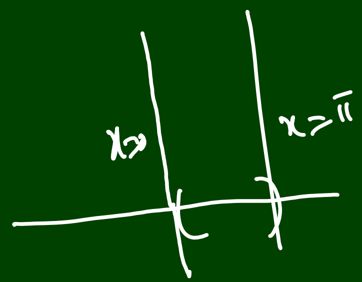
$\Rightarrow$ $v(x) > 0$ if x is taken large enough

$\Rightarrow$ $\phi(x)$ & $\phi'(x)$ have opposite signs if x is sufficiently large

$\Rightarrow$ $\phi'(x) < 0$ ($\because$ $\phi(x) > 0$ by assumption)

which is a contradiction.

Hence the claim.



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Eigenvalues and Eigenfunctions

$x = 0$
$x = \pi$

—o—

BVP $y'' + \lambda y = 0$, $y(0) = 0$, $y(\pi) = 0$

vibrating string $\longrightarrow$ **pde:** $u_{tt} = c^2 u_{xx}$

$c \to$ length of the wave

$\lambda = 0$, $\lambda > 0$, $\lambda < 0$

method of separations of variables.

$u(x,t) = X(x)T(t)$

D. Term
20/11/21

eigenvalue problem

BVP
$$\begin{cases} y'' + \lambda y = 0 \\ y(0) = 0,\ y(\pi) = 0 \end{cases}$$

$u_{tt} = c^2 u_{xx}$ (homogeneous)

MSV: $u(x,t) = X(x)T(t)$

one 2nd order pde → two 2nd order odes

Solution: case(i) $\lambda = 0$

BCs

Then $y'' = 0$ $[y(0) = 0,\ y(\pi) = 0]$

Integrating $y' = c_1$

Again integrating $y = c_1 x + c_2$

$y(x) = c_1 x + c_2$

$y(0) = 0$

$\Rightarrow y(0) = c_1(0) + c_2$

$\Rightarrow 0 = 0 + c_2 \Rightarrow \boxed{c_2 = 0}$

$y(\pi) = 0$

$\Rightarrow y(\pi) = c_1(\pi) + 0$

$\Rightarrow 0 = c_1 \pi \Rightarrow \boxed{c_1 = 0}$ $(\because \pi \neq 0)$

$\therefore y \equiv 0$
which is a trivial solution.

case(ii) $\lambda < 0$ say $\lambda = -\alpha^2$

Then $y'' - \alpha^2 y = 0$

A.E. $m^2 - \alpha^2 = 0$
$m = \pm \alpha$

$\Rightarrow y(x) = c_1 e^{\alpha x} + c_2 e^{-\alpha x}$

Now $y(0) = 0 \Rightarrow y(0) = c_1 + c_2$

$\Rightarrow 0 = c_1 + c_2$ — (i)

$y(\pi) = 0 \Rightarrow y(\pi) = c_1 e^{\alpha\pi} + c_2 e^{-\alpha\pi}$

$\Rightarrow 0 = c_1 e^{\alpha\pi} + c_2 e^{-\alpha\pi}$ — (ii)

from (i) & (ii) $c_1 = c_2 = 0$

$\therefore y(x) \equiv 0$ which is a trivial solution.

Case (iii) $\lambda > 0$, say $\lambda = \alpha^2$

Then $y'' + \alpha^2 y = 0$

$\Rightarrow y(x) = c_1 \cos \alpha x + c_2 \sin \alpha x$

A.E. $m^2 + \alpha^2 = 0$
$m^2 = -\alpha^2$
$m = \pm i\alpha$

$y'' + \lambda y = 0$
A.E. $m^2 + \lambda = 0$
$m^2 = -\lambda$
$m = \pm i\sqrt{\lambda}$

$y(x) = c_1 \cos\sqrt{\lambda}\, x + c_2 \sin\sqrt{\lambda}\, x$

Now, $y(0) = 0$

$\Rightarrow y(0) = c_1(1) + c_2(0)$

$\Rightarrow \boxed{0 = c_1}$

$y(\pi) = 0 \Rightarrow y(\pi) = c_2 \sin \alpha\pi$

$\Rightarrow 0 = c_2 \sin\alpha\pi$

$\sin x = 0$
$x = n\pi$

$\Rightarrow c_2 \sin\alpha\pi = 0 \Rightarrow \sin\alpha\pi = 0$

$\Rightarrow \alpha\pi = n\pi, \quad n = 1, 2, \ldots$

$\Rightarrow \alpha = n$

$y(x) = c_2 \sin nx$

In general $y_n(x) = c_n \sin nx$

$y'' + \alpha^2 y = 0$
$n = 1$, as $\alpha = n$
we have $\alpha = 1$
$y'' + y = 0$
$n = 2$ $\quad y'' + 4y = 0$
$n = 3$ $\quad y'' + 9y = 0$

$\lambda = \alpha^2$

we have $\alpha = n$, $n = 1, 2, 3, \ldots$

$\therefore \lambda = n^2$

$\therefore \lambda = 1, 4, 9, \ldots$ eigenvalues

$\sin x, \sin 2x, \sin 3x \ldots$ eigenfunctions (corresponding)

$\lambda_1 = 1 \longrightarrow \sin x$

$\lambda_2 = 4 \longrightarrow \sin 2x$

$\lambda_3 = 9 \longrightarrow \sin 3x$

$\vdots \qquad \vdots \qquad \vdots$