# Ordinary Differential Equations

Course Code: 21M03CC

UNIT - III

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HW  Prove that $\int_0^1 x^3 J_0(x)\,dx = 2J_0(1) - 3J_1(1)$

Hint: (i) use the above problem : $x=1$

(ii) $J_{p-1}(x) + J_{p+1}(x) = \frac{2p}{x} J_p(x)$ : $p=1$

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## The existence and uniqueness of solutions:

**Picard** → French

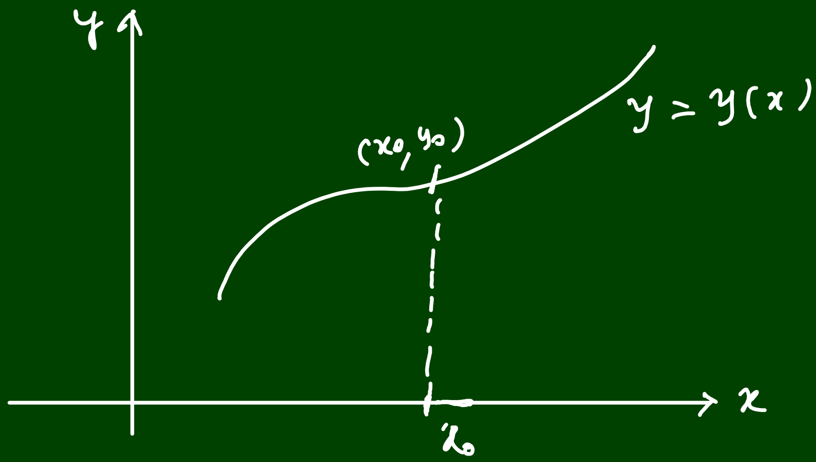
Consider the following first order IVP

$F(x,y,y')=0$
$\Updownarrow$
$y' = f(x,y)$

① $\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases}$ , $f(x,y) \to$ arbitrary function defined and continuous in some neighbourhood of $(x_0, y_0)$

$y(x_0) \to y(x)$ evaluated at $x = x_0$

Aim: To devise a method for constructing a function $y = y(x)$ whose graph passes thro' the point $(x_0, y_0)$ and that satisfies the diff$^l$ eq$^n$ $y' = f(x,y)$ in some nbd of $x_0$.




# Method of successive approximations

Consider ①

$$y' = f(x, y)$$
$$y(x_0) = y_0$$

Key idea: replacing the ivp ① by an equivalent integral equation.

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\,dt$$

[This is called an integral equation because the unknown function occurs under the integral sign]

We write ① as

$$y'(x) = f(x, y(x)) \rightarrow DE$$

Integrating w.r.t $x$

$$\int_{x_0}^{x} y'(x)\,dx = \int_{x_0}^{x} f(t, y(t))\,dt$$

$$\int_{x_0}^{x} d(y(x)) = \int_{x_0}^{x} f(t, y(t))\,dt$$

$$\left[ y(x) \right]_{x_0}^{x} = \int_{x_0}^{x} f(t, y(t))\,dt \qquad y(x_0) = y_0$$

$$y(x) - y(x_0) = \int_{x_0}^{x} f(t, y(t))\,dt$$

$$\boxed{y(x) = y_0 + \int_{x_0}^{x} f[t, y(t)]\,dt} \quad — ② \longleftarrow \text{I.E}$$

$$① \xrightarrow{\text{int}(x)} ②$$

$$② \xrightarrow{\text{diff}} ①$$

① and ② are equivalent.

Diff ② w.r.t $x$

$$y'(x) = 0 + \frac{d}{dx} \int_{x_0}^{x} f[t, y(t)]\,dt$$

$$= \int_{x_0}^{x} \frac{\partial}{\partial x} (f(t, y(t))\,dt + f[x, y(x)] \frac{d}{dx}(x)$$

$$- f[x_0, y(x_0)] \frac{d}{dx}(x_0)$$

$$= 0 + f[x, y(x)](1) - 0$$

$$① \begin{cases} y'(x) = f[x, y(x)] \\ y(x_0) = y_0 \end{cases} \longrightarrow \text{putting } x = x_0 \text{ in } ②.$$

$$\left.\begin{aligned} y' &= f(x, y) \\ y(x_0) &= y_0 \end{aligned}\right\} \overset{\text{equivalent}}{\Longleftrightarrow} y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\,dt$$

Any solution $y(x)$ of ① is a continuous solution of ②

If $y(x)$ is a cont$^{s}$ sol$^{n}$ of ② then it is a sol$^{n}$ of ①

## Successive approximation

We start with our initial approximation as

$$y_0(x) = y_0$$

Then $y_1(x) = y_0 + \int_{x_0}^{x} f[t, y_0(t)]\, dt$

$$y_2(x) = y_0 + \int_{x_0}^{x} f[t, y_1(t)]\, dt$$

$\vdots$

$$y_n(x) = y_0 + \int_{x_0}^{x} f(t, y_{n-1}(t))\, dt$$

$\vdots$

* This procedure is called Picard's method of successive approximation.

<u>Ex</u> $\quad y' = y, \quad y(0) = 1$

$y' = f(x, y)$
$y(x_0) = y_0$

$x_0 = 0, \quad y_0 = 1$
$f(x, y) = y$

$$y_0(x) = y_0$$

i.e., $y_0(x) = 1$

$$y_n(x) = y_0 + \int_{x_0}^{x} f[t, y_{n-1}(t)]\, dt$$

$$y_n(x) = 1 + \int_0^x y_{n-1}(t)\, dt$$

$\therefore \quad y_1(x) = 1 + \int_0^x y_0(t)\, dt$

$$= 1 + \int_0^x 1 \cdot dt$$

$y_0(x) = 1$
$y_0(t) = 1$

$$y_1(x) = 1 + \int_0^x dt = 1 + [t]_0^x = 1 + x$$

$y_1(x) = 1 + x$
$y_1(t) = 1 + t$

$$y_2(x) = 1 + \int_0^x y_1(t)\,dt = 1 + \int_0^x (1+t)\,dt$$

$$= 1 + \left[t + \frac{t^2}{2}\right]_0^x = 1 + x + \frac{x^2}{2}$$

$$y_3(x) = 1 + \int_0^x y_2(t)\,dt = 1 + \int_0^x \left[1 + t + \frac{t^2}{2}\right] dt$$

$$= 1 + \left[t + t^2/2 + \frac{t^3}{6}\right]_0^x$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

$\vdots$

$$y_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$$

$$\lim_{n\to\infty} \{y_n(x)\} \rightarrow e^x \quad (y(x))$$

$\lim_{n\to\infty} \{y_n(x)\} \xrightarrow{\text{uniformly}} y(x)$ ?

$$y = e^x \quad \checkmark$$

Check

$$y' = y$$

$$\frac{dy}{dx} = y \Rightarrow \int \frac{dy}{y} = \int dx$$

$$\log y = x + \log c$$

$$\log y - \log c = x$$

$$\log y/c = x \Rightarrow \frac{y}{c} = e^x$$

$$\Rightarrow \quad y = ce^{x} \quad \Rightarrow \quad y(x) = ce^{x} \qquad \Big|\ y(0) = 1$$

$$y(0) = ce^{0}$$

$$1 = c$$

$$\therefore\ y = e^{x}$$

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HW

Use the method of successive approximations to solve $y' = x + y$, $y(0) = 1$.

[Check your answer by solving the diff$^{l}$ eq$^{n}$ directly]

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D. T. 28/12/2020

# Recall

Picard's method of successive approximation.

IVP $\quad x_0 \in [a,b]$

$$\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases} \iff \text{I.E} \quad y(x) = y_0 + \int_{x_0}^{x} \underbrace{f(t, y(t))}_{\text{integrand}}\, dt$$

PMSA

$$y_n(x) = y_0 + \int_{x_0}^{x} f(t, y_{n-1}(t))\, dt$$

with $y_0(x) = y_0$

$$\lim_{n\to\infty} \{y_n(x)\} \xrightarrow{\text{uniformly}} y(x) \quad (?)$$

$y(x)$ is a sol$^n$ of I.E and hence it is a sol$^n$ of IVP

Ex. 2 $\quad y' = x + y, \quad y(0) = 1.$

Start with $y_0(x) = y_0 = 1$

$$y_1(x) = y_0 + \int_0^x f(t, y_0(t))\, dt$$

$$= 1 + \int_0^x (1+t)\, dt$$

$$y_1(x) = 1 + \left[t + \frac{t^2}{2}\right]_0^x = 1 + x + \frac{x^2}{2}$$

$$= 1 + x + \frac{x^2}{2!}$$

$$y_2(x) = y_0 + \int_0^x f(t, y_1(t))\, dt$$

$$= 1 + \int_0^x \left(1 + 2t + t^2/2\right) dt$$

$y' = f(x,y)$

$y(x_0) = y_0$

$f(x,y) = x + y$

$x_0 = 0, \quad y_0 = 1$

$f(x,y) = x + y$

$f(t, y_0(t)) = t + 1$

$f(x,y) = x + y$

$f(t, y_1(t))$

$= t + 1 + t + \frac{t^2}{2}$

$= 1 + 2t + t^2/2$

$$= 1 + \left[ t + \frac{2t^2}{2} + \frac{t^3}{6} \right]_0^x = 1 + x + x^2 + \frac{x^3}{6}$$

$$= 1 + x + x^2 + \frac{x^3}{3!}$$

$$y_3(x) = y_0 + \int_0^x f(t, y_2(t))\, dt$$

$$= 1 + \int_0^x \left(1 + 2t + t^2 + \frac{t^3}{6}\right) dt$$

$$= 1 + \left[ t + t^2 + \frac{t^3}{3} + \frac{t^4}{24} \right]_0^x$$

$$= 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{4!}$$

$$\boxed{\begin{aligned} &f(x, y) = x + y \\ &f(t, y_2(t)) = \\ &\quad t + 1 + t + t^2 + \frac{t^3}{6} \\ &= 1 + 2t + t^2 + t^3/6 \end{aligned}}$$

$$f(x, y) = x + y$$

$$f(t, y_3(t))$$

$$= t + 1 + t + t^2 + t^3/3 + \frac{t^4}{24}$$

$$y_4(x) = y_0 + \int_0^x f(t, y_3(t))\, dt$$

$$= 1 + \int_0^x \left[ 1 + 2t + t^2 + \frac{t^3}{3} + \frac{t^4}{24} \right] dt$$

$$= 1 + \left[ t + t^2 + t^3/3 + \frac{t^4}{12} + \frac{t^5}{120} \right]_0^x$$

$$y_4(x) = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{3 \cdot 4} + \frac{x^5}{5!}$$

$$\vdots$$

$$y_n(x) = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{3 \cdot 4} + \cdots + \frac{x^n}{3 \cdot 4 \cdots n} + \frac{x^{n+1}}{(n+1)!}$$

$$= 1 + x + 2\left\{ \frac{x^2}{2} + \frac{x^3}{2 \cdot 3} + \frac{x^4}{2 \cdot 3 \cdot 4} + \cdots + \frac{x^n}{2 \cdot 3 \cdots n} \right\} + \frac{x^{n+1}}{(n+1)!}$$

$$y_n(x) = 1 + x + 2\left\{ \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!} \right\} + \frac{x^{n+1}}{(n+1)!}$$

looking for: $\lim_{n\to\infty} \{y_n(x)\} \to y(x)$ (if it exists)

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots$$

$e^x - 1 - x$ :

$\therefore \lim_{n\to\infty} (y_n(x))$ converges to $1 + x + 2(e^x - 1 - x) + 0$

$$\lim_{n\to\infty} (y_n(x)) \to 2e^x - 1 - x \to y(x)$$

ie, $y(x) = 2e^x - x - 1$

Check $y' = x + y,\ y(0) = 1$

$$\frac{dy}{dx} + P(x)y = Q(x)$$

$$y \cdot e^{\int P dx} = \int Q(x) e^{\int P dx} dx + c$$

$$\frac{dy}{dx} = x + y$$

$$\frac{dy}{dx} - y = x \quad \to P(x) = -1,\ Q(x) = x$$

$$e^{\int P(x) dx} = e^{\int -dx} = e^{-x}$$

$$y e^{-x} = \int x e^{-x} dx + c$$

$u = x,\ du = dx$

$\int dv = \int e^{-x} dx$

$v = -e^{-x}$

$$= -x e^{-x} - \int -e^{-x} dx + c$$

$$y e^{-x} = -x e^{-x} - e^{-x} + c$$

$$y = -x - 1 + c e^x$$

$$y(x) = -x - 1 + c e^x$$

$y(0) = 1$

$y(0) = -0 - 1 + c e^0$

$1 = -1 + c \Rightarrow c = 2$

$\therefore y = 2e^x - x - 1$ //////

Ex: Find the exact solution of the initial value problem $y' = y^2$, $y(0) = 1$. Starting with $y_0(x) = 1$ apply Picard's method to calculate $y_1(x), y_2(x), y_3(x)$ and compare these results with the exact solution.

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Exact soln:

$$\frac{dy}{dx} = y^2 \Rightarrow \int \frac{dy}{y^2} = \int dx \Rightarrow \frac{y^{-1}}{-1} = x + c$$

$$\Rightarrow -\frac{1}{y} = x + c \Rightarrow y(x) = -\left(\frac{1}{x+c}\right)$$

$$y(0) = 1 \Rightarrow y(0) = -\left(\frac{1}{0+c}\right)$$

$$1 = -\frac{1}{c} \Rightarrow c = -1$$

$$y(x) = -\frac{1}{x-1} \Rightarrow y = \frac{1}{1-x}, \ |x| < 1$$

$$y = 1 + x + x^2 + x^3 + \ldots, \ |x| < 1$$

$$y_1(x) = 1 + x$$

$$y_2(x) = 1 + x + x^2 + \frac{1}{3}x^3$$

$$y_3(x) = 1 + x + x^2 + x^3 + \frac{2}{3}x^4 + \frac{1}{3}x^5 + \frac{1}{9}x^6 + \frac{1}{63}x^7$$

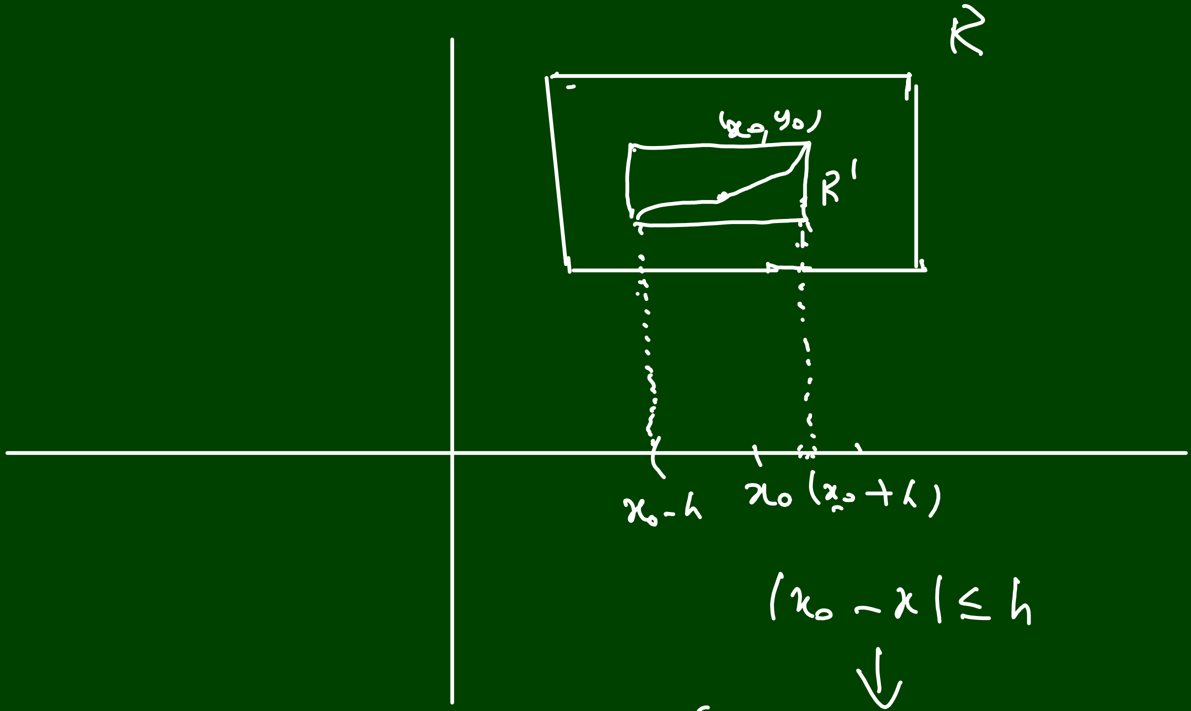
—o—

**Picard's Theorem:** Let $f(x,y)$ and $\frac{\partial f}{\partial y}$ be continuous functions of $x$ and $y$ on a closed rectangle $R$ with sides parallel to the axes. If $(x_0, y_0)$ is any interior point of $R$, then there exists a number $h > 0$ with the property that the initial value problem $y' = f(x,y)$, $y(x_0) = y_0$ has one and only one solution $y = y(x)$ on the interval $|x - x_0| \leq h$.

$$\text{IVP}\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases}$$

existence and uniqueness of the solution for this IVP

$f(x,y)$, $\underline{\underline{\dfrac{\partial f}{\partial y}}}$ → continuous $f^{\underline{ns}}$ of $x$ & $y$ on $R$




$$|x_0 - x| \leq h$$
$$\Downarrow$$
$$(x_0 - h,\ x_0 + h)$$

Condition : $\dfrac{\partial f}{\partial y}$ → cont$^{\underline{s}}$ ( strong condition )

$f(x,y)$ → cont$^{\underline{s}}$ , eventhough not necessary that $\dfrac{\partial f}{\partial y}$ has to exist.

D. Tewi
29/12/2020

Recall

# Picard's theorem

$$\textcircled{1}\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases}$$

When will a unique solution exist? (where?)

Assumptions: $f(x,y)$, $\frac{\partial f}{\partial y}$ continuous functions of $x$ & $y$ on $R$.

$(x_0, y_0) \rightarrow$ interior point of $R$.

$\exists\ h > 0 \ni$ ① has one and only one solution

$y = y(x)$ on the interval $\underbrace{|x - x_0| \leq h}$.

$(x_0 - h,\ x_0 + h)$

$f(x,y)$ is continuous in $R \rightarrow f(x,y)$ is bdd on $R$

$$\therefore |f(x,y)| \leq M$$

$\frac{\partial f}{\partial y}$ is continuous in $R \rightarrow \frac{\partial f}{\partial y}$ is bdd on $R$

$$\therefore \left|\frac{\partial f}{\partial y}\right| \leq k$$

Recall! Mean value theorem $\rightarrow f(x)$

$f$ is cont$^{s}$ on $[a,b]$

$f$ is diff$^{l}$ on $(a,b)$

then $\exists\ c \in (a,b) \ni \dfrac{f(b) - f(a)}{b - a} = f'(c)$

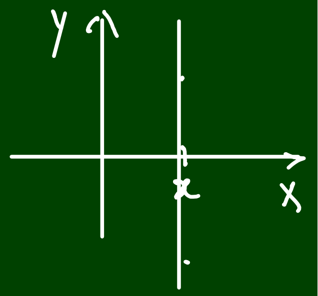
Use MVT for $f(x,y)$ | $f(b) - f(a) = f'(c)(b-a)$

$(x_1, y_1), (x_1, y_2) \in R$

$$|f(x,y_1) - f(x,y_2)| = \left|\frac{\partial}{\partial y} f(x, y^*)\right| |y_1 - y_2| ,$$

$$y_1 < y^* < y_2$$

ie, $$\boxed{|f(x,y_1) - f(x,y_2)| \le K |y_1 - y_2|}$$



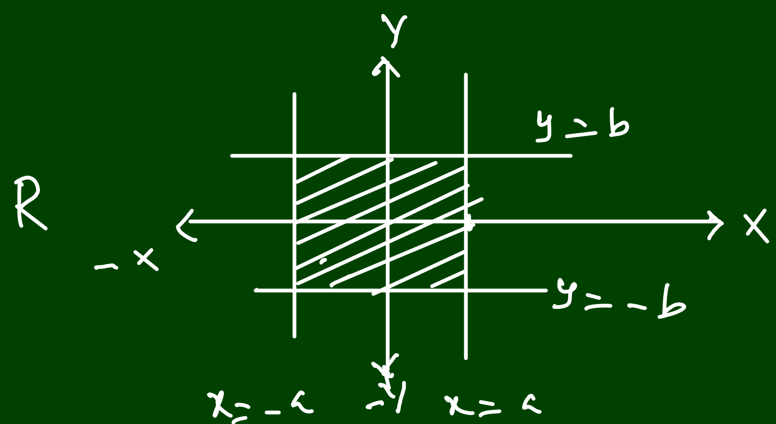

∴ Lipchitz condition

∴ $f(x,y) \rightarrow$ Lipchitz function.

$$\frac{|f(x,y_1) - f(x,y_2)|}{|y_1 - y_2|} \le K \longrightarrow \text{finite constant}$$

∴ (Lipchitz Constant)

Example (i) If $R$ is a rectangle defined by $|x| \le a$, $|y| \le b$, show that $f(x,y) = x^2 + y^2$ satisfies the Lipchitz condition. Find the Lipchitz constant.



$\forall\ (x, y_1), (x, y_2) \in R$ , $f(x, y) = x^2 + y^2$

$$\Rightarrow f(x, y_1) - f(x, y_2) = (x^2 + y_1^2) - (x^2 + y_2^2)$$

$$= y_1^2 - y_2^2$$

$$|f(x, y_1) - f(x, y_2)| = |y_1^2 - y_2^2|$$

$$= |y_1 + y_2||y_1 - y_2|$$

$$\leq (|y_1| + |y_2|)|y_1 - y_2|$$

$$|f(x,y_1) - f(x,y_2)| \leq 2b|y_1 - y_2| \quad (\because |y| < b \text{ in } R)$$

$|ab| = |a||b|$

$y_1^2 - y_2^2$
$= (y_1 - y_2)(y_1 + y_2)$

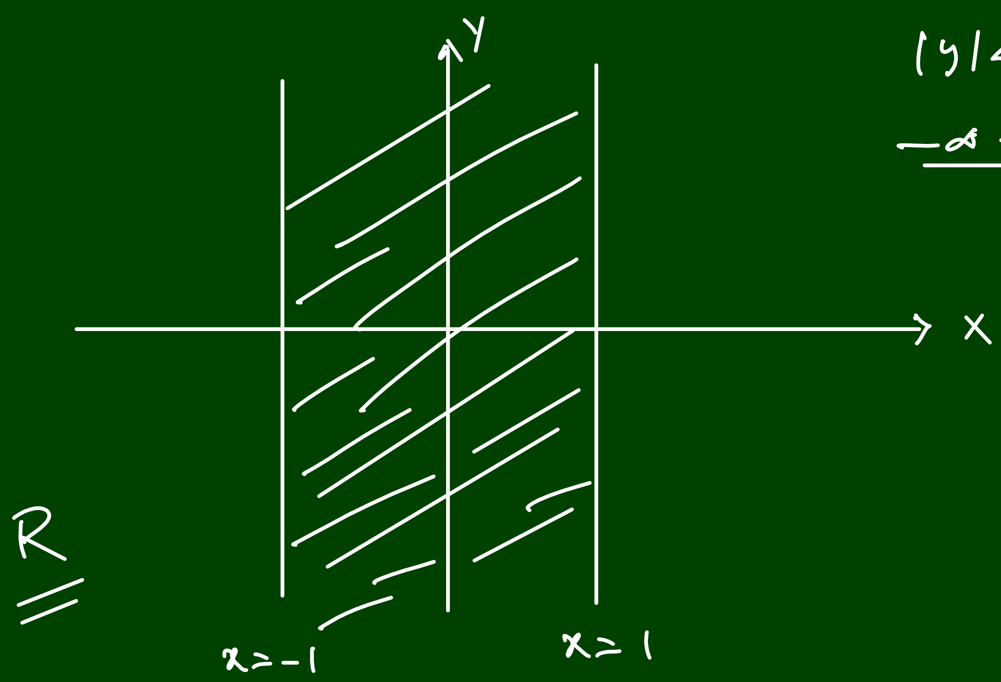
$|a+b| \leq |a| + |b|$

$\therefore f(x, y) = x^2 + y^2$ satisfies the Lipchitz condition

Lipchitz constant is $\underline{\underline{2b}}$

(ii) $f(x, y) = xy^2$ , $R : |x| \leq 1, |y| < \infty$

Test whether $f$ is Lipchitz or not.




$|y| < \infty$

$\underline{-\infty < y < \infty}$

Choose $(x, 0)$ and $(x, y_2) \in R$ $\quad\quad$ $f(x,y) = xy^2$

Then

$$|f(x,0) - f(x,y_2)| = |0 - xy_2^2| = |x||y_2^2|$$

$$|0 - y_2| = |y_2|$$

$$\frac{|f(x,0) - f(x,y_2)|}{|0 - y_2|} = \frac{|x||y_2^2|}{|y_2|}$$

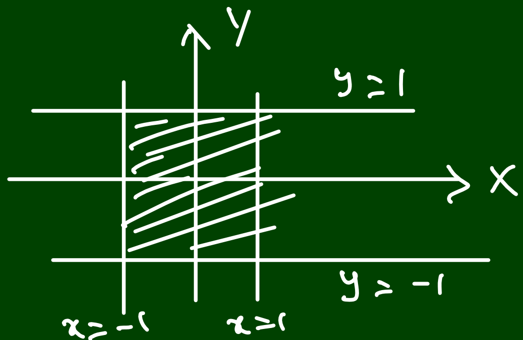
$$\leq |y_2|$$

$$\to \infty \quad \text{as } |y_2| \to \infty$$

$\therefore$ $f(x,y) = xy^2$ on $R$ is not Lipchitz

$*$ Note $f(x,y)$ is continuous $\not\Rightarrow$ $f(x,y)$ is Lipchitz

Example: $f(x,y) = y^{2/3}$, $R: |x| \leq 1, |y| \leq 1$




$$\left|\frac{\partial f}{\partial y}\right| \leq k$$

Here $\frac{\partial f}{\partial y} = \frac{2}{3} y^{-\frac{1}{3}}$

Here $f(x,y) = y^{2/3}$ is continuous on $R$.

But $\left|\frac{\partial f(x,y)}{\partial y}\right| = \left|\frac{2}{3y^{\frac{1}{3}}}\right| \to \infty$ as $y \to 0$

Since $y = 0$ is a point in $R$, the Lipchitz constant is infinite. $\therefore$ $f(x,y) = y^{2/3}$ is not a Lipchitz function in $R$.

*Note* Give an example to show that the existence of partial derivative of $f(x,y)$ is not necessary for $f(x,y)$ to be a Lipchitz function

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Let $f(x,y) = |y|$. Let $R : \{(x,y) \,/\, |x| \le 1, |y| \le 1\}$

Claim (i) $f(x,y)$ is Lipchitz in $R$

$\forall\ (x, y_1), (x, y_2)$ in $R$, we have

$$\frac{|f(x,y_1) - f(x,y_2)|}{|y_1 - y_2|} = \frac{\big|\,|y_1| - |y_2|\,\big|}{|y_1 - y_2|}$$

$$\le \frac{|y_1 - y_2|}{|y_1 - y_2|}$$

$$\le 1$$

$$\big|\,|a| - |b|\,\big| \le |a - b|$$

$\Rightarrow f(x,y) = |y|$ is a Lipchitz function in $R$.

Claim (ii) ($\frac{\partial f}{\partial y}$ does not exist in $R$)

Recall partial derivative of $f(x,y)$ w.r.t $y$ at $(x', y')$ is defined by

$$\left(\frac{\partial f}{\partial y}\right)_{(x',y')} = \lim_{k \to 0} \frac{f(x', y'+k) - f(x', y')}{k}$$

$f(x,y) = |y|$

Use this def$^n$: $\left(\frac{\partial f}{\partial y}\right)_{(x,0)} = \lim_{k \to 0} \frac{f(x, 0+k) - f(x,0)}{k}$

$$= \lim_{k \to 0} \frac{|k|}{k}$$ which does not exist

Thus $\frac{\partial f}{\partial y}$ does not exist at $(x, 0)$

$(x, 0) \in R$

31/12/2020

# Recall Picard's theorem

Existence and uniqueness of the solution

for the IVP $y' = f(x,y)$

$y(x_0) = y_0$

$f(x,y) \to$ continuous
$\frac{\partial f}{\partial y} \to$ continuous
$\Bigg\}$ R

$|f(x,y)| \leq M$

$\left|\frac{\partial f}{\partial y}\right| \leq K$ ✓

$y = y(x), \quad |x - x_0| \leq h$ ✓

Lipschitz condition:

$$\boxed{|f(x,y_1) - f(x,y_2)| \leq K\,|y_1 - y_2|}$$ ✓

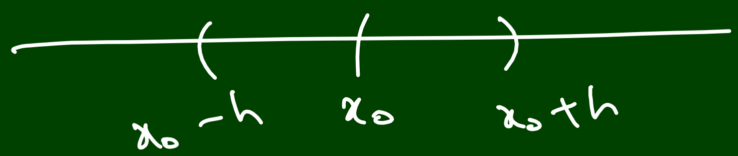
$\forall\ (x,y_1)\ \&\ (x,y_2) \in R$

$$\frac{|f(x,y_1) - f(x,y_2)|}{|y_1 - y_2|} \leq K$$ ✓

✓ $f(x,y)$ is a Lipschitz function

$K$ is called Lipschitz constant

$h > 0,$




$|x - x_0| \leq h$

'R' $\to$ important

Ex: $f(x,y) = x\sin y + y\cos x$

$R : \{(x,y) / |x| \leq a, |y| \leq b\}$

Show that $f$ is Lipschitz.

pf : $x$, $\sin y$, $\cos x$ are cont$^s$ $\to$ $f(x,y)$ cont$^s$

$\frac{\partial f}{\partial y}$ is also cont$^s$

on $R$

✓ $|\cos x| \leq 1$

$|\cos y| \leq 1$

$$\frac{\partial f}{\partial y} = x\cos y + \cos x$$

$$\left|\frac{\partial f}{\partial y}\right| = |x\cos y + \cos x|$$

$$\leq |x||\cos y| + |\cos x|$$

$$\leq |x| + 1$$

$$\leq a+1$$

$\therefore$ $f$ is Lipchitz, the Lipschitz const is $a+1$

f cont$^s$ $\not\Rightarrow$ f is Lipschitz

※ Note: We can't drop the Lipschitz condition in the statement of Picard's theorem.

proof (by an example)

Consider $y' = 3y^{2/3}$, $y(0) = 0$.

Let $R : \{(x,y) \mid |x| \leq 1, |y| \leq 1\}$

clearly $f(x,y) = 3y^{2/3}$, which is cont$^s$ on $R$.

$$y' = 3y^{2/3} \Rightarrow \frac{dy}{y^{2/3}} = 3\,dx$$

$$\Rightarrow \int y^{-2/3}\,dy = 3\int dx$$

$$\Rightarrow \quad \frac{y^{\frac{1}{3}}}{\frac{1}{3}} = 3x + 3c \quad \Big| \quad 3y^{\frac{1}{3}} = 3x + 3c$$

$$y^{1/3} = x + c$$

$y(0) = 0 \quad \Rightarrow \quad 0 = 0 + c \quad \Rightarrow \quad \underline{\underline{c = 0}}$

$\therefore$ one solution is $y^{1/3} = x$

i.e, $\boxed{y = x^3}$

$f(x, y) = 3y^{2/3}$

<u>PMSA</u>

$y_0(x) = 0$

$y_1(x) = y_0 + \int_0^x f(t, y_0(t))\, dt \quad \Big| \quad f(t, y_0(t)) = 0$

$y_1(x) = 0$

$y_2(x) = y_0 + \int_0^x f(t, y_1(t))\, dt \quad \Big| \quad f(t, y_1(t)) = 0$

$\quad = 0$

$\vdots$

$y_n(x) = 0 \qquad \lim_{n \to \infty} \{y_n(x)\} \to 0$

$\vdots$

Another solution is $\boxed{y = 0}$

$\therefore$ The given problem has two solutions.

Becaus $f(x, y)$ is not Lipschitz on R.

for, $(x, y_1), (x, 0) \in R$

$$\frac{|f(x, y_1) - f(x, 0)|}{|y_1 - 0|} = \frac{|3y_1^{2/3}|}{|y_1|} = \frac{3}{|y_1|^{1/3}}$$

which is unbounded when $y_1 \to 0$. (in every Nbd of 0 in $\mathbb{R}$)

---

# Systems of first order equations

$$\text{(1)} \begin{cases} y_1' = f_1(x, y_1, y_2, \dots y_n) \\ y_2' = f_2(x, y_1, y_2, \dots y_n) \\ \vdots \\ y_n' = f_n(x, y_1, y_2, \dots y_n) \end{cases}$$

$y_1(x), y_2(x), \dots y_n(x)$ are unknown functions of a single independent variable $x$.

solve $\left.\begin{array}{l} y' = f(x,y) \\ y(x_0) = y_0 \end{array}\right\} \to$ find $\underline{\underline{y(x)}} \in C^1$ (unknown function) $\ni$

$$\frac{d}{dx}(y(x)) = f(x,y)$$

and $y(x_0) = y_0$

Consider $y^{(n)} = f(x, y, y', \dots y^{(n-1)})$ — (2)

(2) can always be regarded as a special case of (1). || (X)

We put

(3) $y_1 = y, \quad y_2 = y', \; y_3 = y'', \; y_4 = y''' \dots \; y_n = y^{(n-1)}$ -X-

Then (2) is equivalent to the system

$$\text{(4)} \begin{cases} y_1' = y_2 \\ y_2' = y_3 \\ \dots\dots \\ y_n' = f(x, y_1, y_2, \dots y_n) \end{cases}$$

Ex: (i) consider $y'' - x^2 y' - xy = 0$

Hint

$y' = z \rightarrow y'' = z'$

$\left( z' - x^2 z - xy = 0 \right.$

$\left. z' = x^2 z + xy \right)$

system
$$\begin{cases} y' = z \\ z' = x^2 z + xy \end{cases}$$

(ii) consider $y''' = y'' - x^2 y'^2$

Hint

$y' = z$

$z' = w \ (\cancel{z}\, y'')$

$\rightarrow w' = z'' = y'''$

System
$$\begin{cases} y' = z \\ z' = w \\ w' = w - x^2 z^2 \end{cases}$$

A. Tamrakar 04/01/2021

# Linear Systems

$$\begin{cases} \dfrac{dx}{dt} = F(t, x, y) \\ \dfrac{dy}{dt} = G(t, x, y) \end{cases}$$

dept. variables → $x, y$

(unknowns → $x(t), y(t)$)

ind. variable → $t$

find $x(t)$ & $y(t)$

$$(1) \begin{cases} \dfrac{dx}{dt} = a_1(t)x + b_1(t)y + f_1(t) \\ \dfrac{dy}{dt} = a_2(t)x + b_2(t)y + f_2(t) \end{cases}$$

$a_1(t), a_2(t), b_1(t), b_2(t), f_1(t), f_2(t)$ are continuous functions on $[a, b]$ of the $t$-axis.

If $f_1(t)$ and $f_2(t)$ are zero then (2) is called homogeneous; otherwise (2) is nonhomogeneous.

A solution of (1) on $[a, b]$ is a pair of functions $x(t)$ and $y(t)$ that satisfy both equations of (2) throughout $[a, b]$.

Result: If $t_0$ is any point of $[a, b]$ and if $x_0$ and $y_0$ are any numbers whatever, then (1) has one and only one solution

$$\begin{cases} x = x(t) \\ y = y(t) \end{cases}$$ valid throughout $[a, b]$

such that $x(t_0) = x_0$ and $y(t_0) = y_0$.

Consider the homogeneous system obtained from ①:

$$② \begin{cases} \dfrac{dx}{dt} = a_1(t)x + b_1(t)y \\ \dfrac{dy}{dt} = a_2(t)x + b_2(t)y \end{cases}$$

$x(t) \equiv 0$
$y(t) \equiv 0$
trivial sol$^{\underline{n}}$

**Theorem:** If the homogeneous system ② has two solutions $\begin{cases} x = x_1(t) \\ y = y_1(t) \end{cases}$ and $\begin{cases} x = x_2(t) \\ y = y_2(t) \end{cases}$ on $[a, b]$ then

$$\begin{cases} x = c_1 x_1(t) + c_2 x_2(t) \\ y = c_1 y_1(t) + c_2 y_2(t) \end{cases}$$

is also a solution on $[a, b]$ for any constants $c_1$ and $c_2$.

—o—

**proof:** HW

Hint $y'' + p(x)y' + q(x)y = 0$
$y_1, y_2 \Rightarrow$ sol$^{\underline{n}}$
$c_1y_1 + c_2y_2 \Rightarrow$ sol$^{\underline{n}}$

To have the general sol$^{\underline{n}}$

$$x(t_0) = x_0, \quad y(t_0) = y_0, \quad \forall\, t_0 \in [a, b]$$

$$c_1 x_1(t_0) + c_2 x_2(t_0) = x(t_0)$$
$$c_1 y_1(t_0) + c_2 y_2(t_0) = y(t_0)$$

i.e.,
$$c_1 x_1(t_0) + c_2 x_2(t_0) = x_0$$
$$c_1 y_1(t_0) + c_2 y_2(t_0) = y_0$$

$c_1, c_2$ unique: $\begin{vmatrix} x_1(t_0) & x_2(t_0) \\ y_1(t_0) & y_2(t_0) \end{vmatrix} \neq 0$ $\quad W(t) = \begin{vmatrix} x_1(t) & y_1(t) \\ x_2(t) & y_2(t) \end{vmatrix} \neq 0$

Theorem: If $W(t)$ is the Wronskian of the two solutions $\begin{cases} x = x_1(t) \\ y = y_1(t) \end{cases}$ and $\begin{cases} x = x_2(t) \\ y = y_2(t) \end{cases}$ of the homogeneous system (2), then $W(t)$ is either identically zero or nowhere zero on $[a, b]$

— o —

pf: HW

Hint

$$\begin{cases} W(t) = c\, e^{\int [a_1(t) + b_2(t)]\, dt} \\ \Uparrow \\ \dfrac{dW}{dt} = [a_1(t) + b_2(t)]\, W \end{cases}$$

Remark: Higher oder equations are equivalent to systems. (but not the reverse)

$\Downarrow$

SYSTEMS ARE MORE GENERAL

## Homogeneous Linear Systems with constant coeffts

$$(1) \begin{cases} \dfrac{dx}{dt} = a_1 x + b_1 y \\ \dfrac{dy}{dt} = a_2 x + b_2 y \end{cases}$$

, $a_1, a_2, b_1, b_2$ are given constants.

Let $\begin{cases} x = A e^{mt} \\ y = B e^{mt} \end{cases}$ be the solutions of (1)

Then $\dfrac{dx}{dt} = mAe^{mt}$

$\dfrac{dy}{dt} = mBe^{mt}$

$\therefore$ (1) becomes $mAe^{mt} = a_1 Ae^{mt} + b_1 Be^{mt}$

$mBe^{mt} = a_2 Ae^{mt} + b_2 Be^{mt}$

$\Rightarrow (m - a_1) A e^{mt} - b_1 B e^{mt} = 0$

$- a_2 A e^{mt} + (m - b_2) B e^{mt} = 0$

$\Rightarrow$ (*) $\begin{cases} (m - a_1) A - b_1 B = 0 \\ -a_2 A + (m - b_2) B = 0 \end{cases}$ as $e^{mt} \neq 0$

which is a linear algebraic system in the unknowns A and B.

This system has non trivial solution for A and B if $\begin{vmatrix} m - a_1 & -b_1 \\ -a_2 & m - b_2 \end{vmatrix} = 0$

$\Rightarrow (m - a_1)(m - b_2) - a_2 b_1 = 0$

$\Rightarrow \boxed{m^2 - \underbrace{(a_1 + b_2)}_{\text{trace of } A} m + \underbrace{a_1 b_2 - a_2 b_1}_{\det A} = 0} \rightarrow$ auxiliary equation.

$\begin{cases} \dfrac{dx}{dt} = a_1 x + b_1 y \\ \dfrac{dy}{dt} = a_2 x + b_2 y \end{cases} \Rightarrow \dot{X} = AX,$

where, $X = \begin{pmatrix} x \\ y \end{pmatrix}$, $A = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$

Let $m_1$ and $m_2$ be the roots of the a.e

If we replace $m$ by $m_1$ in $(*)$, we get nontrivial solutions for A and B (say $A_1$ & $B_1$)

So $\begin{cases} x = A_1 e^{m_1 t} \\ y = B_1 e^{m_1 t} \end{cases}$ is a nontrivial solⁿ of ①

III<sup></sup>ly if we replace $m$ by $m_2$ in $(*)$, we get nontrivial solⁿ for A and B (say $A_2$, & $B_2$)

So $\begin{cases} x = A_2 e^{m_2 t} \\ y = B_2 e^{m_2 t} \end{cases}$ is another nontrivial solⁿ of ①

——o——

D. Tamuli 05/01/2021

Recall

$t \in [a, b]$

Linear homogeneous system with constant coeff.

$$(1)\quad \begin{cases} \dfrac{dx}{dt} = a_1 x + b_1 y \\[2mm] \dfrac{dy}{dt} = a_2 x + b_2 y \end{cases}, \quad a_1, b_1, a_2, b_2 \text{ are known real constants}$$

Let $\begin{cases} x = A e^{mt} \\ y = B e^{mt} \end{cases}$ be the solution of (1)

Then a.e. $\quad m^2 - (a_1 + b_2) m + a_1 b_2 - a_2 b_1 = 0 \quad$ ∵

$m = m_1,\ m = m_2$

$$(*)\ \begin{cases} (m - a_1) A - b_1 B = 0 \\ -a_2 A + (m - b_2) B = 0 \end{cases} \quad ∵$$

$m = m_1 \to A_1 \,\&\, B_1 \qquad \begin{cases} x_1 = A_1 e^{m_1 t} \\ y_1 = B_1 e^{m_1 t} \end{cases}$

$m = m_2 \to A_2 \,\&\, B_2 \qquad \begin{cases} x_2 = A_2 e^{m_2 t} \\ y_2 = B_2 e^{m_2 t} \end{cases}$

General sol^n of (1)

$$\begin{cases} x = c_1 x_1 + c_2 x_2 \\ y = c_1 y_1 + c_2 y_2 \end{cases}$$

(1) Can be written in the matrix form as:

$$\dot{X} = AX$$

$$X = \begin{pmatrix} x \\ y \end{pmatrix} \quad \dot{X} = \begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix}$$

$$A = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$$

trace of A

$$\underbrace{m^2 - (a_1 + b_2) m}_{} + \underbrace{a_1 b_2 - a_2 b_1}_{\det A} = 0 \xleftarrow{\text{a.e}}$$

$$W(t) = \begin{vmatrix} x_1(t) & x_2(t) \\ y_1(t) & y_2(t) \end{vmatrix} \neq 0$$

Case(i) The a.e has distinct real roots (say $m_1$ and $m_2$ are real, $m_1 \neq m_2$)

Example Solve: $\begin{cases} \dfrac{dx}{dt} = x+y \\ \dfrac{dy}{dt} = 4x-2y \end{cases}$

Solution: The given system is of the form

$$\begin{cases} \dfrac{dx}{dt} = a_1 x + b_1 y \\ \dfrac{dy}{dt} = a_2 x + b_2 y \end{cases}, \quad \text{where } \begin{array}{l} a_1 = 1,\ b_1 = 1 \\ a_2 = 4,\ b_2 = -2 \end{array}$$

Let $\begin{cases} x = Ae^{mt} \\ y = Be^{mt} \end{cases}$ be a solⁿ of the given system.

Then we have, $m^2 - (a_1+b_2)m + a_1b_2 - a_2b_1 = 0$

, and $(*) \begin{cases} (m-a_1)A - b_1 B = 0 \\ -a_2 A + (m-b_2)B = 0 \end{cases}$.

Now the a.e becomes

$$m^2 - (1-2)m + (-2) - (4) = 0$$

i.e, $m^2 + m - 6 = 0 \Rightarrow (m+3)(m-2) = 0$

$\Rightarrow m_1 = -3,\ m_2 = 2$

✓ When we replace $m$ with $m_1 = -3$ in $(*)$, we have

$$\left.\begin{array}{l} (-3-1)A - B = 0 \\ -4A + (-3+2)B = 0 \end{array}\right\} \Rightarrow \begin{cases} -4A - B = 0 \\ -4A - B = 0 \end{cases}$$

both the equations reduces to a single eqⁿ as $-4A-B=0$ ie, $4A+B=0$

A simple nontrivial solution of this system is

$A=1,\ B=-4$

$\checkmark$ $\boxed{\begin{array}{l} B=1 \\ A=-\frac{1}{4} \end{array}}$

$\therefore$ The first set of solution is

$$\begin{cases} x_1 = e^{-3t} \\ y_1 = -4e^{-3t} \end{cases}$$

$x = Ae^{mt}$
$y = Be^{mt}$
$m = m_1 = -3$
$A=1,\ B=-4$

Again if we replace $m$ with $m_2 = 2$ in $(*)$, we have

$$\begin{cases} (2-1)A - B = 0 \\ -4A + (2+2)B = 0 \end{cases} \Rightarrow \begin{cases} A - B = 0 \\ -4A + 4B = 0 \end{cases}$$

This algebraic system reduces to a single equation as: $A - B = 0$
ie $A = B$

$\therefore$ a simple nontrivial solution is $A=1,\ B=1$

$\therefore$ the second set of solution is

$$\begin{cases} x_2 = e^{2t} \\ y_2 = e^{2t} \end{cases}$$

$\therefore$ The general sol$^n$ of the given system of equations is

$$\begin{cases} x = c_1 e^{-3t} + c_2 e^{2t} \\ y = -4c_1 e^{-3t} + c_2 e^{2t} \end{cases}$$

$$W(t) = \begin{vmatrix} x_1(t) & x_2(t) \\ y_1(t) & y_2(t) \end{vmatrix} = \begin{vmatrix} e^{-3t} & e^{2t} \\ -4e^{-3t} & e^{2t} \end{vmatrix} = e^{-t} + 4e^{-t} = 5e^{-t} \neq 0.$$

Case (ii) The a.e has distinct complex roots

Let $m_1$ and $m_2$ be distinct complex numbers.

say $m_1 = a+ib$, $m_2 = a-ib$ ($a, b$ real numbers and $b \neq 0$)

In this case we expect the values of A and B obtained from (*) to be complex numbers.

The two linearly independent solutions will be of the form

$$\begin{cases} x_1 = A_1^* e^{(a+ib)t} \\ y_1 = B_1^* e^{(a+ib)t} \end{cases}, \quad \begin{cases} x_2 = A_2^* e^{(a-ib)t} \\ y_2 = B_2^* e^{(a-ib)t} \end{cases}$$

where $A_1^* = A_1 + iA_2$, $B_1^* = B_1 + iB_2$ etc.

Consider the first set of solution:

$$\begin{cases} x = (A_1 + iA_2)\, e^{at} \cdot e^{ibt} \\ y = (B_1 + iB_2)\, e^{at} \cdot e^{ibt} \end{cases}$$

ie,
$$\begin{cases} x = (A_1 + iA_2)\, e^{at} \{\cos bt + i \sin bt\} \\ y = (B_1 + iB_2)\, e^{at} \{\cos bt + i \sin bt\} \end{cases}$$
(Euler's formula)

$$\begin{cases} x = e^{at}\{\underbrace{(A_1 \cos bt - A_2 \sin bt)} + i\underbrace{(A_1 \sin bt + A_2 \cos bt)}\} \\ y = e^{at}\{\underbrace{(B_1 \cos bt - B_2 \sin bt)} + i\underbrace{(B_1 \sin bt + B_2 \cos bt)}\} \end{cases}$$

we have two real parts and two imaginary parts but which are real valued functions.

Hence the two real valued solutions are

$$\begin{cases} x_1 = e^{at}(A_1 \cos bt - A_2 \sin bt) \\ y_1 = e^{at}(B_1 \cos bt - B_2 \sin bt) \end{cases}$$

and $$\begin{cases} x_2 = e^{at}(A_1 \sin bt + A_2 \cos bt) \\ y_2 = e^{at}(B_1 \sin bt + B_2 \cos bt) \end{cases}$$

$$\begin{cases} x = c_1 x_1 + c_2 x_2 \\ y = c_1 y_1 + c_2 y_2 \end{cases}$$ is the general solution.

—o—

Example: solve: $$\begin{cases} \dfrac{dx}{dt} = 4x - 2y \\ \dfrac{dy}{dt} = 5x + 2y \end{cases}$$

—o—

The given system is of the form $$\begin{cases} \dfrac{dx}{dt} = a_1 x + b_1 y \\ \dfrac{dy}{dt} = a_2 x + b_2 y \end{cases}$$

where $a_1 = 4$, $b_1 = -2$
$a_2 = 5$, $b_2 = 2$.

Let $$\begin{cases} x = Ae^{mt} \\ y = Be^{mt} \end{cases}$$ be a solution of the given system.

Then we have

$$m^2 - (a_1 + b_2)m + a_1 b_2 - a_2 b_1 = 0$$ and

$$(*) \begin{cases} (m - a_1)A - b_1 B = 0 \\ -a_2 A + (m - b_2)B = 0 \end{cases}$$

The a.e is $m^2-(4+2)m+8+10=0$

ie, $m^2-6m+18=0$

$$m=\frac{6\pm\sqrt{36-72}}{2} \Rightarrow m=\frac{6\pm\sqrt{-36}}{2}$$

$$m=\frac{6\pm 6i}{2} \Rightarrow m=3\pm i3$$

$m_1=3+i3$, $m_2=3-i3$

If we replace $m$ with $m_1=3+i3$ in (*), we have

$$\begin{cases}(3+i3-4)A+2B=0\\ -5A+(3+i3-2)B=0\end{cases}$$

$$\Rightarrow \begin{cases}(-1+i3)A+2B=0\\ -5A+(1+i3)B=0\end{cases}$$

Consider

$-5A+(1+i3)B=0$

Take $A=1$

$B=5/(1+i3)$

$$B=\frac{5(1-i3)}{(1+i3)(1-i3)}=\frac{5-i15}{10}$$

$$\boxed{B=\frac{1}{2}-\frac{i3}{2}}$$

$$\therefore \begin{cases}x=e^{(3+i3)t}\\ y=\left(\frac{1}{2}-i\frac{3}{2}\right)e^{(3+i3)t}\end{cases}$$

$$\begin{cases}x=e^{3t}\cdot e^{i3t}\\ y=\left(\frac{1}{2}-i\frac{3}{2}\right)e^{3t}\cdot e^{i3t}\end{cases}$$

$$\begin{cases} x = e^{3t}\{\cos 3t + i\sin 3t\} \\ y = \left(\frac{1}{2} - i\frac{3}{2}\right) e^{3t}\{\cos 3t + i\sin 3t\} \end{cases}$$

$$\begin{cases} x = e^{3t}\{\cos 3t + i\sin 3t\} \\ y = e^{3t}\left\{\left(\frac{1}{2}\cos 3t + \frac{3}{2}\sin 3t\right) + i\left(\frac{1}{2}\sin 3t - \frac{3}{2}\cos 3t\right)\right\} \end{cases}$$

$\therefore$ The two set of solutions are

$$\begin{cases} x_1 = e^{3t}\cos 3t \\ y_1 = e^{3t}\left(\frac{1}{2}\cos 3t + \frac{3}{2}\sin 3t\right) \end{cases}$$

and $\begin{cases} x_2 = e^{3t}\sin 3t \\ y_2 = e^{3t}\left(\frac{1}{2}\sin 3t - \frac{3}{2}\cos 3t\right) \end{cases}$

$\therefore$ The general solution is

$$\begin{cases} x = c_1 e^{3t}\cos 3t + c_2 e^{3t}\sin 3t \\ y = c_1 e^{3t}\left(\frac{1}{2}\cos 3t + \frac{3}{2}\sin 3t\right) + c_2 e^{3t}\left(\frac{1}{2}\sin 3t - \frac{3}{2}\cos 3t\right) \end{cases}$$

---

<u>Hw</u>

(i) Solve:

$$\begin{cases} \frac{dx}{dt} = -3x + 4y \\ \frac{dy}{dt} = -2x + 3y \end{cases}$$

(ii) Solve:

$$\begin{cases} \frac{dx}{dt} = x - 2y \\ \frac{dy}{dt} = 4x + 5y \end{cases}$$

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D. Tcmc
06/01/2021

**Recall**

**HW**

(i) $\begin{cases} \frac{dx}{dt} = -3x+4y \\ \frac{dy}{dt} = -2x+3y \end{cases}$

**Ans:** $\begin{cases} x = 2c_1 e^{-t} + c_2 e^{t} \\ y = c_1 e^{-t} + c_2 e^{t} \end{cases}$

(ii) $\begin{cases} \frac{dx}{dt} = x-2y \\ \frac{dy}{dt} = 4x+5y \end{cases}$

**Ans:** $\begin{cases} x = e^{3t}\left[c_1\cos 2t + c_2\sin 2t\right] \\ y = e^{3t}\left\{c_1(\sin 2t-\cos 2t) - c_2(\sin 2t+\cos 2t)\right\} \end{cases}$

## Case (iii) Equal real roots ($m_1 = m_2 = m$, say)

Here $\begin{cases} x = A_1 e^{m_1 t} \\ y = B_1 e^{m_1 t} \end{cases}$ and $\begin{cases} x = A_2 e^{m_2 t} \\ y = B_2 e^{m_2 t} \end{cases}$

are not linearly independent (X)

As $m_1 = m_2 = m$, we have only one solution $\begin{cases} x = Ae^{mt} \\ y = Be^{mt} \end{cases}$. ✓

$y'' - 4y' + 4y = 0$

a.e $m^2 - 4m + 4 = 0$

$(m-2)^2 = 0$

$m = 2$ twice

$y_1 = e^{2x}$

$y_2 = xe^{2x}$

$W(y_1, y_2) \neq 0$

$y = (c_1 + c_2 x)e^{2x}$

To have another linearly independent solution we can expect it of the form

will not work (Notable difference) $\begin{cases} x = Ate^{mt} \\ y = Bte^{mt} \end{cases}$.

But unfortunately this is not quite simple.

we look for a second solution of the form

$$\because \left\{ \begin{array}{l} x = (A_1 + A_2 t) e^{mt} \\ y = (B_1 + B_2 t) e^{mt} \end{array} \right. \quad \text{—} (a)$$

$\therefore$ The general solution is

$$\left\{ \begin{array}{l} x = c_1 A e^{mt} + c_2 (A_1 + A_2 t) e^{mt} \\ y = c_1 B e^{mt} + c_2 (B_1 + B_2 t) e^{mt} \end{array} \right. \quad // \underline{\underline{\because}}$$

The constants $A_1, A_2, B_1, B_2$ are found by substituting (a) into the system.

o

**Example** solve

$$\left\{ \begin{array}{l} \frac{dx}{dt} = 3x - 4y \\ \frac{dy}{dt} = x - y . \end{array} \right.$$

— o —

The given system is of the form

$$\left\{ \begin{array}{l} \frac{dx}{dt} = a_1 x + b_1 y \\ \frac{dy}{dt} = a_2 x + b_2 y , \end{array} \right. \quad \text{where} \quad \begin{array}{l} a_1 = 3, \ b_1 = -4 \\ a_2 = 1, \ b_2 = -1 . \end{array}$$

Let $\left\{ \begin{array}{l} x = A e^{mt} \\ y = B e^{mt} \end{array} \right.$ be a solution of the given system.

Then $m^2 - (a_1 + b_2) m + a_1 b_2 - a_2 b_1 = 0$ and

$$(*) \left\{ \begin{array}{l} (m - a_1) A - b_1 B = 0 \\ - a_2 A + (m - b_2) B = 0 \end{array} \right.$$

a.e is $m^2 - (3-1)m + (-3) - (-4) = 0 \Rightarrow m^2 - 2m + 1 = 0$

$\Rightarrow m = 1$ twice

Replace $m = 1$ in $(*)$, $\begin{cases} -2A + 4B = 0 \\ -A + 2B = 0 \end{cases}$

Both the equations reduces to $-A + 2B = 0$

The simple nontrivial solution is $B = 1, A = 2$

$\therefore$ The one sol$^n$ is $\begin{cases} x = 2e^t \\ y = e^t \end{cases}$

Choose the second solution as

$$\begin{cases} x = (A_1 + A_2 t)\, e^t \\ y = (B_1 + B_2 t)\, e^t \end{cases} \qquad (m = 1)$$

Then, $\begin{cases} \dfrac{dx}{dt} = (A_1 + A_2 t)e^t + A_2 e^t \\ \dfrac{dy}{dt} = (B_1 + B_2 t)e^t + B_2 e^t \end{cases}$

Substituting the above into the given system we have

$$(A_1 + A_2 t)e^t + A_2 e^t = 3(A_1 + A_2 t)e^t - 4(B_1 + B_2 t)e^t$$

$$(B_1 + B_2 t)e^t + B_2 e^t = (A_1 + A_2 t)e^t - (B_1 + B_2 t)e^t$$

$$\Rightarrow \begin{cases} (A_2 - 3A_2 + 4B_2)t + (A_1 + A_2 - 3A_1 + 4B_1) = 0 \\ (B_2 - A_2 + B_2)t + (B_1 + B_2 - A_1 + B_1) = 0 \end{cases}$$

ie, $\begin{cases} (-2A_2 + 4B_2)t + (-2A_1 + A_2 + 4B_1) = 0 \\ (2B_2 - A_2)t + (2B_1 + B_2 - A_1) = 0 \end{cases}$

$\Rightarrow$ $-2A_2 + 4B_2 = 0$ (i), $-2A_1 + A_2 + 4B_1 = 0$ (iii)

$2B_2 - A_2 = 0$ (ii) $2B_1 + B_2 - A_1 = 0$ (iv)

(i) & (ii) reduces to a single equation $2B_2 - A_2 = 0$

$B_2 = 1$, $A_2 = 2$

(iii) & (iv) becomes

$-2A_1 + 4B_1 = -2$

$-A_1 + 2B_1 = -1$

both eqn reduces to $-A_1 + 2B_1 = -1$

$A_1 = 1$, $B_1 = 0$

$\therefore$ the second solution is

$$\begin{cases} x = (1+2t)e^t \\ y = (0+t)e^t \end{cases} \quad \text{i.e.,} \quad \begin{cases} x = (1+2t)e^t \\ y = te^t \end{cases}$$

$\therefore$ The general soln of the given system is

$$\begin{cases} x = 2c_1 e^t + c_2 (1+2t)e^t \\ y = c_1 e^t + c_2 t e^t \end{cases}$$

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Non-homogeneous system

Consider,

$$(1) \begin{cases} \dfrac{dx}{dt} = a_1(t)x + b_1(t)y + f_1(t) \\ \dfrac{dy}{dt} = a_2(t)x + b_2(t)y + f_2(t) \end{cases}$$

$y'' + P(x)y' + Q(x)y = R(x)$

Corresponding homogeneous eqn

$y'' + P(x)y' + Q(x)y = 0$

$\downarrow$

$y = c_1 y_1 + c_2 y_2$

PI $\rightarrow$ MVP

$y = v_1 y_1 + v_2 y_2$

Corresponding homogeneous system of (1) is

$$\text{(2)} \begin{cases} \dfrac{dx}{dt} = a_1(t)x + b_1(t)y \\ \dfrac{dy}{dt} = a_2(t)x + b_2(t)y \end{cases}$$

Let $\begin{cases} x = x_1(t) \\ y = y_1(t) \end{cases}$ and $\begin{cases} x = x_2(t) \\ y = y_2(t) \end{cases}$

be linearly independent solutions of (2)

so that $\begin{cases} x = c_1 x_1(t) + c_2 x_2(t) \\ y = c_1 y_1(t) + c_2 y_2(t) \end{cases}$ is its general solution.

Let the particular solution of (1) be

$$\begin{cases} x = v_1(t) x_1(t) + v_2(t) x_2(t) \\ y = v_1(t) y_1(t) + v_2(t) y_2(t) \end{cases}$$ i.e. $\begin{cases} x = v_1 x_1 + v_2 x_2 \\ y = v_1 y_1 + v_2 y_2 \end{cases}$

Then $\dfrac{dx}{dt} = v_1 x_1' + v_1' x_1 + v_2 x_2' + v_2' x_2$

$\dfrac{dy}{dt} = v_1 y_1' + v_1' y_1 + v_2 y_2' + v_2' y_2$

Substitute the above in (1)

$$\text{(3)} \begin{cases} v_1 x_1' + v_1' x_1 + v_2 x_2' + v_2' x_2 = a_1(v_1 x_1 + v_2 x_2) + b_1(v_1 y_1 + v_2 y_2) + f_1 \\ v_1 y_1' + v_1' y_1 + v_2 y_2' + v_2' y_2 = a_2(v_1 x_1 + v_2 x_2) + b_2(v_1 y_1 + v_2 y_2) + f_2 \end{cases}$$

$\left.\begin{matrix} x = x_1(t) \\ y = y_1(t) \end{matrix}\right\}$ sub in (2)

$\dfrac{dx}{dt} = \dfrac{dx_1}{dt} = x_1' = a_1 x_1 + b_1 y_1$

$\dfrac{dy}{dt} = \dfrac{dy_1}{dt} = y_1' = a_2 x_1 + b_2 y_1$

$$\left.\begin{array}{l} x = x_2(t) \\ y = y_2(t) \end{array}\right\} \text{sub in (2)} \quad \frac{dx}{dt} = \frac{dx_2}{dt} = x_2' = a_1 x_2 + b_1 y_2$$

$$\frac{dy}{dt} = \frac{dy_2}{dt} = y_2' = a_2 x_2 + b_2 y_2$$

(3) becomes

$$v_1[a_1x_1 + b_1y_1] + v_1'x_1 + v_2[a_1x_2 + b_1y_2] + v_2'x_2 = a_1[v_1x_1 + v_2x_2] + b_1[v_1y_1 + v_2y_2] + f_1$$

$$v_1[a_2x_1 + b_2y_1] + v_1'y_1 + v_2[a_2x_2 + b_2y_2] + v_2'y_2 = a_2(v_1x_1 + v_2x_2) + b_2(v_1y_1 + v_2y_2) + f_2$$

$$W(t) = \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} = x_1y_2 - y_1x_2$$

$$\Rightarrow \quad v_1'x_1 + v_2'x_2 = f_1 \quad \text{— (i)}$$

$$v_1'y_1 + v_2'y_2 = f_2 \quad \text{— (ii)}$$

$$\text{(i) } y_1 \rightarrow \quad v_1'x_1\cancel{y_1} + v_2'x_2y_1 = f_1y_1$$

$$\text{(ii) } x_1 \rightarrow \quad v_1'\cancel{x_1}y_1 + v_2'x_1y_2 = f_2x_1$$

$$(-) \qquad (-)$$

$$v_2'[y_1x_2 - x_1y_2] = y_1f_1 - x_1f_2$$

$$-v_2'[x_1y_2 - y_1x_2] = y_1f_1 - x_1f_2$$

$$\Rightarrow \quad v_2' = \frac{x_1f_2 - y_1f_1}{W(t)}$$

$$\Rightarrow \quad \boxed{v_2 = \int \frac{x_1f_2 - y_1f_1}{W(t)}\, dt}$$

$$\text{(i)} \times y_2 \rightarrow \quad v_1'x_1y_2 + v_2'\cancel{x_2y_2} = y_2f_1$$

$$\text{(ii)} \times x_2 \rightarrow \quad v_1'y_1x_2 + v_2'\cancel{y_2x_2} = x_2f_2$$

$$v_1'(x_1y_2 - y_1x_2) = y_2f_1 - x_2f_2$$

$$v_1' = \frac{y_2f_1 - x_2f_2}{W(t)}$$

$$\Rightarrow \boxed{v_1 = \int \frac{y_2f_1 - x_2f_2}{W(t)}\,dt}$$

HW: Solve:

$$\begin{cases} \dfrac{dx}{dt} = x + y - 5t + 2 \\ \dfrac{dy}{dt} = 4x - 2y - 8t - 8 \end{cases}$$

Hint

$f_1 = -5t + 2$

$f_2 = -8t - 8$

D. T[illegible] 07/01/2021

Recall:

NHLS
$$\begin{cases} \dfrac{dx}{dt} = a_1(t)x + b_1(t)y + f_1(t) \\ \dfrac{dy}{dt} = a_2(t)x + b_2(t)y + f_2(t) \end{cases}$$

PI: $x = v_1 x_1 + v_2 x_2$, $y = v_1 y_1 + v_2 y_2$, where $v_1$ and $v_2$ are functions of 't'.

$$\begin{cases} v_1' x_1 + v_2' x_2 = f_1 \\ v_1' y_1 + v_2' y_2 = f_2 \end{cases}$$

$$v_1 = \int \frac{y_2 f_1 - x_2 f_2}{W(t)}\, dt$$

$$v_2 = \int \frac{x_1 f_2 - y_1 f_1}{W(t)}\, dt$$

$y'' + P(x)y' + Q(x)y = R(x)$

HE: $y'' + P(x)y' + Q(x)y = 0$

$y_1, y_2$

CF: $c_1 y_1 + c_2 y_2$

PI:

$y = CF + PI$

HW Solve:

① $\begin{cases} \dfrac{dx}{dt} = x + y - 5t + 2 \\ \dfrac{dy}{dt} = 4x - 2y - 8t - 8 \end{cases}$

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The corresponding homogeneous eq$^n$ of ① is

② $\begin{cases} \dfrac{dx}{dt} = x + y \\ \dfrac{dy}{dt} = 4x - 2y \end{cases}$, which is of the form $\begin{cases} \dfrac{dx}{dt} = a_1 x + b_1 y \\ \dfrac{dy}{dt} = a_2 x + b_2 y \end{cases}$

where $a_1 = 1$, $b_1 = 1$
$a_2 = 4$, $b_2 = -2$.

Let $\begin{cases} x = Ae^{mt} \\ y = Be^{mt} \end{cases}$ be a sol$^n$ of ②

Then we have $m^2 - (a_1 + b_2)m + a_1 b_2 - a_2 b_1 = 0$

and $(*)\begin{cases} (m - a_1)A - b_1 B = 0 \\ -a_2 A + (m - b_2)B = 0 \end{cases}$

$\therefore$ The a.e is $m^2 - (1-2)m + (-2) - (4) = 0$

i.e, $m^2 + m - 6 = 0 \Rightarrow (m+3)(m-2) = 0$
$m = -3$, $m = 2$

Example ①

$\begin{cases} x_1 = e^{-3t} \\ y_1 = -4e^{-3t} \end{cases}$, $\begin{cases} x_2 = e^{2t} \\ y_2 = e^{2t} \end{cases}$

Let the particular solution be
$$\begin{cases} x = v_1 x_1 + v_2 x_2 \\ y = v_1 y_1 + v_2 y_2 \end{cases}$$

Then $v_1 = \int \frac{y_2 f_1 - x_2 f_2}{W(t)} dt$

and $v_2 = \int \frac{x_1 f_2 - y_1 f_1}{W(t)} dt$.

$$W(t) = \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} = \begin{vmatrix} e^{-3t} & e^{2t} \\ -4e^{-3t} & e^{2t} \end{vmatrix}$$
$$= e^{-t} + 4e^{-t} = 5e^{-t}$$

$$y_2 f_1 - x_2 f_2 = e^{2t}(-5t+2) - e^{2t}(-8t-8)$$
$$= -5te^{2t} + 2e^{2t} + 8te^{2t} + 8e^{2t}$$
$$= 3te^{2t} + 10e^{2t}$$

$$x_1 f_2 - y_1 f_1 = e^{-3t}(-8t-8) - (-4e^{-3t})(-5t+2)$$
$$= -8te^{-3t} - 8e^{-3t} - 20te^{-3t} + 8e^{-3t}$$
$$= -28te^{-3t}$$

$$\therefore v_1 = \int \frac{y_2 f_1 - x_2 f_2}{W(t)}\, dt$$
$$= \int \frac{(3te^{2t} + 10e^{2t})}{5e^{-t}}\, dt = \frac{1}{5}\int (3te^{3t} + 10e^{3t})\, dt$$

$$= \frac{1}{5}\left[te^{3t} - \frac{e^{3t}}{3} + 10\frac{e^{3t}}{3}\right]$$
$$= \frac{1}{5}\left[te^{3t} + 3e^{3t}\right]$$
$$= \frac{e^{3t}}{5}\left[t+3\right]$$

$$3\int te^{3t}\, dt$$

$u = t$
$du = dt$
$\int dv = \int e^{3t}\, dt$
$v = \frac{e^{3t}}{3}$

$$3\left\{\left[\frac{te^{3t}}{3} - \int \frac{e^{3t}}{3}\, dt\right]\right\}$$
$$3\left\{\frac{te^{3t}}{3} - \frac{e^{3t}}{9}\right\}$$
$$te^{3t} - \frac{e^{3t}}{3}$$

$$v_2 = \int \frac{x_1 f_2 - y_1 f_1}{W(t)}\, dt = \int -\frac{28te^{-3t}}{5e^{-t}}\, dt$$

$$= -\frac{28}{5}\int te^{-2t}\, dt$$

$u = t$
$du = dt$
$\int dv = \int e^{-2t}\, dt$
$v = e^{-2t}/(-2$

$$= -\frac{28}{5}\left\{ \frac{te^{-2t}}{-2} - \int \frac{e^{-2t}}{-2}\,dt \right\}$$

$$= -\frac{28}{5}\left\{ \frac{te^{-2t}}{-2} + \frac{1}{2}\left(\frac{e^{-2t}}{-2}\right) \right\}$$

$$= -\frac{28}{5}\left\{ -\frac{te^{-2t}}{2} - \frac{e^{-2t}}{4} \right\}$$

$$v_2 = \frac{28}{10} e^{-2t}\left[t + \frac{1}{2}\right]$$

$\therefore$ The particular integral is

$$x = \frac{e^{3t}}{5}(t+3)e^{-3t} + \frac{28}{10}e^{-2t}\left(t+\frac{1}{2}\right)e^{2t}$$

$$y = \frac{e^{3t}}{5}(t+3)(-4e^{-3t}) + \frac{28}{10}e^{-2t}\left(t+\frac{1}{2}\right)e^{2t}$$

ie, $x = \frac{1}{5}(t+3) + \frac{28}{10}\left(t+\frac{1}{2}\right)$

$y = -\frac{4}{5}(t+3) + \frac{28}{10}\left(t+\frac{1}{2}\right)$

$$\frac{t}{5} + \frac{3}{5} + \frac{28}{10}t + \frac{28}{20}$$

$$\frac{2t+28t}{10} + \frac{12+28}{20}$$

$$\underline{\underline{3t+2}}$$

$$-\frac{4}{5}t - \frac{12}{5} + \frac{28}{10}t + \frac{28}{20}$$

$$\frac{-8t+28t}{10} + \frac{-48+28}{20}$$

$$\underline{\underline{2t-1}}$$

ie, $\boxed{\begin{aligned} x &= 3t+2 \\ y &= 2t-1 \end{aligned}}$

$\therefore$ The solution of (1) is

$$x = c_1 e^{-3t} + c_2 e^{2t} + 3t + 2$$

$$y = -4c_1 e^{-3t} + c_2 e^{2t} + 2t - 1$$

HW: (1) Solve:

$$\begin{cases} \dfrac{dx}{dt} = x + 2y + t - 1 \\ \dfrac{dy}{dt} = 3x + 2y - 5t - 2 \end{cases}$$

—o—

(2) Solve:
$$\begin{cases} \dfrac{dx}{dt} = 7x + 6y \\ \dfrac{dy}{dt} = 2x + 6y \end{cases}$$

—o—

(3) Solve:
$$\begin{cases} \dfrac{dx}{dt} = 4x - 3y \\ \dfrac{dy}{dt} = 8x - 6y \end{cases}$$

—o—

(4) Solve:
$$\begin{cases} \dfrac{dx}{dt} = -4x - y \\ \dfrac{dy}{dt} = x - 2y \end{cases}$$

(5) Solve:
$$\begin{cases} \dfrac{dx}{dt} = -3x + 4y \\ \dfrac{dy}{dt} = -2x + 3y \end{cases}$$