Ordinary Differential Equations

Course Code: 21M03CC

UNIT - V

Dr. A. Tamilselvan
Professor and Chair
School of Mathematical Sciences

BHARATHIDASAN UNIVERSITY
Tiruchirappalli- 620024, Tamil Nadu, India

HW: Find the eigenvalues (λ_n) and eigenfunctions ($y_n(x)$) for the equation

(i) $y'' + \lambda y = 0$, $y(0) = 0$ and $y(\pi/2) = 0$

(ii) $y'' + \lambda y = 0$, $y(0) = 0$ and $y(2\pi) = 0$

— o —

Non Linear Equations

i) Autonomous systems

Qualitative theory of nonlinear equations was introduced by Poincare around 1880

Why nonlinear equations?

Many physical systems — and the equations describes them — are nonlinear.

Linearization → approximating technique.

Van der Pol equation:

$$\frac{d^2x}{dt^2} + \mu(x^2-1)\frac{dx}{dt} + x = 0$$

Second order nonlinear eqn: $\quad \dfrac{d^2x}{dt^2} = f\left(x, \dfrac{dx}{dt}\right)$ — (1)

a simple dynamical system consisting of a particle of unit mass moving on the x-axis and if $f(x, \frac{dx}{dt})$ is the force acting on it then (1) is the equation of motion.

$F = ma$

$m = 1$ ✓

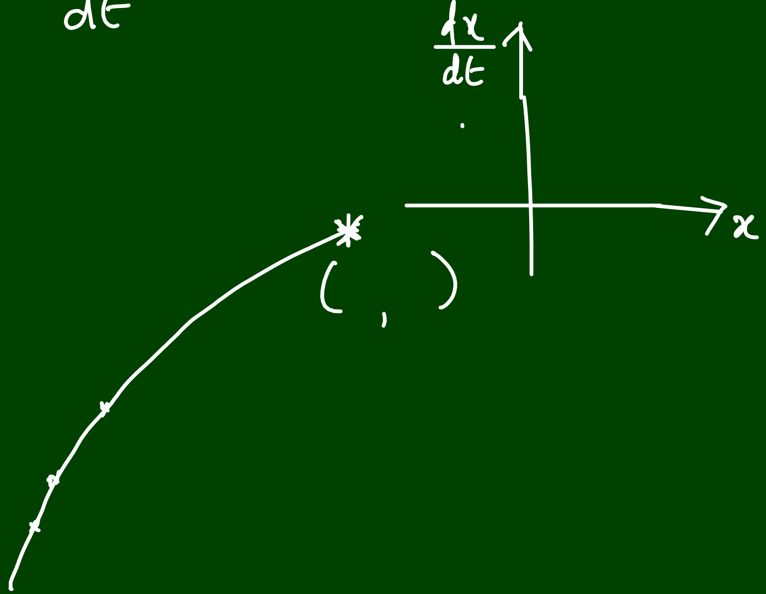
✓ $F = f(x, \frac{dx}{dt})$

$a = \frac{d^2x}{dt^2}$

$$\frac{d^2x}{dt^2} = f(x, \frac{dx}{dt})$$

The values of $x(t)$ (position) and $\frac{dx}{dt}$ (velocity) which at each instant characterize the state of the system, are called its phases and the plane of the variables x ($x(t)$) and $\frac{dx}{dt}$ is the called the

phase plane.



Consider ① : $\frac{d^2x}{dt^2} = f\left(x, \frac{dx}{dt}\right)$

Let $y = \frac{dx}{dt}$ Then $\frac{dy}{dt} = \frac{d^2x}{dt^2}$

Then ① can be written as an equivalent system

$$② \begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = f(x, y) \end{cases} \rightarrow \underline{\text{solution}}: \quad \begin{matrix} x(t) \\ y(t) \end{matrix}$$

We study a system of general form as

$$③ \begin{cases} \frac{dx}{dt} = F(x, y) \\ \frac{dy}{dt} = G(x, y) \end{cases}$$

, where $F(x,y)$ & $G(x,y)$ are continuous functions and have continuous first partial derivatives (throughout the plane)

③ is an <u>autonomous system</u> as F and G are independent of 't'.

A. Tamr
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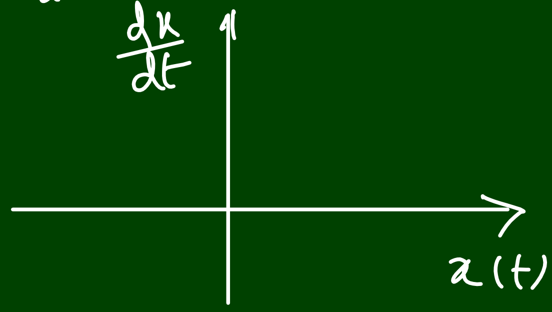
NLE: $\dfrac{d^2x}{dt^2} = f\left(x, \dfrac{dx}{dt}\right)$

$x(t) \to$ position

$\dfrac{dx}{dt}$ – velocity

$$\begin{cases} \dfrac{dx}{dt} = y \\[2mm] \dfrac{dy}{dt} = f(x,y) \end{cases}$$



$\left(x(t), \dfrac{dx}{dt}\right)$

phases

Phase plane

$$① \begin{cases} \dfrac{dx}{dt} = F(x,y) \\[2mm] \dfrac{dy}{dt} = G(x,y) \end{cases}$$

Autonomous system (F & G are independent of 't')

uniqueness thm

If t_0 is any number and (x_0, y_0) is any point in the phase plane, then $\exists$ a unique solution

$$② \begin{cases} x = x(t) \\ y = y(t) \end{cases}$$

of ① such that

$x(t_0) = x_0, \quad y(t_0) = y_0$

Df: If $x(t)$ and $y(t)$ are not both constant functions then ② defines a curves in the phase plane called a path of the system ①.

$\begin{cases} x = t+1 \\ y = e^t \end{cases}$ $\begin{cases} x = t \\ y = 1 \end{cases}$ $\begin{cases} x = 2 \\ y = t^2 \end{cases}$ $\begin{cases} x = 0 \\ y = 1 \end{cases}$ X

If (2) is a solution of (1) then

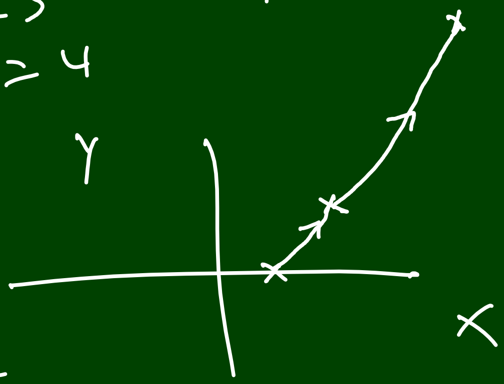
$$\begin{cases} x = x(t+c) \\ y = y(t+c) \end{cases}$$ is a solution for any constant c.

$$\begin{cases} x = t+1 \\ y = t^2 \end{cases}$$

$t=0$: $x=1$, $y=0$ $(1,0)$

$t=1$: $x=2$, $y=1$ $(2,1)$

$t=2$: $x=3$, $y=4$ $(3,4)$



$\therefore$ A path is a directed curve.

critical point (equilibrium point)

A point (x_0, y_0) is called a critical point if both $F(x_0, y_0) = 0$ and $G(x_0, y_0) = 0$.

velocity

$$\begin{cases} \dfrac{dx}{dt} = F(x,y) \\ \dfrac{dy}{dt} = G(x,y) \end{cases}$$

acc^n

(x_0, y_0) :
$$\begin{cases} F(x,y) = 0 \\ G(x,y) = 0 \end{cases}$$

$F = 0 \rightarrow$ velocity : 0

$G = 0 \rightarrow$ acc^n : 0

$\Downarrow$

$\therefore$ particle is at rest

✓ velocity $\leftarrow \dfrac{dx}{dt} = y$

✓ acc^n $\leftarrow \dfrac{dy}{dt} = \dfrac{d^2x}{dt^2}$

ie, no force acting on the particle
it is in the state of equilibrium

We consider only isolated critical points:

(x_0, y_0) is an isolated critical point
if there exists a circle centered on (x_0, y_0)
that contains no other critical point

Ex:

① $\begin{cases} \dfrac{dx}{dt} = -x \\ \dfrac{dy}{dt} = -y \end{cases} \rightarrow \begin{array}{l} F = -x \\ G = -y \end{array}$

$F = 0$ & $G = 0$
$\Rightarrow x = 0$ & $y = 0$
$(0,0)$ is the only critical point. and hence it is isolated.

Soln:

$\begin{cases} x = c_1 e^{-t} \\ y = c_2 e^{-t} \end{cases}$

$\dfrac{x}{y} = \dfrac{c_1}{c_2} = k$

$y = kx$

$$\frac{dy}{dx} = \frac{y}{x} \Rightarrow \int \frac{dy}{y} = \int \frac{dx}{x}$$

$\log y = \log x + \log c$

$\log \dfrac{y}{x} = \log c$

$y = cx$

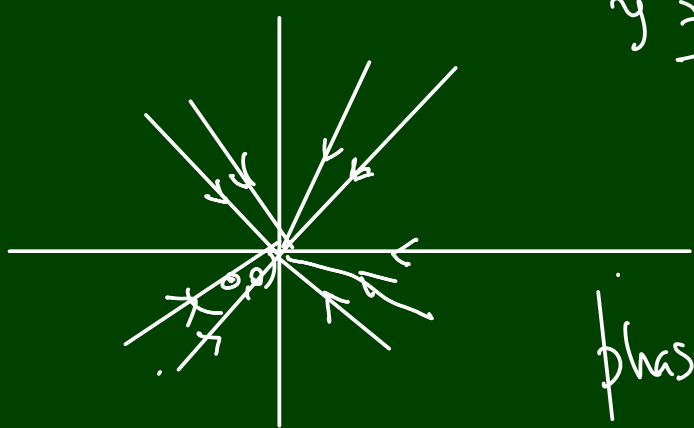
$c = 0$, $y = 0$

$c = 1$, $y = x$

$c = -1$, $y = -x$

$c = 2$, $y = 2x$

$t \uparrow \quad x \to 0$
$\quad\quad y \to 0$



phase portraits

(2) $\begin{cases} \dfrac{dx}{dt} = 1 \\[2ex] \dfrac{dy}{dt} = 2 \end{cases}$

$F = 1$
$G = 2$

$F = 0$ & $G = 0$
$1 = 0$ & $2 = 0$,
'absurd'

This system has no critical point

$$\frac{dy}{dx} = \frac{2}{1} = 2$$

$\begin{cases} x = t + c_1 \\ y = 2t + c_2 \end{cases}$

eliminate 't'

$t = x - c_1$

$y = 2(x - c_1) + c_2$

$y = 2x - 2c_1 + c_2$

$y = 2x + c$

$dy = 2dx$

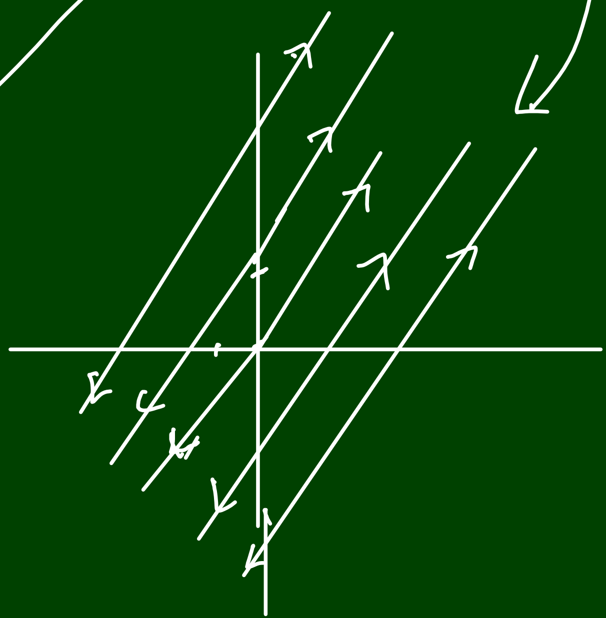
$y = 2x + c$, a st. line with slope 2 and y-intercept c

$y = 2x + c$

$c = 0 \quad y = 2x$

$c = 1 \quad y = 2x + 1$

x	0	$-\frac{1}{2}$
y	1	0



$\begin{cases} t \uparrow \ x \uparrow \\ \quad\ \ y \uparrow \end{cases}$ away from (0, 0)

(3) $\begin{cases} \dfrac{dx}{dt} = x \\[2ex] \dfrac{dy}{dt} = 0 \end{cases}$

$F = x$
$G = 0$

$F = 0$
$G = 0$ $\Rightarrow$ $x = 0$
y-axis

Every point on the y axis is a critical point.

(no isolated critical point)

$\frac{dy}{dx} = 0 \Rightarrow dy = 0$

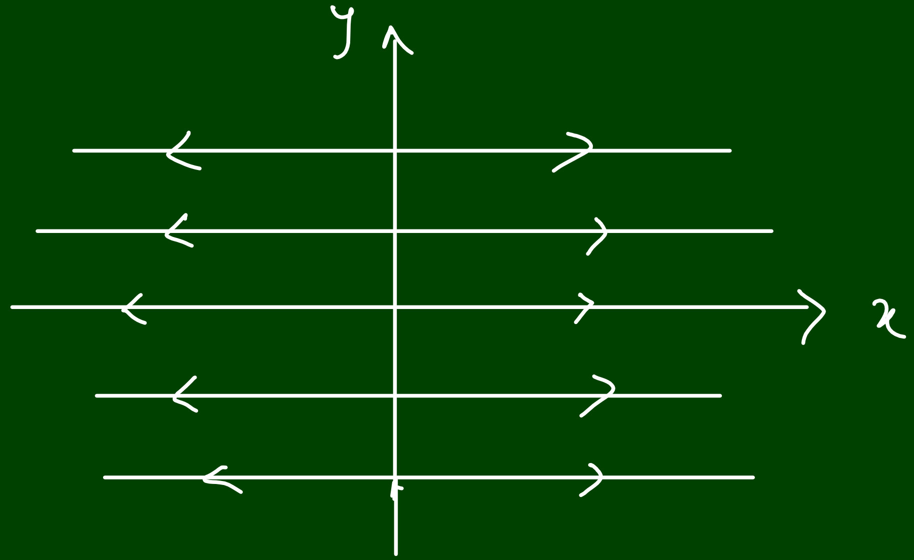
$y = \text{const}$

$\begin{cases} x = c_1 e^t \\ y = c_2 \end{cases}$

paths are horizontal half lines $\quad y = c$

$t \uparrow$

$x \uparrow$



④ $\begin{cases} \dfrac{dx}{dt} = 0 \\ \dfrac{dy}{dt} = 0 \end{cases}$ Every point is a critical point.

$\begin{cases} x = c_1 \\ y = c_2 \end{cases} \longrightarrow$ There are no paths

* (Both x and y are constants)

⑤ Find the critical points of

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} - (x^3 + x^2 - 2x) = 0$$

— o — if $\dfrac{dx}{dt} = y$, then

$$\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = x^3 + x^2 - 2x - y \end{cases}$$

critical points $\begin{cases} y = 0 \\ x^3 + x^2 - 2x - y = 0 \end{cases}$

$\therefore \; x^3 + x^2 - 2x = 0$

$x(x^2 + x - 2) = 0$

$x(x+2)(x-1) = 0$

$x = 0, \; x = 1, \; x = -2$

$\therefore$ critical points are: $(0,0)$, $(1,0)$, $(-2,0)$

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(6) Find the critical points of

$$\begin{cases} \dfrac{dx}{dt} = y^2 - 5x + 6 \\ \dfrac{dy}{dt} = x - y \end{cases}$$

$(2, 2)$
$(3, 3)$

$y^2 - 5x + 6 = 0$

$x - y = 0 \Rightarrow x = y$

$x^2 - 5x + 6 = 0$

$(x-3)(x-2) = 0$

$x = 3,\ x = 2$

(7) Find all solutions of

HW

$$\begin{cases} \dfrac{dx}{dt} = x \\ \dfrac{dy}{dt} = x + e^t \end{cases}$$

and sketch (in xy plane) some of the curves defined by these solutions

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A. Tumr
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HW
$$\begin{cases} \dfrac{dx}{dt} = x & \text{(i)} \\ \dfrac{dy}{dt} = x + e^t & \text{(ii)} \end{cases}$$

(i) $\dfrac{dx}{dt} = x \Rightarrow \displaystyle\int \frac{dx}{x} = \int dt \Rightarrow \log x = t + \log c_1$

$\Rightarrow \boxed{x = c_1 e^t}$

$t \uparrow \; x \to \infty$ $(t \to \infty)$

Sub the value of x in (ii)

$$\frac{dy}{dt} = c_1 e^t + e^t$$

$$\int dy = \int (c_1 e^t + e^t)\, dt$$

$$y = c_1 e^t + e^t + c_2$$

$$\boxed{y = (c_1 + 1) e^t + c_2}$$

$t \uparrow \; y \to \infty$ $(t \to \infty)$

$x = c_1 e^t$

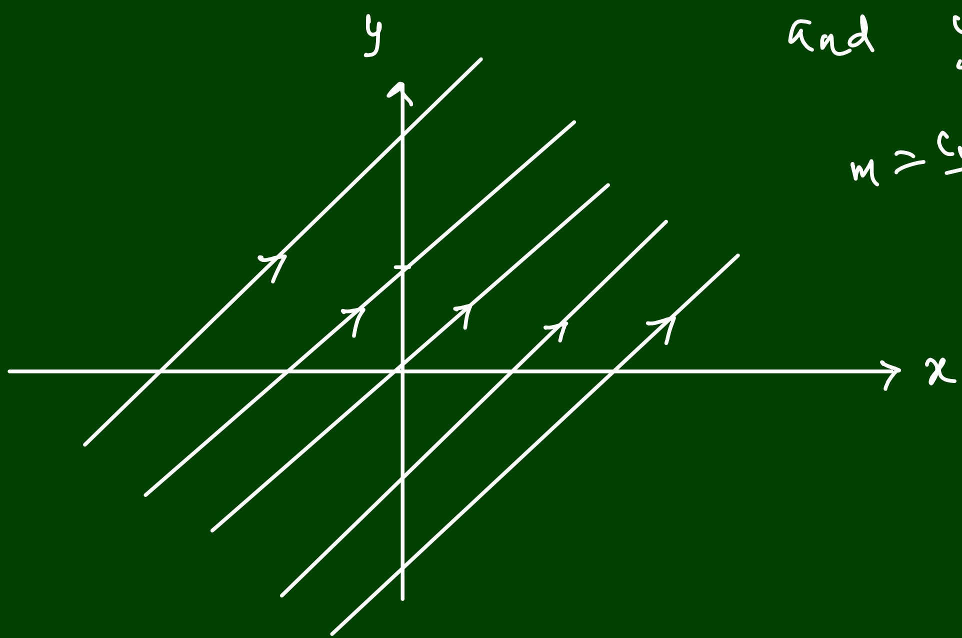
$\Rightarrow \dfrac{x}{c_1} = e^t$

$\Rightarrow y = (c_1 + 1)\dfrac{x}{c_1} + c_2$

$y = \left(\dfrac{c_1 + 1}{c_1}\right) x + c_2$

$\boxed{y = \underline{m} x + c_2}$ a strline with slope m and y intercept c_2

$m = \dfrac{c_1 + 1}{c_1}$ $m = 1$ $y = c_2$



Types of critical points

Stability:

$(x_0, y_0) \to$ CP

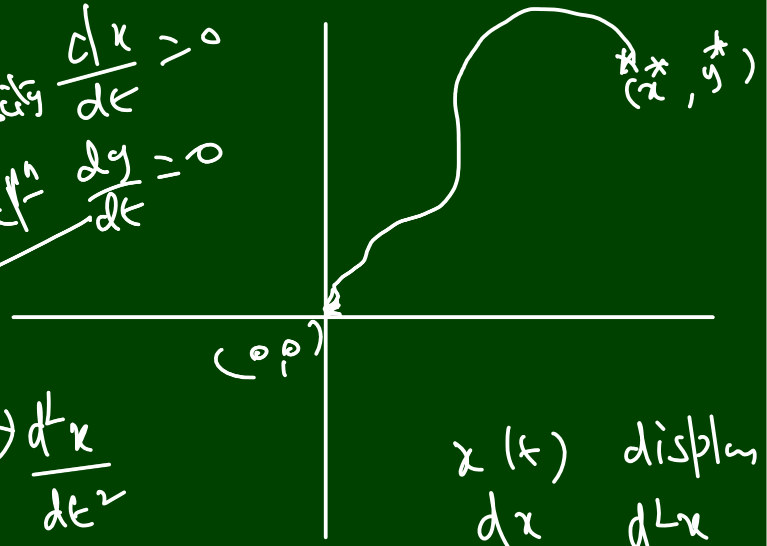
iff $F(x_0, y_0) = 0$

and $G(x_0, y_0) = 0$

velocity $\dfrac{dx}{dt} = 0$

accⁿ $\dfrac{dy}{dt} = 0$

$\dfrac{d^2x}{dt^2}$



$x(t)$ displacement

$\dfrac{dx}{dt}$, $\dfrac{d^2x}{dt^2}$

Let (x_0, y_0) be an isolated critical point of

$$(1)\begin{cases} \dfrac{dx}{dt} = F(x,y) \\[2mm] \dfrac{dy}{dt} = G(x,y). \end{cases}$$

If $C = [x(t), y(t)]$ is a path of (1), then we say that C approaches (x_0, y_0) as $t \to \infty$ if

$$\lim_{t\to\infty} x(t) = x_0 \quad \text{and}$$

$$\lim_{t\to\infty} y(t) = y_0 .$$

Consider the autonomous linear system with constant coeffⁿ

$$(*)\begin{cases} \dfrac{dx}{dt} = a_1x + b_1y \\[2mm] \dfrac{dy}{dt} = a_2x + b_2y \end{cases}$$

which has $(0,0)$ as an isolated critical point.

Here we assume that $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$, so that

(0,0) is the <u>only critical point</u>.

[For example

$$\begin{cases} \dfrac{dx}{dt} = x+y \\ \dfrac{dy}{dt} = 2x+2y \end{cases}$$

CP $\begin{cases} x+y=0 \\ 2x+2y=0 \end{cases} \Rightarrow x+y=0$

$\Rightarrow y=-x$

infinitely many critical points

$$\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} = 2-2 = \underline{\underline{0}}$$]

Let the nontrivial solution of (*) be

$$\begin{cases} x = Ae^{mt} \\ y = Be^{mt} \end{cases},$$ then m is the root of

the Quadratic eq^n

$$m^2 - (a_1+b_2)m + (a_1b_2 - a_2b_1) = 0 \qquad \text{(a.e)}$$

and A and B can be found from

$$\begin{cases} (m-a_1)A - b_1B = 0 \\ -a_2A + (m-b_2)B = 0 \end{cases}.$$

<u>Major cases</u>

<u>Case I</u> (node) The roots m_1 and m_2 of the a.e are real and distinct.

<u>Same sign:</u>

(i) both are −ve → Asymptotically stable

(ii) both are +ve → unstable

Case II (Saddle point)

The roots m_1 and m_2 of the a-e are real and distinct.

Opposite signs $m_1 > 0, m_2 < 0$

$m_1 < 0, m_2 > 0$

Unstable

Case III (Spiral)

The roots m_1 and m_2 of the a-e are Complex conjugate (**but not pure imaginary**)

$\alpha \pm i\beta$

$\alpha > 0$ unstable

$\alpha < 0$ Asymptotically stable

Borderline cases

Case IV (Node)

The roots m_1 and m_2 of the a-e are **real and equal**

(i) both are +ve → unstable

(ii) both are −ve → asymptotically stable

Case V (center)

The roots m_1 and m_2 of the a-e are **pure imaginary** $(\pm i\beta)$

'**Stable**'

Problem 1

$$\begin{cases} \frac{dx}{dt} = x \\ \frac{dy}{dt} = -x+2y \end{cases}$$

Solⁿ: The given system is of the form

$$\begin{cases} \frac{dx}{dt} = a_1 x + b_1 y \\ \frac{dy}{dt} = a_2 x + b_2 y \end{cases}, \quad \text{where} \quad a_1 = 1,\ b_1 = 0;\ a_2 = -1,\ b_2 = 2$$

$$\begin{vmatrix} 1 & 0 \\ -1 & 2 \end{vmatrix} = 2 \neq 0$$

CP: $x = 0$ and $-x+2y = 0$ $\Rightarrow x = y = 0$ $\therefore (0,0)$ is the isolated C.P.

A.C: $m^2 - (a_1+b_2)m + (a_1 b_2 - a_2 b_1) = 0$

$m^2 - (1+2)m + (2-0) = 0$

$m^2 - 3m + 2 = 0$

$(m-2)(m-1) = 0 \Rightarrow m_1 = 2,\ m_2 = 1$

$(0,0)$ is a node but unstable

($(0,0)$ is an unstable node) ✓

Phase portraits

$$\begin{cases} \frac{dx}{dt} = x \\ \frac{dy}{dt} = -x+2y \end{cases} \quad \Rightarrow x = c_1 e^t$$

$$\therefore \frac{dy}{dt} = -c_1 e^t + 2y$$

$$\frac{dy}{dt} - 2y = -c_1 e^t$$

$$p(t) = -2 \qquad Q(t) = -c_1 e^t$$

$$IF = e^{\int -2\,dt} = e^{-2t}$$

$$y \cdot e^{-2t} = \int -c_1 e^t \cdot e^{-2t}\,dt + c_2$$

$$y \cdot e^{-2t} = \int -c_1 e^{-t}\,dt + c_2$$

$$y e^{-2t} = c_1 e^{-t} + c_2$$

$$\boxed{y = c_1 e^t + c_2 e^{2t}} \qquad \boxed{x = c_1 e^t}$$

$$\frac{dy}{dx} + P(x)y = Q(x)$$

$$IF = e^{\int P\,dx}$$

$$y(IF) = \int Q(IF)\,dx + c$$

$$\frac{dy}{dt} + p(t)y = Q(t)$$

$$IF = e^{\int p(t)\,dt}$$

$$y(IF) = \int Q(IF)\,dt + c$$

$t \to \infty$ then $x \to \infty$, $y \to \infty$

D. Tiwari
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$$\left.\begin{aligned} x &= c_1 e^t \\ y &= c_1 e^t + c_2 e^{2t} \end{aligned}\right\}$$

eliminate 't'

$$y = c_1 e^t + c_2 (e^t)^2$$

$$\therefore y = x + c_2 \left(\frac{x}{c_1}\right)^2$$

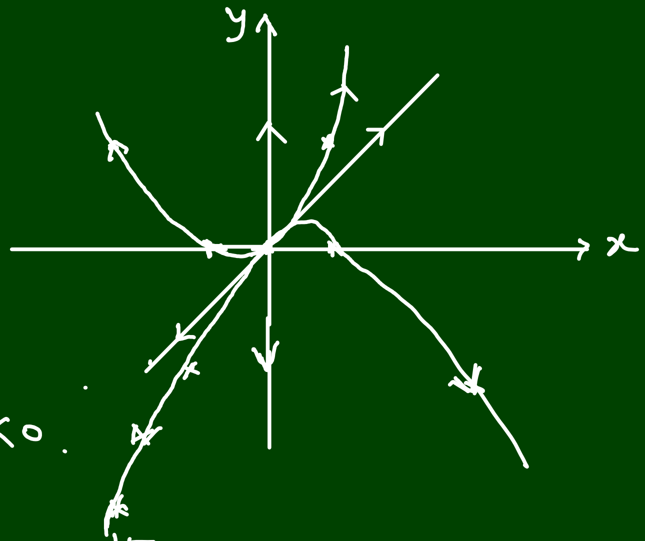
$$\Rightarrow y = x + \frac{c_2}{c_1^2} x^2 \quad \Rightarrow \underline{\underline{y = x + kx^2}}, \quad k = \frac{c_2}{c_1^2}$$

$\underline{\underline{c_1 = 0}}$ $\qquad x = 0$ & $y = c_2 e^{2t}$

In this case the path is +ve y axis when $c_2 > 0$ and −ve y axis when $c_2 < 0$.

$\underline{c_2 = 0}$

$$\left.\begin{array}{l} x = c_1 e^t \\ y = c_1 e^t \end{array}\right\} \Rightarrow y = x$$



In this case the path is the half line $y = x,\ x > 0$ when $c_1 > 0$ and the half line $y = x,\ x < 0$ when $c_1 < 0$.

$\underline{c_1 \neq 0,\ c_2 \neq 0}$ The paths lie on the parabolas $y = x + \left(\frac{c_2}{c_1^2}\right)x^2$

(which go thro' the origin with slope 1)

$y = x + kx^2$

$k > 0$, $k = 1$: $y = x + x^2$

x	-1	0	1	-2
y	0	0	2	2

$k < 0$, $k = -1$: $y = x - x^2$

x	-2	-1	0	1	2
y	-6	-2	0	0	-2

—— o ——

$\underline{\text{Problem 2}}$

$$\left\{\begin{array}{l} \dfrac{dx}{dt} = -y \\[2ex] \dfrac{dy}{dt} = x \end{array}\right.$$

$a_1 = 0,\ b_1 = -1$

$a_2 = 1,\ b_2 = 0$

$$\left.\begin{array}{r} -y = 0 \\ \text{\&}\ x = 0 \end{array}\right\} \Rightarrow (0,0) \text{ is the isolated } \underline{\text{cp}}$$

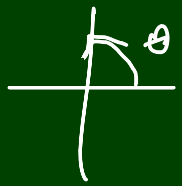
$\underline{\text{Q.e}}$ $\quad m^2 - (a_1 + b_2)m + a_1 b_2 - a_2 b_1 = 0$

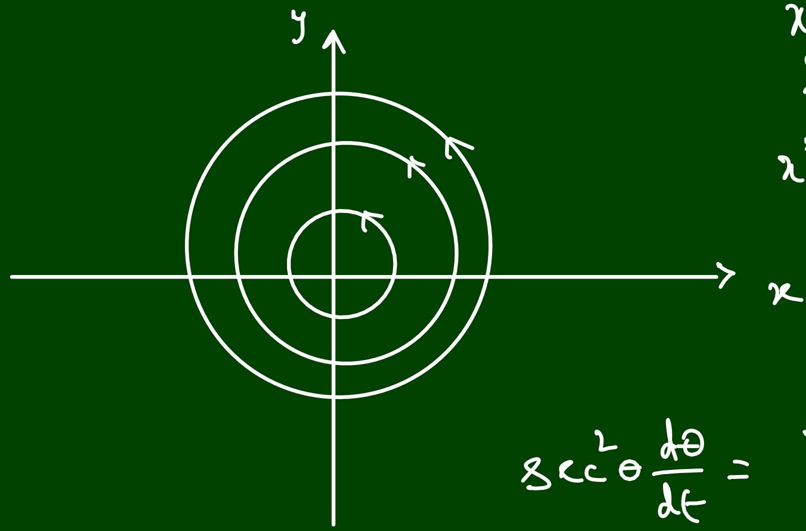
$m^2 + 1 = 0 \Rightarrow m = \pm i$ pure imaginary

$\therefore (0,0)$ is a $\underline{\text{stable center}}$

$$\frac{dy}{dx} = -\frac{x}{y} \quad \Rightarrow \quad x\,dx + y\,dy = 0$$

integrating $\underline{x^2 + y^2 = c^2}$





$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$x^2+y^2 = r^2$$
$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$
$$\tan\theta = \frac{y}{x}$$

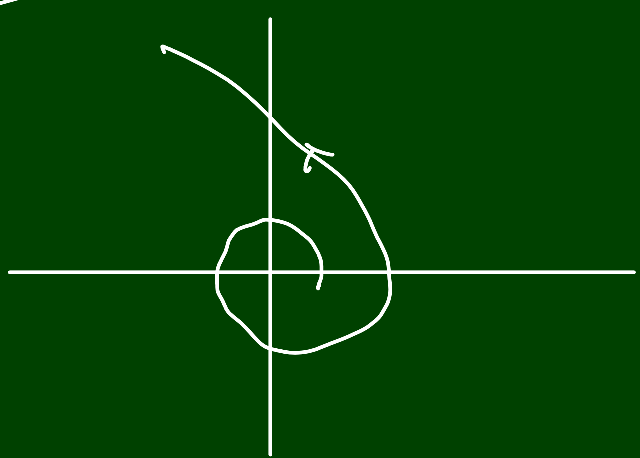
$$\sec^2\theta \frac{d\theta}{dt} = \frac{x\frac{dy}{dt} - y\frac{dx}{dt}}{x^2}$$
$$= \frac{x(x) - y(-y)}{x^2}$$
$$= \frac{x^2+y^2}{x^2} = \frac{r^2}{r^2\cos^2\theta}$$

$$\sec^2\theta \frac{d\theta}{dt} = \sec^2\theta$$

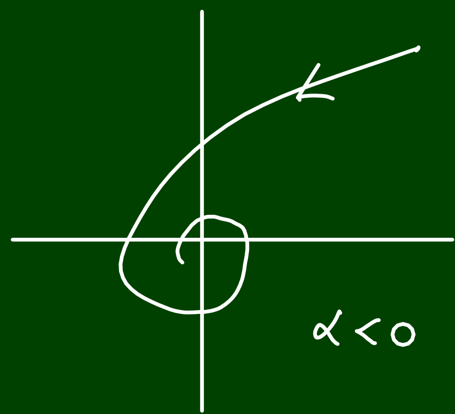
$$\frac{d\theta}{dt} = 1 \Rightarrow \int d\theta = \int dt$$
$$\theta = t$$

when $t\uparrow$, $\theta\uparrow$

Spiral



$\alpha > 0$



$\alpha < 0$

Stability by Liapunov's direct method

Idea: If the total energy of a physical system has a local minimum at a certain equilibrium point, then that point is stable

—

$$(1)\begin{cases} \dfrac{dx}{dt} = F(x,y) \\ \dfrac{dy}{dt} = G(x,y) \end{cases}$$

$(0,0)$ isolated c.p

$C : [x(t), y(t)]$ path of (1)

Consider a function $E(x,y)$, continuous first order partial derivatives $\frac{\partial E}{\partial x}, \frac{\partial E}{\partial y}$

$E(x,y) = E(t)$

$$\frac{dE}{dt} = \frac{\partial E}{\partial x}\frac{dx}{dt} + \frac{\partial E}{\partial y}\frac{dy}{dt}$$

$$= \frac{\partial E}{\partial x}F(x,y) + \frac{\partial E}{\partial y}G(x,y)$$

(1) $E(0,0) = 0$, $E(x,y) > 0$ $\forall (x,y) \neq (0,0)$

$E(x,y)$ is +ve definite

(2) $E(0,0) = 0$, $E(x,y) < 0$ $\forall (x,y) \neq (0,0)$

$E(x,y)$ is −ve definite

(3) $E(0,0) = 0$, $E(x,y) \geq 0$, $\forall (x,y) \neq (0,0) \rightarrow E(x,y)$ is +ve semidefinite

(4) $E(0,0) = 0$, $E(x,y) \leq 0$, $\forall (x,y) \neq (0,0) \rightarrow E(x,y)$ is −ve semidefinite

$E(x,y) = ax^{2m} + by^{2n}$, $a>0$, $b>0$ m and n are +ve integers

$\searrow$ positive definite

<u>Liapunov function</u> : A positive definite function $E(x,y)$ with the property that

$$(*) \quad \frac{dE}{dt} = \frac{\partial E}{\partial x} F + \frac{\partial E}{\partial y} G$$

is <u>negative semidefinite</u>

is called a Liapunov function for the system (1).

<u>Result</u> ① If $\exists$ a Liapunov function $E(x,y)$ for the system ①, then the critical point $(0,0)$ is stable.

② If this function $[E(x,y)]$ has the additional property that $(*)$ is negative definite then the critical point $(0,0)$ is asymptotically stable

<u>Problem</u> Test the stability properties for the critical point of the system

$$\begin{cases} \dfrac{dx}{dt} = -2xy \\[2ex] \dfrac{dy}{dt} = x^2 - y^3 \end{cases}$$

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<u>Soln</u>

$F(x,y) = -2xy \qquad G(x,y) = x^2 - y^3$

$$\left.\begin{aligned} -2xy &= 0 \\ x^2 - y^3 &= 0 \end{aligned}\right\} \quad x = y = 0$$

$(0,0)$ is an isolated critical point.

$$E(x,y) = ax^{2m} + by^{2n}, \quad a>0, \; b>0,$$ m & n are +ve integers

$$\frac{\partial E}{\partial x} = 2max^{2m-1}, \quad \frac{\partial E}{\partial y} = 2nby^{2n-1}$$

$$\frac{\partial E}{\partial x}F + \frac{\partial E}{\partial y}G = 2max^{2m-1}(-2xy) + 2nby^{2n-1}(x^2 - y^3)$$

$$= -4max^{2m} \cdot y + 2nbx^2y^{2n-1} - 2nby^{2n+2}$$

$$= \left(-4max^{2m} \cdot y + 2nbx^2y^{2n-1}\right) - 2nby^{2n+2}$$

make the expression in parenthesis zero.

$$\left.\begin{aligned} 2m = 2 &\Rightarrow m = 1 \\ 2n - 1 = 1 &\Rightarrow n = 1 \end{aligned}\right\} \left(-4ax^2y + 2bx^2y\right)$$

$$\Rightarrow a = 1, \; b = 2$$

$$\therefore E(x,y) = x^2 + 2y^2$$

$E(0,0) = 0$, $E(x,y) > 0 \;\; \forall (x,y) \neq (0,0)$

$E(x,y)$ is +ve definite

$(1,0)$

$$\frac{\partial E}{\partial x}F + \frac{\partial E}{\partial y}G = -4y^4 \leq 0$$ which is -ve semidefinite

$E(x,y) = x^2 + 2y^2$ is a Liapunov function.

$\therefore$ $(0,0)$ is stable.