Ordinary Differential Equations

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UNIT - II

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Recall

Gauss's hypergeometric equation

$$x(1-x)y'' + [c-(a+b+1)x]y' - aby = 0 \quad \text{— (1)}$$

where a, b, c are constants.

$x=0$ and $x=1$ are r.s. points

Case(i) WR consider $x=0$, the r.s. point : $\left[y = x^m \sum_{n=0}^{\infty} a_n x^n, \; a_0 \neq 0 \right]$

$m = m_1 \to y_1$

$m = m_2 \to y_2$

$m_1 - m_2$ is not an integer

$y = a_0 y_1 + a_1 y_2$

$m_1 = m_2 = m$(say)

one solution is assured

$y_1 \leftarrow m$

$y_2 = v y_1$

Std Form of (1).

$$P(x) = \frac{c-(a+b+1)x}{x(1-x)} \to p_0 = c$$

$$Q(x) = \frac{-ab}{x(1-x)} \to q_0 = 0$$

Indicial equation : $m(m-1) + m p_0 + q_0 = 0$

$\Rightarrow m_1 = 0, \; m_2 = 1-c$

we find the solution (y_1) when $m = m_1 = 0$.

Let $y = x^0 \sum_{n=0}^{\infty} a_n x^n$ be a solution of ①.

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$$

$$a_{n+1} = \frac{(a+n)(b+n)}{(n+1)(c+n)} a_n, \quad \forall n \geq 0$$

$$y = 1 + \sum_{n=1}^{\infty} \frac{a(a+1)\cdots(a+n-1)\, b(b+1)\cdots(b+n-1)}{n!\, c(c+1)\cdots(c+n-1)} x^n \qquad -(*)$$

∗ hypergeometric series → $F(a,b,c,x)$

Note: (i) $F(a,b,c,x) = F(b,a,c,x)$

(ii) when $a=1$, $c=b$

$$F(1,b,b,x) = 1 + \sum_{n=1}^{\infty} x^n$$

$$= 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}, \quad |x|<1$$

(iii) when a or b is zero or negative integer

The series $(*)$ breaks off and is a polynomial.

Check: (i) $F(-p,b,b,-x) =$

(ii) $x F(1,1,2,-x) =$

(iii) $x F(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, x^2) =$

(iv) $x F(\frac{1}{2}, 1, \frac{3}{2}, -x^2) =$

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When $m = m_2 = 1-c$: [$1-c$ is not zero or negative integer ie, c is not a +ve integer]

The solution corresponding to $m = m_2 = 1-c$ can be found directly, as follows:

Let $y = x^{1-c} \sum_{n=0}^{\infty} a_n x^n$.

We change the dependent variable in ① from y to z by writing

$$y = x^{1-c} z$$

$y \to y(x)$

$y' = \frac{dy}{dx}$ in ①

but in the transformed equation $z' = \frac{dz}{dx}$

$z = z(x)$

$$y' = x^{1-c} \frac{dz}{dx} + (1-c) x^{1-c-1} \cdot z$$

$$y' = x^{1-c} z' + (1-c) x^{-c} \cdot z$$

$$y'' = x^{1-c} z'' + (1-c) x^{1-c-1} z' + (1-c) \left[x^{-c} z' - c x^{-c-1} z\right]$$

$$y'' = x^{1-c} z'' + (1-c) x^{-c} z' + (1-c) x^{-c} z' - c(1-c) x^{-c-1} z$$

Sub in ①, we get

$$x(1-x)z'' + \left[(2-c) - \{(a-c+1)+(b-c+1)+1\}x\right]z'$$
$$-(a-c+1)(b-c+1)z = 0$$

— ② (check)

which is the hypergeometric equation with the constants a, b and c replaced by $a-c+1$, $b-c+1$ and $2-c$

$$\left\{ x(1-x)y'' + [c-(a+b+1)x]y' - aby = 0 \quad \text{— ①} \right\}$$

We know that ② has a power series solution

$$z = F(a-c+1, b-c+1, 2-c, x)$$ near the origin.

$\therefore$ our required second solution is

$$y = x^{1-c} F(a-c+1, b-c+1, 2-c, x)$$

When c is not an integer, we have

$$y = c_1 F(a,b,c,x) + c_2 x^{1-c} F(a-c+1, b-c+1, 2-c, x)$$

as the general solution of the hypergeometric equation near the regular singular point $x=0$.

Now, we solve (1) near the regular singular point $x=1$.

We introduce a new independent variable $t = 1-x$. [when $x=1 \to t=0$]

(1) becomes

$$t(1-t)y'' + [(a+b-c+1) - (a+b+1)t]y' - aby = 0$$
— (3)

(check) put $x = 1-t$

(derivatives with respect to t)

clearly (3) is a hypergeometric equation. [$a \to a$, $b \to b$, $c \to a+b-c+1$ $x \to t$]

$\therefore$ The general solution is

$$y = c_1 F(a, b, a+b-c+1, t) + c_2 t^{1-c} F[(a-(a+b-c+1)+1, b-(a+b-c+1)+1, 2-(a+b-c+1), t]$$

$\therefore$ The general solution of (1) near $x=1$ is

$$y = c_1 F(a, b, a+b-c+1, 1-x) + c_2 (1-x)^{c-a-b} F[c-b, c-a, c-a-b+1, 1-x]$$

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Recall

(i) $F(-p, b, b, -x) = (1+x)^p$

(ii) $xF(1, 1, 2, -x) = \log(1+x)$

(iii) $xF(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, x^2) = \sin^{-1}x$

(iv) $xF(\frac{1}{2}, 1, \frac{3}{2}, -x^2) = \tan^{-1}x$

Check (v) $\lim_{b\to\infty} F(a, b, a, \frac{x}{b}) = e^x$

$$x(1-x)y'' + [c-(a+b+1)x]y' - aby = 0$$

(i) coeff^{t} of y'' → polynomial of degree 2
coeff^{t} of y' → polynomial of degree 1
coeff^{t} of y → polynomial of degree 0

(ii) polynomial of degree 2 → it has distinct real roots.

Any diff^{l} eq^{n} with these properties can be rewritten as a hypergeometric form (eq^{n}), by a linear <u>change of the independent variable</u>, and hence it can be solved near its singular points interms of hypergeometric function.

Consider the general equation of the type

$$(x-A)(x-B)\,y'' + [C + Dx]\,y' + Ey = 0 \qquad -(1),$$

where $A \neq B$.

Change the independent variable from x to t by means of

$$t = \frac{x-A}{B-A}, \quad -(2)$$

then $x = A$ corresponds to $t = 0$, and $x = B$ to $t = 1$.

By (2), (1) becomes

$$t(1-t)\,y'' + (F + Gt)\,y' + Hy = 0 \qquad -(3),$$

where F, G and H are certain combinations of constants in (1).

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$$x(1-x)\,y'' + \left(c - (a+b+1)\,x\right] y' - ab\,y = 0$$

$x=0$, $x=1$ $\rightarrow$ $t = 1-x$

$$t = \frac{x-A}{B-A}$$

$x=0$ $\quad$ $x=1$

B $\quad$ A

$$t = \frac{x-1}{0-1}$$

$$t = 1-x$$

$A \neq B$

$\boxed{x = A \rightarrow t = 0}$

$x = 1 \rightarrow t = 0$

$\boxed{x = B \rightarrow t = 1}$

$x = 0 \rightarrow t = 1$

Recall: Chebyshev's equation.

We need the general solution near $x = 1$

$$(1-x^2)y'' - xy' + p^2 y = 0, \quad \text{— (1)}$$

p is a non negative constant.

(i)
- coeff^t of $y'' \to 1-x^2 \to$ polynomial of degree 2
- coeff^t of $y' \to -x \to$ polynomial of degree 1
- coeff^t of $y \to p^2 \to$ polynomial of degree 0

(ii) polynomial of degree 2 $\to$ distinct zeros.

$$1-x^2 = (1-x)(1+x) \qquad \underline{x=1, \ x=-1}$$

As (1) satisfies the above characteristics it can be written as a hypergeometric eq^n using the substitution

$$t = \frac{x-A}{B-A}, \quad (A \neq B)$$

Here,

$x = 1 \to A$, $\quad x = -1 \to B$

$$t = \frac{x-1}{(-1)-1}$$

$$\boxed{t = \frac{1-x}{2}}$$

$x = A \to t = 0$

i.e, $x = 1 \to t = 0$

$x = B \to t = 1$

i.e, $x = -1 \to t = 1$

$$y' = \frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx}$$

$$= \frac{dy}{dt}\cdot\left(-\frac{1}{2}\right)$$

$$= -\frac{1}{2}y'$$

$$t = \frac{1-x}{2}$$

$$2t = 1-x$$

$$x = 1-2t$$

$$1+x = 1+1-2t = 2-2t = 2(1-t)$$

$$1-x = 1-(1-2t) = 2t$$

eq^n (1), $(1-x)(1+x)y'' - xy' + p^2 y = 0$

becomes

$$(2t)\,2(1-t)\frac{1}{4}y'' - (1-2t)\left(-\frac{1}{2}y'\right) + p^2 y = 0$$

$$y'' = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dx}\left(\frac{dy}{dt}\left(-\frac{1}{2}\right)\right)$$

$$= \frac{d}{dt}\left(-\frac{1}{2}\frac{dy}{dt}\right)\frac{dt}{dx}$$

$$= -\frac{1}{2}\frac{d^2y}{dt^2}\left(-\frac{1}{2}\right)$$

$$= \frac{1}{4}\frac{d^2y}{dt^2} = \frac{1}{4}y''$$

$$t(1-t)y'' + \left[\frac{1}{2} - t\right]y' + p^2 y = 0 \quad \text{--- (2)}$$

(which is a hypergeometric eqn)

Compare this with

$$x(1-x)y'' + [c-(a+b+1)x]y' - aby = 0$$

$c = \frac{1}{2}$, $a+b+1 = 1$, $ab = -p^2$

$a+b = 0$ $\qquad ab = -p^2 \Rightarrow a = p,\ b = -p$

WKT: $F(a,b,c,x) = F(b,a,c,x)$

* The general soln of (1) near $x=1$ is nothing but the general of (2) near $t=0$.

WKT the general solution of (2) near $t=0$ is

$$y = c_1 F(a,b,c,t) + c_2 t^{1-c} F(a-c+1, b-c+1, 2-c, t)$$

ie, $$y = c_1 F\left(p, -p, \frac{1}{2}, t\right) + c_2 t^{1-\frac{1}{2}} F\left(p-\frac{1}{2}+1, -p-\frac{1}{2}+1, 2-\frac{1}{2}, t\right)$$

$$y = c_1 F\left(p, -p, \frac{1}{2}, t\right) + c_2 t^{\frac{1}{2}} F\left(p+\frac{1}{2}, -p+\frac{1}{2}, \frac{3}{2}, t\right)$$

Hence the general solution of (1) near $x=1$ is

$$y = c_1 F\left(p, -p, \frac{1}{2}, \frac{1-x}{2}\right) +$$

$$c_2 \left(\frac{1-x}{2}\right)^{\frac{1}{2}} F\left(p+\frac{1}{2}, -p+\frac{1}{2}, \frac{3}{2}, \frac{1-x}{2}\right)$$

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Question: What is the general solution of $(x^2-x-6)y''+(5+3x)y'+y=0$ near the singular point $x=3$. ?

(i) coeff of y'' → 2^{nd} order polynomial
coeff of y' → 1^{st} order polynomial
coeff of y → zero order polynomial

(ii) roots of of the 2^{nd} ~~order~~ degree polynomial must be distinct ..

$$x^2-x-6=(x-3)(x+2)$$

$x=3$ ✓
$x=-2$

$$A \neq B$$

$$(x-A)(x-B)y''+(C+Dx)y'+Ey=0$$

We can rewrite the gn eqn in the form of hypergeometric equation as follows:

$$(x-3)(x+2)y''+(5+3x)y'+y=0 \quad (*) \rightarrow \left[y'=\frac{dy}{dx}\right]$$

$$\therefore \; t=\frac{x-A}{B-A}, \quad (A\neq B)$$

$$\therefore \begin{cases} x=A \text{ corresponds to } t=0 \checkmark \\ x=B \text{ corresponds to } t=1 \end{cases}$$

$$t=\frac{x-3}{-2-3}$$

✓$A=3$
$B=-2$

$$t=\frac{3-x}{5} \rightarrow \frac{dt}{dx}=-\frac{1}{5}$$

$$y' = \frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = -\frac{1}{5}\frac{dy}{dt} = -\frac{1}{5}y'$$

$$y'' = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx}$$

$$= \frac{d}{dt}\left(-\frac{1}{5}y'\right)\left(-\frac{1}{5}\right) = \frac{1}{25}y''$$

$$t = \frac{3-x}{5} \Rightarrow 5t = 3-x \Rightarrow x = 3-5t$$

$$x(1-x)y'' + [c-(a+b+1)x]y' - aby = 0$$

$$x-3 = 3-5t-3 = -5t$$

$$x+2 = 3-5t+2 = 5-5t = 5(1-t)$$

(*) becomes

$$(-5t)5(1-t)\frac{1}{25}y'' + \left[5+3(3-5t)\right]\left(-\frac{1}{5}y'\right) + y = 0$$

$$-t(1-t)y'' + [5+9-15t]\left(-\frac{1}{5}y'\right) + y = 0$$

$$-t(1-t)y'' + [14-15t]\left[-\frac{1}{5}y'\right] + y = 0$$

$$t(1-t)y'' + \left(\frac{14}{5} - 3t\right)y' - y = 0 \quad (**)$$

which is the transformed hypergeometric eq^n with a, b and c are replaced with

$$c = \frac{14}{5} \qquad \begin{cases} ab = 1 \\ a=1,\ b=1 \end{cases} \qquad \begin{cases} a+b+1 = 3 \\ a+b = 2 \end{cases}$$

The general solution of (**) near $t=0$ (which corresponds to $x=3$)

$$\checkmark\ y = c_1 F(a,b,c,t) + c_2 t^{1-c} F(a-c+1, b-c+1, 2-c, t)$$

$\therefore$ The general solution of the given equation near the singular point $x=3$ is given by

$$y = c_1 F\left(1, 1, \frac{14}{5}, \frac{3-x}{5}\right) + c_2 \left(\frac{3-x}{5}\right)^{1-\frac{14}{5}} F\left(1-\frac{14}{5}+1, 1-\frac{14}{5}+1, 2-\frac{14}{5}, \frac{3-x}{5}\right)$$

ie, $$y = c_1 F\left(1, 1, \frac{14}{5}, \frac{3-x}{5}\right) + c_2 \left(\frac{3-x}{5}\right)^{-\frac{9}{5}} F\left(-\frac{4}{5}, \frac{-4}{5}, \frac{-4}{5}, \frac{3-x}{5}\right)$$

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HW Find the general solution of the differential equations near the indicated singular point :

(i) $x(1-x)y'' + \left(\frac{3}{2} - 2x\right)y' + 2y = 0$, $x=0$

(ii) $(x^2-1)y'' + (5x+4)y' + 4y = 0$, $x=-1$

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The Point at Infinity

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Consider,

$$y'' + p(x)y' + q(x)y = 0 \quad -(1) \longrightarrow \left[y' = \frac{dy}{dx}\right]$$

What about the solutions of (1) for large values of the independent variable? (ie, when $x \to \infty$) If the variable is time, we want to know how the physical system described by (1) behaves in the distant future. $(t \to \infty)$

we want to study about the solutions near the point at infinity.

We change the independent variable from x to $t = \frac{1}{x}$.

[when $x \to \infty$, $t \to 0$]

$$y' = \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$= \frac{dy}{dt}\left(-\frac{1}{x^2}\right) = (-t^2)\frac{dy}{dt} = (-t^2 y')$$

$t = \frac{1}{x}$

$\frac{dt}{dx} = -\frac{1}{x^2}$

$$y'' = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx}$$

$$= \frac{d}{dt}\left(-t^2\frac{dy}{dt}\right)\left(-t^2\right)$$

$$= \left(-t^2\frac{d^2y}{dt^2} + \frac{dy}{dt}(-2t)\right](-t^2)$$

$$= \left[-t^2 y'' - 2ty'\right]\left[-t^2\right]$$

$t = \frac{1}{x}$

$x = \frac{1}{t}$

$P(x) = P(\frac{1}{t})$

① becomes

$$\left[-t^2y'' - 2ty'\right]\left[-t^2\right] + P\left(\frac{1}{t}\right)\left(-t^2y'\right] + Q\left(\frac{1}{t}\right)y = 0$$

$$t^2y'' + 2ty' - P\left(\frac{1}{t}\right)y' + \frac{1}{t^2}Q\left(\frac{1}{t}\right)y = 0 \quad (\div t^2)$$

$$t^2y'' + \left[2t - P\left(\frac{1}{t}\right)\right]y' + \frac{1}{t^2}Q\left(\frac{1}{t}\right)y = 0$$

$\div t^2$ $$y'' + \left[\frac{2}{t} - \frac{P(1/t)}{t^2}\right]y' + \frac{1}{t^4}Q\left(\frac{1}{t}\right)y = 0 \quad ②$$

Equation (1) has $x=\infty$ as an ordinary point, a regular singular point (with exponents m_1 & m_2) or an irregular singular point, if the point $t=0$ has the corresponding character for the transformed equation (2).

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Example: Determine the nature of the point $x=\infty$ for the Legendre's equation $(1-x^2)y''-2xy'+p(p+1)y=0$.

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Solution The given equation can be written as

$$y''-\frac{2x}{1-x^2}y'+\frac{p(p+1)}{1-x^2}y=0. \quad \text{—— (1)}$$

Here $P(x)=-\frac{2x}{1-x^2}$, $Q(x)=\frac{p(p+1)}{1-x^2}$

Put $t=\frac{1}{x}$.

$$\boxed{P\left(\tfrac{1}{t}\right)=\frac{-2\left(\frac{1}{t}\right)}{1-\frac{1}{t^2}}=\frac{-2\left(\frac{1}{t}\right)t^2}{t^2-1}=\frac{-2t}{t^2-1}}$$

Then the equation

$$y''+\left(\frac{2}{t}-\frac{P\left(\frac{1}{t}\right)}{t^2}\right]y'+\frac{1}{t^4}Q\left(\tfrac{1}{t}\right)y=0 \text{ becomes}$$

$$y''+\left[\frac{2}{t}-\frac{-2(1/t)\Big/1-\left(\frac{1}{t}\right)^2}{t^2}\right]y'+\frac{1}{t^4}\frac{p(p+1)}{1-\left(\frac{1}{t}\right)^2}y=0$$

$$y''+\left(\frac{2}{t}-\frac{-2t/t^2-1}{t^2}\right]y'+\frac{1}{t^4}\frac{p(p+1)t^2}{t^2-1}y=0$$

$$y'' + \left[\frac{2}{t} + \frac{2t}{t^2(t^2-1)}\right] y' + \frac{p(p+1)}{t^2(t^2-1)} y = 0$$

$$y'' + \left(\frac{2t(t^2-1) + 2t}{t^2(t^2-1)}\right] y' + \frac{p(p+1)}{t^2(t^2-1)} y = 0$$

$$y'' + \frac{2t}{t^2-1} y' + \frac{p(p+1)}{t^2(t^2-1)} y = 0 \quad \text{--- (2)}$$

What is the nature of $t=0$ for (2) ?

$$p(t) = \frac{2t}{t^2-1} \qquad q(t) = \frac{p(p+1)}{t^2(t^2-1)}$$

at $\underline{t=0}$, $q(t)$ is not analytic

But $t\,p(t)$, $t^2 q(t)$ are analytic at $t=0$

$\therefore$ $t=0$ is a r.s.p for equation (2)

Hence $x=\infty$ is a r.s.p for equation (1).

<u>Hw</u>: Determine the nature of the point $x=\infty$ for Bessel's equation (of order p)

$$x^2 y'' + x y' + (x^2 - p^2) y = 0.$$

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Special functions

Mathematical physics:

(i) Legendre polynomials ✓

(ii) Bessel functions.

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Recall Legendre's equation:

$$\boxed{(1-x^2)y'' - 2xy' + p(p+1)y = 0, \quad p \text{ is const.}}$$

$|x| < 1$, $-1 < x < 1$

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0, \quad (1)$$

where n is understood to be a non-negative integer.

(i) Coeff^t of $y'' \rightarrow$ polynomial of degree 2
 " $y' \rightarrow$ polynomial of degree 1
 " $y \rightarrow$ polynomial of degree 0

(ii) $1-x^2 = (1-x)(1+x)$ distinct roots: $x = 1,\ x = -1$

(1) Can be written as a hypergeometric eq^n as follows:

put $t = \dfrac{x-A}{B-A}$ $A = 1$, $B = -1$

$x = A \rightarrow t = 0$
$x = B \rightarrow t = 1$

$$t = \frac{x-1}{-1-1}$$

$$t = \frac{1-x}{2} \quad \therefore 2t = 1-x$$

$$x = 1-2t$$

$$1-x = 1-(1-2t) = 2t$$

$$1+x = 1-2t+1 = 2-2t = 2(1-t)$$

$\checkmark\ t = \dfrac{1-x}{2} \Rightarrow \dfrac{dt}{dx} = -\dfrac{1}{2}$

$$y' = \frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{dy}{dt}\left(-\frac{1}{2}\right) = -\frac{1}{2}y'$$

$$y'' = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(-\frac{1}{2}\frac{dy}{dt}\right)\frac{dt}{dx}$$

$$= \frac{1}{4}\frac{d^2y}{dt^2} = \frac{1}{4}y''$$

(1) $\Rightarrow (1-x)(1+x)y'' - 2xy' + n(n+1)y = 0$

$\Rightarrow (2t)2(1-t)\frac{1}{4}y'' - 2(1-2t)\left(-\frac{1}{2}y'\right) + n(n+1)y = 0$

$\Rightarrow t(1-t)y'' + [1-2t]y' + n(n+1)y = 0$, which is — (2)

a hypergeometric equation with

$a = -n$
$b = (n+1)$
$c = 1$

$x(1-x)y'' + [c-(a+b+1)x]y' - aby = 0$

compare this with eqⁿ (2).

$c = 1$
$a+b+1 = 2$
$a+b = 1$ ✓
$ab = -n(n+1)$ ✓
$a = -n,\ b = n+1$

W.k.T the exponents of (2) at the origin are $m_1 = 0,\ m_2 = 1-c$

∴ Here $m_1 = m_2 = 0$.

Equation (2) has the following polynomial solution near $t = 0$ as

$$y_1 = F(-n, n+1, 1, t)$$

As $m_1 = m_2 = 0$, we seek a second solution (y_2) by assuming $y_2 = v y_1$ where v is a function of t.

Now $y_2 = v y_1 \Rightarrow v' = \frac{1}{y_1^2} e^{-\int p(t)dt}$

$$\Rightarrow v' = \frac{1}{y_1^2}\left[\frac{1}{t(1-t)}\right]$$

$$\Rightarrow v' = \frac{1}{t}\left[\frac{1}{y_1^2(1-t)}\right]$$

$p(t) = \frac{1-2t}{t(1-t)}$

$$e^{-\int p(t)dt} = e^{\int \frac{2t-1}{t(1-t)}dt}$$

$$= e^{\int \frac{2t-1}{t-t^2}dt}$$

$$= e^{-\log t(1-t)}$$

$$= \frac{1}{t(1-t)}$$

Since y_1^2 is a polynomial with constant term 1, the bracketed expression on the r.h.s is an analytic function of the form $1 + a_1 t + a_2 t^2 + \cdots$

$\therefore$ we have

$$v' = \frac{1}{t}\left[1 + a_1 t + a_2 t^2 + \cdots\right]$$

$$v' = \frac{1}{t} + a_1 + a_2 t + \cdots$$

$$\Rightarrow v = \log t + a_1 t + \frac{a_2}{2} t^2 + \cdots$$

$$\therefore y_2 = y_1 \cdot \left[\log t + a_1 t + \cdots\right]$$

$\therefore$ The general solution of (2) near $t=0$ is

$$y = c_1 y_1 + c_2 y_2 \quad \text{--- (3)}$$

Because of the presence of the term $\log t$ in y_2, it is clear that (3) is bdd near $t=0$

if and only if $c_2 = 0$.

$\therefore$ we have only one solution y_1 for ② near $t = 0$.

$\therefore$ The solution of ① bdd near $x = 1$ are constant multiples of the polynomial

$$F\left(-n, n+1, 1, \frac{1-x}{2}\right).$$

Hence the n^{th} Legendre polynomial denoted by $P_n(x)$ is defined by

$$P_n(x) = F\left(-n, n+1, 1, \frac{1-x}{2}\right)$$

$$= 1 + \frac{(-n)(n+1)}{1 \cdot 1}\left(\frac{1-x}{2}\right)$$

$$+ \frac{(-n)(-n+1)(n+1)(n+2)}{1 \cdot 2 \cdot 1 \cdot (1+1)}\left(\frac{1-x}{2}\right)^2$$

$$+ \cdots\cdots$$

$$+ \frac{(-n)(-n+1)\cdots(-n+(n-1))(n+1)(n+2)\cdots(2n)}{1 \cdot 2 \cdots n \cdot 1(1+1)\cdots(1+n-1)}\left(\frac{1-x}{2}\right)^n$$

ie, $P_n(x) = 1 + \dfrac{n(n+1)}{(1!)^2 \cdot 2}(x-1) + \dfrac{n(n-1)(n+1)(n+2)}{(2!)^2\, 2^2}(x-1)^2$

$$+ \cdots + \frac{(2n)!}{(n!)^2 2^n}(x-1)^n$$

$\underline{n=0}$ $\quad P_0(x) = 1$

$\underline{n=1}$ $\quad P_1(x) = 1 + \frac{1\cdot 2}{1\cdot 2}(x-1) = 1 + x - 1 = x$

$\underline{n=2}$ $\quad P_2(x) = 1 + \frac{2\cdot 3}{1\cdot 2}(x-1) + \frac{2\cdot 1\cdot 3\cdot 4}{4\cdot 4}(x-1)^2$

$$= 1 + 3(x-1) + \frac{3}{2}(x-1)^2$$

$$= 1 + 3x - 3 + \frac{3}{2}(x^2 - 2x + 1)$$

$$= 1 + \cancel{3x} - 3 + \frac{3}{2}x^2 - \cancel{3x} + \frac{3}{2}$$

$$= \frac{3}{2}x^2 - \frac{1}{2} = \frac{1}{2}(3x^2 - 1)$$

$1 + \frac{3}{2}$

$\frac{5}{2} - 3$

$-\frac{1}{2}$

$\underline{n=3}$
$\underline{\text{(check)}}$ $\quad P_3(x) = \frac{1}{2}(5x^3 - 3x)$

<u>Note</u>: $\checkmark P_0(x) = 1$, $\times P_1(x) = x$, $\checkmark P_2(x) = \frac{1}{2}(3x^2 - 1)$

$\times P_3(x) = \frac{1}{2}(5x^3 - 3x)$

$P_n(x)$ is a polynomial of degree n that contains <u>only even</u> or <u>only odd</u> powers of x according as 'n' is even or odd.

Hence $P_n(x)$ can be written in the form

$$P_n(x) = a_n x^n + a_{n-2} x^{n-2} + a_{n-4} x^{n-4} + \cdots, \quad (*)$$

where this sum ends with a_0 if n is even and $a_1 x$ if n is odd.

$n=0$ in $(*)$ $\quad P_0(x) = a_0$

$n=4$ in $(*)$ $\quad P_4(x) = a_4 x^4 + a_2 x^2 + a_0$

$n=5$ in $(*)$ $\quad P_5(x) = a_5 x^5 + a_3 x^3 + a_1 x$

Rodrigue's formula:

* page: 338–339

$$P_n(x) = \frac{1}{2^n . n!} \frac{d^n}{dx^n} (x^2-1)^n$$

$$P_0(x) = 1, \quad P_1(x) = \frac{1}{2^1 1!} \frac{d}{dx}(x^2-1) = \frac{1}{2}(2x) = x$$

$$P_2(x) = \frac{1}{2^2 . 2!} \frac{d^2}{dx^2}(x^2-1)^2 = \frac{1}{4 . 2} \frac{d^2}{dx^2}\left[x^4 - 2x^2 + 1\right]$$

$$= \frac{1}{4 . 2}\left[12x^2 - 4\right] = \frac{3}{2}x^2 - \frac{1}{2}$$

$$= \frac{1}{2}(3x^2 - 1)$$

Check $P_3(x) =$

Generating function:

The function on the left side of

$$\frac{1}{\sqrt{1-2xt+t^2}} = P_0(x) + P_1(x)t + P_2(x)t^2 + \cdots + P_n(x)t^n + \cdots$$

is called the generating function of the Legendre Polynomials.

— o —

$$\frac{1}{\sqrt{1-2xt+t^2}} = \frac{1}{\sqrt{1-t(2x-t)}}$$

$$= \left[1-t(2x-t)\right]^{-\frac{1}{2}}$$

$$= 1 + \frac{1}{2}t(2x-t) + \frac{1\cdot 3}{2^2\cdot 2!}t^2(2x-t)^2 +$$

$$\cdots + \frac{1\cdot 3\cdots(2n-3)}{2^{n-1}(n-1)!}t^{n-1}(2x-t)^{n-1}$$

(check)

$$+ \frac{1\cdot 3\cdots(2n-1)}{2^n\cdot n!}t^n(2x-t)^n + \cdots$$

Coeff^t of t^0 : $1 \to P_0(x)$

Coeff^t of t : $\frac{1}{2}(2x) = x \to P_1(x)$

Coeff^t of t^2 : $-\frac{1}{2} + \frac{1\cdot 3}{4\cdot 2}4x^2 = -\frac{1}{2} + \frac{3}{2}x^2$

$= \frac{1}{2}(3x^2-1) \to P_2(x)$

$$\boxed{\begin{aligned} &\frac{1}{2}\left[2xt - t^2\right] \\ &t^2\left[4x^2 - 4xt + t^2\right] \\ &4x^2t^2 - 4xt^3 + 4t^4 \end{aligned}}$$

$P_n(1) = 1$, $P_n(-1) = (-1)^n$ (check)

— o —

Consider
$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n\geq 0}^{\infty} P_n(x)t^n$$

Differentiating both sides w.r.t 't'

$$-\frac{1}{2}(1-2xt+t^2)^{-\frac{3}{2}}(-2x+2t) = \sum_{n\geq 1}^{\infty} n P_n(x) t^{n-1}$$

$$(1-2xt+t^2)^{-\frac{1}{2}}$$

$$-\frac{1}{2}(1-2xt+t^2)^{-\frac{1}{2}-1}\cdot(-2x+2t)$$

$$(x-t)(1-2xt+t^2)^{-\frac{3}{2}} = \sum_{n=1}^{\infty} n P_n(x) t^{n-1}$$

$$(x-t)(1-2xt+t^2)^{-3/2}\cdot(1-2xt+t^2) = (1-2xt+t^2)\sum_{n\geq 1}^{\infty} n P_n(x) t^{n-1}$$

$$(x-t)(1-2xt+t^2)^{-\frac{1}{2}} = (1-2xt+t^2)\sum_{n\geq 1}^{\infty} n P_n(x) t^{n-1}$$

$$(x-t)\sum_{n\geq 0}^{\infty} P_n(x)t^n = (1-2xt+t^2)\sum_{n\geq 1}^{\infty} n P_n(x) t^{n-1}$$

$$(x-t)\left[P_0(x) + P_1(x)t + \cdots + P_{n-1}(x)t^{n-1} + P_n(x)t^n + \cdots\right]$$

$$= \left[1-2xt+t^2\right]\left[P_1(x) + 2P_2(x)t + 3P_3(x)t^2 + \cdots + (n-1)P_{n-1}(x)t^{n-2} + nP_n(x)t^{n-1} + (n+1)P_{n+1}(x)t^n + \cdots\right]$$

Equating the coeff^{t} of t^n on both sides

$$x P_n(x) - P_{n-1}(x) = (n+1) P_{n+1}(x) - 2xn P_n(x) + (n-1) P_{n-1}(x)$$

$$x P_n(x) + 2xn P_n(x) = (n+1) P_{n+1}(x) + P_{n-1}(x) + (n-1) P_{n-1}(x)$$

$$x P_n(x)[2n+1] = (n+1) P_{n+1}(x) + P_{n-1}(x)[1+n-1]$$

$$x P_n(x)(2n+1) = (n+1) P_{n+1}(x) + n P_{n-1}(x)$$

$$\boxed{(n+1) P_{n+1}(x) = (2n+1) x P_n(x) - n P_{n-1}(x)}$$

recursion formula

WKT

$P_0(x) = 1$

$P_1(x) = x$

$P_2(x) = \frac{1}{2}(3x^2-1)$

$\underline{\underline{n=1}}$ $\quad 2 P_2(x) = 3x P_1(x) - P_0(x)$

$$\Rightarrow P_2(x) = \frac{3}{2} x P_1(x) - \frac{1}{2} P_0(x)$$

$$= \frac{3}{2} x [x] - \frac{1}{2}(1)$$

$$= \frac{3}{2} x^2 - \frac{1}{2} = \underline{\underline{\frac{1}{2}(3x^2-1)}}$$

$\underline{\underline{n=2}}$ $\quad 3 P_3(x) = 5x P_2(x) - 2 P_1(x)$

A. Tcm
10/12/2020

Legendre's equation

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0,$$

n, nonnegative integer.

$$\frac{d}{dx}\left[(1-x^2)y'\right] + n(n+1)y = 0$$

$p(p+1)y$

p is a const

$y = a_0 \{\underset{y_1}{\text{even}}\} + a_1 \{\underset{y_2}{\text{odd}}\}$

$P_0(x)$, $P_1(x)$, $P_2(x)$ .... who are they?

$P_n(x) \rightarrow n^{th}$ Legendre polynomial

particular solution of Legendre eq^n.

$n=1 \Rightarrow n(n+1) = 2$

$n=1 \rightarrow P_1(x)$ is the solution of

✓ $(1-x^2)y'' - 2xy' + 2y = 0$

ie, $(1-x^2)P_1''(x) - 2xP_1'(x) + 2P_1(x) = 0$ —(*)

W.K.T $P_1(x) = x \Rightarrow P_1'(x) = 1$, $P_1''(x) = 0$

Sub in (*)

$$(1-x^2)(0) - 2x(1) + 2(x) = -2x + 2x = 0$$

$n = 2$, $n(n+1) = 6$

$n=2 \rightarrow P_2(x)$ is the solution of

$$(1-x^2)y'' - 2xy' + 6y = 0$$

ie, $(1-x^2)P_2''(x) - 2xP_2'(x) + 6P_2(x) = 0$ —(*)

W.K.T $P_2(x) = \frac{1}{2}(3x^2-1)$, $P_2'(x) = 3x$, $P_2''(x) = 3$

Sub in (*) $(1-x^2)(3) - 2x(3x) + 6 \cdot \frac{1}{2}(3x^2-1)$

$$= 3 - 3x^2 - 6x^2 + 9x^2 - 3$$

$$= 0$$

$$|x| < 1$$

$$-1 < x < 1$$

Orthogonality : $\int_{-1}^{1} P_m(x) P_n(x)\, dx = \begin{cases} 0 & \text{if } m \neq n \\ \frac{2}{2n+1} & \text{if } m = n \end{cases}$

Proof:

Case (i) $m \neq n$

Since $P_m(x)$ and $P_n(x)$ satisfy Legendre's equation, we have

$$(1-x^2) P_m''(x) - 2x P_m'(x) + m(m+1) P_m(x) = 0 \quad -①$$

and $(1-x^2) P_n''(x) - 2x P_n'(x) + n(n+1) P_n(x) = 0 \quad -②$

$$① P_n(x) \Rightarrow (1-x^2) P_m''(x) P_n(x) - 2x P_m'(x) P_n(x) + m(m+1) P_m(x) P_n(x) = 0$$

$$② P_m(x) \Rightarrow (1-x^2) P_n''(x) P_m(x) - 2x P_n'(x) P_m(x) + n(n+1) P_n(x) P_m(x) = 0$$

$$(-) \quad (+) \quad (-)$$

$$(1-x^2)[P_m'' P_n - P_n'' P_m] - 2x[P_m' P_n - P_n' P_m] + [m(m+1) - n(n+1)] P_m P_n = 0 \quad -③$$

Now,

$$\frac{d}{dx}\left[(1-x^2)(P_n P_m' - P_n' P_m)\right] = (1-x^2)\left[P_n P_m'' + \cancel{P_n' P_m'} - \cancel{P_n' P_m'} - P_n'' P_m\right] + \left[P_n P_m' - P_n' P_m\right](-2x)$$

$\therefore$ ③ becomes $\frac{d}{dx}\left[(1-x^2)(P_n P_m' - P_n' P_m)\right] + [m^2 + m - n^2 - n] P_m P_n = 0$

$$\Rightarrow \frac{d}{dx}\left\{(1-x^2)(P_n P_m' - P_n' P_m)\right] + (m-n)(m+n+1) P_m P_n = 0$$

$m^2 - n^2 = (m-n)(m+n)$

$(m^2-n^2) + (m-n)$

$= (m-n)(m+n) + (m-n)$

$= (m-n)(m+n+1)$

$$\Rightarrow \frac{d}{dx}(1-x^2)(P_n P_m' - P_n' P_m) = (n-m)(m+n+1) P_m P_n$$

Integrating both sides w.r.t x from -1 to 1, we get

$$\left[(1-x^2)(P_n P_m' - P_n' P_m)\right]_{-1}^{1} = (n-m)(m+n+1)\int_{-1}^{1} P_m(x) P_n(x)\, dx$$

$$\Rightarrow (n-m)(m+n+1)\int_{-1}^{1} P_m(x) P_n(x)\, dx = 0$$

$$\Rightarrow \int_{-1}^{1} P_m(x) P_n(x)\, dx = 0 \quad \text{as } m \neq n$$

Check

$P_0(x) = 1$
$P_1(x) = x$ | $0 \neq 1$

$$\int_{-1}^{1} P_0(x) P_1(x)\, dx$$

$$= \int_{-1}^{1} 1 \cdot x\, dx = \int_{-1}^{1} x\, dx$$

$$= \left[\frac{x^2}{2}\right]_{-1}^{1} = 0$$

Check

$P_2(x)$ & $P_3(x)$

<u>Case (ii)</u> $m = n$

$$\int_{-1}^{1} P_m(x) P_n(x)\, dx = \frac{2}{2n+1}$$

i.e., $$\int_{-1}^{1} \left[P_n(x)\right]^2 dx = \frac{2}{2n+1}$$

<u>Check</u> $n=1$

$$\int_{-1}^{1} P_1(x) P_1(x)\, dx$$

$$= \int_{-1}^{1} x \cdot x\, dx$$

$$= \int_{-1}^{1} x^2\, dx$$

$$= \left[\frac{x^3}{3}\right]_{-1}^{1}$$

$$= \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

$n=1$ $$\frac{2}{2n+1} = \frac{2}{2+1} = \frac{2}{3}$$

A. Tewari
15/12/2020

Recall Test

Question 2

$x, \; x^2$

$y_1 \quad y_2$

$y'' + p(x)y' + q(x)y = 0$ ✓

✓ $p(x) = -\dfrac{y_1 y_2'' - y_2 y_1''}{W(y_1, y_2)}$

$W[y_1, y_2] \neq 0$

✓ $q(x) = \dfrac{y_1' y_2'' - y_2' y_1''}{W(y_1, y_2)}$

Question 3

$(1-x^2)y'' - 2xy' + 2y = 0$ ✓ $(p=1)$ nonnegative integer

$P_n(x)$: Particular soln

$\therefore x$ ✓ $P_1(x) = x$

$\boxed{\text{Given one soln is } x}$

Asked: to find the general soln

$y = c_1 y_1 + c_2 y_2$, y_1 & y_2 are l.i.

✓ Given: $y_1 = x$

To find y_2: $y_2 = \int y_1$

$v = \int \frac{1}{y_1^2} e^{-\int p(x)dx} dx$

Q^n: 4

Method of variation of parameters.

Q^n: 5

✓ R, $(x_0 - R, \; x_0 + R)$ $\quad x_0 \neq 0$, $x_0 = 1$

$x^n \to x_0 = 0$

$(x-1)^n \to x_0 = 1$

$$\sum_{n=0}^{\infty} (-1)^n n (x-1)^n$$

$a_n = (-1)^n \cdot n$

$\dfrac{(-1)^n}{(-1)^n(-1)}$

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} \left| \frac{(-1)^n n}{(-1)^{n+1}(n+1)} \right|$$

$$= \lim_{n\to\infty} \left(\frac{n}{n(1+\frac{1}{n})} \right) = 1$$

$(x_0 - R, \; x_0 + R) = (0, 2)$

Q^n: 6 direct question.

Q^n: 7 Is $x=0$ an ordinary point?

$$(1+x^2)y'' + 2xy' - 2y = 0$$

do this as a H.W

Std form:

$$y'' + \frac{2x}{1+x^2}y' - \frac{2}{1+x^2}y = 0$$

$$P(x) = \frac{2x}{1+x^2} \qquad Q(x) = -\frac{2}{1+x^2}$$

At $x=0$, $P(x)=0$, $Q(x)=-2$, both are analytic at $x=0$

$\therefore$ $x=0$ is an ordinary pt.

$$y = \sum_{n=0}^{\infty} a_n x^n \text{ as a sol}^n.$$

$$y = a_0\{1 + x\tan^{-1}x\} + a_1 x \checkmark$$

Case (ii) $\int_{-1}^{1} P_m(x) P_n(x)\, dx = \frac{2}{2n+1}$, $n = m$

Pf: To prove $\int_{-1}^{1} [P_n(x)]^2\, dx = \frac{2}{2n+1}$

We know that

$$[1-2xt+t^2]^{-\frac{1}{2}} = \sum_{n\geq 0}^{\infty} t^n P_n(x) \quad -(1)$$

Also (1) can be rewritten as

$$[1-2xt+t^2]^{-\frac{1}{2}} = \sum_{m\geq 0}^{\infty} t^m P_m(x) \quad -(2)$$

Multiplying (1) & (2)

$$[1-2xt+t^2]^{-1} = \sum_{n\geq 0}^{\infty} \sum_{m\geq 0}^{\infty} P_n(x) P_m(x)\, t^{n+m}$$

ie, $$\sum_{n\geq 0}^{\infty} \sum_{m\geq 0}^{\infty} P_n(x) P_m(x)\, t^{n+m} = \frac{1}{1-2xt+t^2}$$

Integrating both sides w.r.t x, we get

$$\sum_{n\geq 0}^{\infty} \sum_{m\geq 0}^{\infty} \left\{ \int_{-1}^{1} P_n(x) P_m(x)\, dx \right\} t^{n+m} = \int_{-1}^{1} \frac{dx}{1-2xt+t^2}$$

$$\sum_{n\geq 0}^{\infty} \left\{ \int_{-1}^{1} [P_n(x)]^2 dx \right\} t^{2n} = \int_{-1}^{1} \frac{dx}{1-2xt+t^2} \quad \text{(using case (i))}$$

$\boxed{-2t\, dx}$

$$= \left[-\frac{1}{2t} \log\left[1-2xt+t^2\right] \right]_{-1}^{1}$$

$$= -\frac{1}{2t} \left\{ \log\left[1-2t+t^2\right] - \log\left[1+2t+t^2\right] \right\}$$

$$= -\frac{1}{2t}\left\{\log(1-t)^2 - \log(1+t)^2\right\}$$

$$= -\frac{1}{2t}\left\{2\log(1-t) - 2\log(1+t)\right\}$$

$$= -\frac{1}{t}\left\{\log(1-t) - \log(1+t)\right\}$$

$$= \frac{1}{t}\left\{\log(1+t) - \log(1-t)\right\}$$

$$= \frac{1}{t}\left\{\left[t - \frac{t^2}{2} + \frac{t^3}{3} - \cdots\right] - \left[-t - \frac{t^2}{2} - \frac{t^3}{3} - \cdots\right]\right\}$$

$$= \frac{1}{t}\left\{2t + 2\frac{t^3}{3} + 2\frac{t^5}{5} + \cdots\right\}$$

$$= \frac{2}{t}\left\{t + t^3/3 + t^5/5 + \cdots\right\}$$

$$\sum_{n=0}^{\infty}\left\{\int_{-1}^{1}[P_n(x)]^2\,dx\right\} t^{2n} = \frac{2}{t}\sum_{n=0}^{\infty}\frac{t^{2n+1}}{2n+1}$$

$$\sum_{n=0}^{\infty}\left\{\int_{-1}^{1}[P_n(x)]^2\,dx\right\} t^{2n} = \sum_{n=0}^{\infty}\frac{2t^{2n}}{2n+1}$$

Equating the coeff^{t} of t^{2n} on both sides

$$\boxed{\int_{-1}^{1}[P_n(x)]^2\,dx = \frac{2}{2n+1}}$$

Hence $$\int_{-1}^{1}P_n(x)P_m(x)\,dx = \begin{cases} 0, & \text{if } n \neq m \\ \dfrac{2}{2n+1}, & \text{if } n = m \end{cases}$$

Note Expanding an arbitrary function $f(x)$ in a Legendre series:

$$\text{ie,} \quad f(x) = \sum_{n=0}^{\infty} a_n P_n(x) \quad \text{---} \quad (*)$$

$$\text{ie,} \quad f(x) = a_0 P_0(x) + a_1 P_1(x) + a_2 P_2(x) + \cdots$$

For this we have to find $a_0, a_1, a_2, \ldots$

Multiply $(*)$ by $P_m(x)$ and integrate (term by term) from -1 to 1 w.r.t x, we get

$$\int_{-1}^{1} f(x) P_m(x)\, dx = \sum_{n=0}^{\infty} a_n \int_{-1}^{1} P_m(x) P_n(x)\, dx$$

$$\Rightarrow \int_{-1}^{1} f(x) P_m(x)\, dx = \frac{2 a_m}{2m+1}$$

$\underline{m=1}$

$$\int_{-1}^{1} P_1(x) P_1(x) dx = \frac{2}{2n+1}$$

$\underline{n=1}$ $\quad \frac{2}{3}$

$\underline{m=2}$

$$\int_{-1}^{1} P_2(x) P_2(x) dx = \frac{2}{5}$$

$$\text{ie,} \quad \int_{-1}^{1} f(x) P_n(x)\, dx = \frac{2 a_n}{2n+1}$$

$$\Rightarrow \boxed{a_n = \left[\frac{2n+1}{2}\right] \int_{-1}^{1} f(x) P_n(x)\, dx}$$

D. Tewari 16/12/2020

Recall: Legendre series (so called)

$f(x) \rightarrow$ arbitrary function

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x), \quad \text{where}$$

$$a_n = \left(\frac{2n+1}{2}\right)\int_{-1}^{1} f(x) P_n(x)\, dx .$$

Problem: Expand $f(x)$ in a series of Legendre polynomials if $f(x) = \begin{cases} 0, & -1 < x < 0 \\ 1, & 0 < x < 1 \end{cases}$

—o—

Given that $f(x) = \begin{cases} 0, & -1 < x < 0 \\ 1, & 0 < x < 1. \end{cases}$

Let $f(x) = \sum_{n=0}^{\infty} a_n P_n(x)$, where $a_n = \left(\frac{2n+1}{2}\right)\int_{-1}^{1} f(x) P_n(x)\, dx$

ie, $f(x) = a_0 P_0(x) + a_1 P_1(x) + \cdots + a_n P_n(x) + \cdots$

Now, $a_0 = \frac{1}{2}\int_{-1}^{1} f(x) P_0(x)\, dx = \frac{1}{2}\int_{-1}^{1} f(x) \cdot 1\, dx$

$$= \frac{1}{2}\left\{\int_{-1}^{0} 0 \cdot 1\, dx + \int_{0}^{1} 1 \cdot 1\, dx\right\}$$

$$a_0 = \frac{1}{2}\left[x\right]_0^1 = \frac{1}{2}$$

$$a_1 = \frac{3}{2}\int_{-1}^{1} f(x) P_1(x)\, dx = \frac{3}{2}\left\{\int_{-1}^{0} f(x) P_1(x)\, dx + \int_{0}^{1} f(x) P_1(x)\, dx\right\}$$

$$= \frac{3}{2}\left\{\int_{-1}^{0} 0 \cdot x\, dx + \int_{0}^{1} 1 \cdot x\, dx\right\} = \frac{3}{2}\left[\frac{x^2}{2}\right]_0^1 = \frac{3}{4}$$

$$a_2 = \frac{5}{2}\int_{-1}^{1} f(x)\,P_2(x)\,dx = \frac{5}{2}\left\{\int_{-1}^{0} f(x)\,P_2(x)\,dx + \int_{0}^{1} f(x)\,P_2(x)\,dx\right\}$$

$$= \frac{5}{2}\left\{\int_{-1}^{0} 0\cdot\frac{1}{2}(3x^2-1)\,dx + \int_{0}^{1} 1\cdot\frac{1}{2}(3x^2-1)\,dx\right\}$$

$$= \frac{5}{2}\left(\frac{1}{2}\left[\frac{3x^3}{3} - x\right]_0^1\right) = \frac{5}{4}\left[0\right] = 0$$

$$a_3 = \frac{7}{2}\int_{-1}^{1} f(x)\,P_3(x)\,dx = \frac{7}{2}\left\{\int_{-1}^{0} f(x)\,P_3(x)\,dx + \int_{0}^{1} f(x)\,P_3(x)\,dx\right\}$$

$$= \frac{7}{2}\left\{\int_{-1}^{0} 0\cdot\frac{1}{2}(5x^3-3x)\,dx + \int_{0}^{1} 1\cdot\frac{1}{2}(5x^3-3x)\,dx\right\}$$

$$= \frac{7}{2}\left\{\frac{1}{2}\left[\frac{5x^4}{4} - \frac{3x^2}{2}\right]_0^1\right\} = \frac{7}{4}\left(\frac{5}{4} - \frac{3}{2}\right)$$

$$\vdots \qquad\qquad = -\frac{7}{16}$$

$$\therefore f(x) = \frac{1}{2}P_0(x) + \frac{3}{4}P_1(x) - \frac{7}{16}P_3(x) + \cdots + a_n P_n(x) + \cdots$$

where $a_n = \frac{2n+1}{2}\int_{-1}^{1} f(x)\,P_n(x)\,dx$

HW find the first three terms of the Legendre series of

$$f(x) = \begin{cases} 0, & -1 \le x < 0 \\ x, & 0 \le x \le 1 \end{cases}$$

Hint $f(x) = a_0 P_0(x) + a_1 P_1(x) + a_2 P_2(x)$; find a_0, a_1 and a_2.

GATE Let $P_n(x)$ be the Legendre polynomial of degree $n \geq 0$. If $1 + x^{10} = \sum_{n=0}^{10} a_n P_n(x)$, then a_5 equals :

✓(a) 0 (b) $2/11$ (c) 1 (d) $11/2$

—o—

GATE Let $P_n(x)$ denote the Legendre polynomial of degree n. If $f(x) = \begin{cases} x, & -1 \leq x < 0 \\ 0, & 0 \leq x \leq 1 \end{cases}$

and $f(x) = a_0 P_0(x) + a_1 P_1(x) + a_2 P_2(x) + \cdots$ then

(a) $a_0 = -\frac{1}{4}$, $a_1 = -\frac{1}{2}$ (b) $a_0 = -\frac{1}{4}$, $a_1 = 1/2$

(c) $a_0 = 1/2$, $a_1 = -\frac{1}{4}$ (d) $a_0 = -\frac{1}{2}$, $a_1 = -\frac{1}{4}$

—o—

Bessel functions

The differential equation (1)

$$x^2 y'' + x y' + (x^2 - p^2) y = 0,$$ where p is a non-negative constant, is called Bessel's equation, (of order p) and its solutions are called Bessel functions

Std form of (1) is :

$$y'' + \frac{1}{x} y' + \frac{(x^2 - p^2)}{x^2} y = 0$$

$P(x) = \frac{1}{x}$, $Q(x) = \frac{x^2 - p^2}{x^2}$ $x = 0$ is a singular point.

But both $xP(x) = 1$ and $x^2 Q(x) = x^2 - p^2$ are analytic at $x=0$. $\therefore$ $x=0$ is a r.s.p.

$y = x^m \sum_{n=0}^{\infty} a_n x^n$ $(a_0 \neq 0)$ Frobenius series

Now $xp(x) = 1 \Rightarrow p_0 = 1$

$x^2 q(x) = x^2 - p^2 \Rightarrow q_0 = -p^2$

$\therefore$ The indicial equation $m(m-1) + p_0 m + q_0 = 0$ becomes

$$m(m-1) + m - p^2 = 0$$

$$\Rightarrow m^2 - m + m - p^2 = 0 \Rightarrow m^2 - p^2 = 0 \Rightarrow m^2 = p^2 \Rightarrow m = \pm p$$

$m_1 = p, \ m_2 = -p$

$\because m_1 - m_2 = 2p$ if p is an integer then we have a problem

$\therefore$ equation ① has a solution of the form $y = x^p \sum_{n=0}^{\infty} a_n x^n$, $(a_0 \neq 0)$

i.e, $y = \sum_{n=0}^{\infty} a_n x^{p+n}$, $y' = \sum_{n=0}^{\infty} (p+n) a_n x^{p+n-1}$

$$y'' = \sum_{n=0}^{\infty} (p+n)(p+n-1) a_n x^{p+n-2}$$

$\therefore$ ① becomes

$$x^2 \sum_{n=0}^{\infty} (p+n)(p+n-1) a_n x^{p+n-2} + x \sum_{n=0}^{\infty} (p+n) a_n x^{p+n-1}$$

$$+ x^2 \sum_{n=0}^{\infty} a_n x^{p+n} - p^2 \sum_{n=0}^{\infty} a_n x^{p+n} = 0$$

$$\sum_{n=0}^{\infty} (p+n)(p+n-1) a_n x^{p+n} + \sum_{n=0}^{\infty} (p+n) a_n x^{p+n}$$

$$+ \sum_{n=0}^{\infty} a_n x^{p+n+2} - p^2 \sum_{n=0}^{\infty} a_n x^{p+n} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty}(p+n)(p+n-1)a_n x^{p+n} + \sum_{n=0}^{\infty}(p+n)a_n x^{p+n}$$

$$+ \sum_{n=2}^{\infty} a_{n-2}\, x^{p+n} - \sum_{n=0}^{\infty} p^2 a_n x^{p+n} = 0$$

$$\Rightarrow p(p-1)a_0 x^p + (p+1)(p)a_1 x^{p+1} + \sum_{n=2}^{\infty}(p+n)(p+n-1)a_n x^{p+n}$$

$$+ p a_0 x^p + (p+1)a_1 x^{p+1} + \sum_{n=2}^{\infty}(p+n)a_n x^{p+n}$$

$$+ \sum_{n=2}^{\infty} a_{n-2}\, x^{p+n}$$

$$- p^2 a_0 x^p - p^2 a_1 x^{p+1} - \sum_{n=2}^{\infty} p^2 a_n x^{p+n} = 0$$

$$\Rightarrow \left(p(p-1)+p-p^2\right)a_0 = 0 \qquad \text{—(i)}$$

$$\left(p(p+1)+p+1-p^2\right)a_1 = 0 \qquad \text{—(ii)}$$

$$\left\{(p+n)(p+n-1)a_n + (p+n)a_n + a_{n-2} - p^2 a_n\right\} = 0 \qquad \text{(iii)}$$

$$\left[(p+n)(p+n-1) + (p+n) - p^2\right]a_n + a_{n-2} = 0$$

$$\left[(p+n)\left[p+n-1+1\right] - p^2\right]a_n + a_{n-2} = 0$$

$$\left[(p+n)^2 - p^2\right]a_n + a_{n-2} = 0$$

$$\left[p^2+n^2+2pn-p^2\right]a_n + a_{n-2} = 0$$

$$n(n+2p)a_n + a_{n-2} = 0$$

$$a_n = -\frac{a_{n-2}}{n(2p+n)}, \quad \forall\ n \geq 2$$

$\boxed{\text{(ii)} \Rightarrow (2p+1)a_1 = 0 \Rightarrow a_1 = 0}$

$\underline{n=2}$

$$a_2 = -\frac{a_0}{2(2p+2)}$$

$\underline{n=3}$

$$a_3 = -\frac{a_1}{3(2p+3)} = 0 \quad (\because a_1 = 0)$$

$\underline{n=4}$

$$a_4 = -\frac{a_2}{4(2p+4)} = \frac{a_0}{2\cdot 4(2p+2)(2p+4)}$$

$\underline{n=5}$

$$a_5 = 0$$

$\underline{n=6}$

$$a_6 = -\frac{a_0}{2\cdot 4\cdot 6(2p+2)(2p+4)(2p+6)}$$

$$\vdots$$

$$\therefore\ y = x^p\left\{a_0 - \frac{a_0}{2(2p+2)}x^2 + \frac{a_0}{2\cdot 4(2p+2)(2p+4)}x^4 - \frac{a_0}{2\cdot 4\cdot 6\cdot(2p+2)(2p+4)(2p+6)}x^6 + \cdots\right\}$$

$$y = a_0 x^p\left\{1 - \frac{x^2}{2^2(p+1)} + \frac{x^4}{2^4\cdot 2!(p+1)(p+2)} - \frac{x^6}{2^6\cdot 3!\,(p+1)(p+2)(p+3)} + \cdots\right\}$$

Δ. Tcmr
21/12/2020

[Note that $m_1 = p$] [$m_2 = -p$]

$$y = a_0 x^p \left\{ 1 - \frac{x^2}{2^2 (p+1)} + \frac{x^4}{2^4 \cdot 2! (p+1)(p+2)} - \frac{x^6}{2^6 \cdot 3! (p+1)(p+2)(p+3)} + \cdots \right\}$$

ie,
$$y = a_0 x^p \sum_{n\geq 0}^{\infty} (-1)^n \frac{x^{2n}}{2^{2n} n! (p+1)\cdots(p+n)} \quad \text{—} (*)$$

The Bessel function of the first kind of order p (denoted by $J_p(x)$) is defined by putting $a_0 = \frac{1}{2^p \cdot p!}$ in $(*)$, so that

$$J_p(x) = \frac{x^p}{2^p p!} \sum_{n\geq 0}^{\infty} (-1)^n \frac{x^{2n}}{2^{2n} n! (p+1)\cdots(p+n)}$$

$$\boxed{J_p(x) = \sum_{n\geq 0}^{\infty} (-1)^n \frac{(x/2)^{2n+p}}{n!(p+n)!}}$$

$x^{2n} \cdot x^p = x^{2n+p}$
$2^{2n} \cdot 2^p = 2^{2n+p}$
$(x/2)^{2n+p}$
$p! = 1 \cdot 2 \cdots p$

Can we replace $-p$ in place of p to get the second sol^n?

$(p+n)!$? $p \to -p$ **Ans:** Gamma function

The gamma function $\Gamma(p)$ is defined by

$$\Gamma(p) = \int_0^\infty t^{p-1} e^{-t}\, dt \,, \quad p>0$$

Note: $\Gamma(p+1) = p\,\Gamma(p)$

pf:

$$\Gamma(p+1) = \int_0^\infty t^{p+1-1} e^{-t}\, dt = \int_0^\infty t^{p} e^{-t}\, dt$$

$u = t^p$
$du = p\,t^{p-1}\,dt$
$dv = e^{-t}\,dt$
$\int dv = \int e^{-t}\,dt$
$v = -e^{-t}$

$$= \left[-t^{p} e^{-t}\right]_0^\infty - \int_0^\infty - e^{-t}\, p\, t^{p-1}\, dt$$

$$= p\int_0^\infty t^{p-1} e^{-t}\, dt = p\,\Gamma(p)$$

Note: $\Gamma(1) = 1$

$$\Gamma(1) = \int_0^\infty t^{1-1} e^{-t}\, dt = \int_0^\infty e^{-t}\, dt$$

$$= \left[-e^{-t}\right]_0^\infty = -\left[0 - 1\right] = 1$$

Note:

use $\Gamma(p+1) = p\,\Gamma(p)$ to prove $\Gamma(2) = 1$

$$\Gamma(2) = \Gamma(1+1) = 1\,\Gamma(1) = 1$$

$$\Gamma(3) = \Gamma(2+1) = 2\,\Gamma(2) = 2\cdot 1 \,, \quad \Gamma(4) = \Gamma(3+1) = 3\,\Gamma(3) = 3\cdot 2\cdot 1$$

In general $\Gamma(n+1) = n!$, $\forall$ integer $n \geq 0$

To find the general solution of $x^2y'' + xy' + (x^2 - p^2)y = 0$

$m_1 = p \rightarrow J_p(x)$

when $m_2 = -p$; $m_1 - m_2 = 2p$ is not zero or positive integer

[whenever the nonnegative constant p is an integer or half an odd integer] ✗.

Assume p is not an integer (we can replace p by $-p$)

$$J_{-p}(x) = \sum_{n\geq 0}^{\infty} \frac{(-1)^n (x/2)^{2n-p}}{n!\,(-p+n)!}$$

Hence the general solution of the Bessel's equation (1) is given by

$$\boxed{y = c_1 J_p(x) + c_2 J_{-p}(x), \quad p \text{ not an integer}}$$

D. Tamu
23/12/2020

Recall

Bessel's equation of order p (non-negative constant) : $x^2y'' + xy' + (x^2 - p^2)y = 0$ — ①

$x = 0 \rightarrow$ a r.s.p of ①.

indicial equation $\rightarrow m^2 - p^2 = 0 \Rightarrow m_1 = p, \; m_2 = -p$

$m_1 = p$ $\quad y = x^p \sum_{n=0}^{\infty} a_n x^n, \quad a_n = -\frac{a_{n-2}}{n(2p+n)}, \; \forall n \geq 2$

$$y = a_0 x^p \sum_{n=0}^{\infty} (-1) \frac{x^{2n}}{2^{2n} n! (p+1) \cdots (p+n)}$$

$$a_0 = \frac{1}{2^p p!} \Rightarrow J_p(x) = \sum_{n=0}^{\infty} (-1)^n \frac{(x/2)^{2n+p}}{n!(p+n)!}$$

Bessel function of the first kind of order p.

$m_2 = -p$ $\quad m_1 - m_2 = 2p$ $\quad (p+n)! \rightarrow$ Gamma function.

$m_1 - m_2 = 2p \neq 0$
$\neq$ positive integer

$$\Gamma(p) = \int_0^{\infty} t^{p-1} e^{-t} \, dt, \quad p > 0$$

$$J_{-p}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n-p}}{n!(-p+n)!}$$

$\therefore$ general solution of ①

$$\boxed{y = c_1 J_p(x) + c_2 J_{-p}(x), \quad p \text{ not an integer}}$$

Suppose p is an integer, say $p = m \geq 0$

Then $J_{-m}(x) = \sum_{n\geq 0}^{\infty} \frac{(-1)^n (x/2)^{2n-m}}{n!\,(-m+n)!}$

$(-m+n)!$?

$n=0$ $(-m+0)!$
$n=1$ $(-m+1)!$
$n=2$ $(-m+2)!$
$\vdots$
$n=m-1$ $(-m+m-1)!$

$\underline{n=m}$ $\underline{(-m+m)!}$

$\downarrow$

$\boxed{n=n+m}$

$\frac{1}{(-m+n)!} \rightarrow n=0,1,2,\ldots m-1.$

i.e, $J_{-m}(x) = \sum_{n=m}^{\infty} \frac{(-1)^n (x/2)^{2n-m}}{n!\,(-m+n)!}$

$$= \sum_{n+m=m}^{\infty} \frac{(-1)^{n+m} (x/2)^{2(n+m)-m}}{(n+m)!\,(-m+n+m)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n(-1)^m (x/2)^{2n+m}}{n!\,(m+n)!}$$

$$= (-1)^m \sum_{n\geq 0}^{\infty} \frac{(-1)^n (x/2)^{2n+m}}{n!\,(m+n)!}$$

$$\underline{J_{-m}(x) = (-1)^m J_m(x)} \checkmark$$

$\boxed{f = cg \qquad \frac{f}{g} = c}$

$\underline{\quad (-1)^m \quad}$

$\therefore \Rightarrow J_{-m}(x)$ and $J_m(x)$ are linearly dependent

$\checkmark$ $J_p(x) \rightarrow$? $\quad x^2y'' + xy' + (x^2-p^2)y = 0$, p nonnegative const.

$\hookrightarrow$ solution.

$\times$? $J_{-p}(x) \rightarrow$? $\quad \hookrightarrow$ solution $(m_1 = m_2 = 2p \neq 0$
$\neq$ integer$)$

[p is not an integer (or) half an odd integer]

$P_n(x) \to$? $(1-x^2)y'' - 2xy' + n(n+1)y = 0$

n, nonnegative constant

— o —

$J_m(x) \to y_1$ then what is y_2 ? (p is an integer say m.)

$y_2 = v y_1 \to v = \int \frac{1}{y_1^2} e^{-\int P(x)dx} dx$

$x^2y'' + xy' + (x^2 - p^2)y = 0$

$y'' + \frac{1}{x}y' + \left(\frac{x^2 - p^2}{x^2}\right)y = 0 \to P(x) = \frac{1}{x}$

$e^{-\int P(x)dx} = e^{-\int \frac{1}{x}dx}$

$= e^{-\log x} = \frac{1}{x}$.

$y_2 = J_m(x) \int \frac{1}{(J_m(x))^2} \cdot \frac{1}{x} dx$

---

* When p is not integer any function of the form $y = c_1 J_p(x) + c_2 J_{-p}(x)$ with $c_2 \neq 0$ is a Bessel function of second kind (including $J_{-p}(x)$)

* Neumann (N_p), Weber (Y_p) Standard Bessel function of the second kind is defined by

$$Y_p(x) = \frac{J_p(x)\cos p\pi - J_{-p}(x)}{\sin p\pi}$$

$$Y_m(x) = \lim_{p \to m} Y_p(x) \rightarrow \text{exists}$$

<u>Conclusion</u>: In all cases, whether 'p' is an integer or not, the general solution of Bessel's equation is given by

$$y = c_1 J_p(x) + c_2 Y_p(x)$$

<u>Test</u>: What is the general solution of

$$x^2 y'' + x y' + \left(x^2 - \frac{1}{4}\right) y = 0 \; ?$$

Hint 1 $\rightarrow$ Bessel's equation of order $\frac{1}{2}$ (p)

Hint 2 $\rightarrow$ not an integer

$$y = c_1 J_{\frac{1}{2}}(x) + c_2 J_{-\frac{1}{2}}(x)$$

Properties of Bessel Functions

$$J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n+p}}{n!\,(p+n)!}$$

(i) $\frac{d}{dx}\left[x^p J_p(x)\right] = x^p J_{p-1}(x)$

—

pf $\frac{d}{dx}\left[x^p J_p(x)\right] = \frac{d}{dx}\left\{x^p \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n+p}}{n!\,(p+n)!}\right\}$

$$= \frac{d}{dx} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2p}}{2^{2n+p}\, n!\,(p+n)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (2n+2p)\, x^{2n+2p-1}}{2^{2n+p}\, n!\,(p+n)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 2(p+n)\, x^{2n+p-1} \cdot x^p}{2^{2n+p} \cdot n!\,(p+n)!}$$

$$= x^p \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+p-1}}{2^{2n+p-1}\, n!\,(p-1+n)!}$$

$$= x^p \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n+p-1}}{n!\,(p-1+n)!}$$

$x^{2n+p-1} \cdot x^p$

$2n+2p$
$= 2(n+p)$

$\frac{n}{n!} = \frac{n}{1\cdot 2 \cdots (n-1)n}$
$= \frac{1}{(n-1)!}$

$\frac{(p+n)}{(p+n)!}$
$= \frac{1}{(p-1+n)!}$

$\frac{d}{dx}\left[x^p J_p(x)\right] = x^p J_{p-1}(x)$

---

llly (ii) $\frac{d}{dx}\left[x^{-p} J_p(x)\right] = -x^{-p} J_{p+1}(x)$

— o —

But: $\frac{d}{dx}\left[x^p J_p(x)\right] = x^p J_p'(x) + J_p(x)\, p x^{p-1}$

from (i) $\frac{d}{dx}\left(x^p J_p(x)\right] = x^p J_{p-1}(x)$

$\therefore \quad x^p J_p'(x) + p x^{p-1} J_p(x) = x^p J_{p-1}(x)$

$\div x^p \quad J_p'(x) + \frac{p}{x} J_p(x) = J_{p-1}(x) \quad —(*)$

$|||^{ly} \quad \frac{d}{dx}\left[x^{-p} J_p(x)\right] = x^{-p} J_p'(x) + J_p(x)\left(-p x^{-p-1}\right)$

$$= x^{-p} J_p'(x) - p x^{-p-1} J_p(x)$$

from (ii) $\frac{d}{dx}\left(x^{-p} J_p(x)\right] = -x^{-p} J_{p+1}(x)$

$\therefore \quad x^{-p} J_p'(x) - p x^{-p-1} J_p(x) = -x^{-p} J_{p+1}(x)$

$\div x^{-p} \quad J_p'(x) - \frac{p}{x} J_p(x) = -J_{p+1}(x) \quad —(**)$

$(*) + (**)$

$$\Rightarrow \boxed{2 J_p'(x) = J_{p-1}(x) - J_{p+1}(x)} \quad — \mathrm{I}$$

$(*) - (**)$

$$\Rightarrow \boxed{2 \frac{p}{x} J_p(x) = J_{p-1}(x) + J_{p+1}(x)} \quad — \mathrm{II}$$

$p = \frac{3}{2}$ in II

$\frac{3}{2} - 1 = \frac{1}{2}$

$\frac{3}{2} + 1 = 5/2$

$$\frac{2\left(\frac{3}{2}\right)}{x} J_{3/2}(x) = J_{\frac{1}{2}}(x) + J_{\frac{5}{2}}(x)$$

$$\frac{3}{x} J_{\frac{3}{2}}(x) = J_{\frac{1}{2}}(x) + J_{\frac{5}{2}}(x)$$

$$\Rightarrow \quad \underline{\underline{J_{\frac{5}{2}}(x) = \frac{3}{x} J_{\frac{3}{2}}(x) - J_{\frac{1}{2}}(x)}}$$

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24/12/2020

Recall

$$\begin{cases} \dfrac{d}{dx}\left[x^{p} J_p(x)\right] = x^{p} J_{p-1}(x) & \text{--- (1)} \\ \dfrac{d}{dx}\left[x^{-p} J_p(x)\right] = -x^{-p} J_{p+1}(x) & \text{--- (2)} \end{cases}$$

Integrating (1) w.r.t x,

$$x^{p} J_p(x) = \int x^{p} J_{p-1}(x)\,dx + c$$

$$\boxed{\int_0^x x^{p} J_{p-1}(x)\,dx = x^{p} J_p(x)} \quad \text{--- (3)}$$

||ly Integrating (2)

$$\boxed{\int_0^x x^{-p} J_{p+1}(x)\,dx = -x^{-p} J_p(x)} \quad \text{--- (4)}$$

GATE Show that $\displaystyle\int_0^x x^3 J_0(x)\,dx = x^3 J_1(x) - 2x^2 J_2(x)$

WKT $\displaystyle\int_0^x x^{p} J_{p-1}(x)\,dx = x^{p} J_p(x)$ ✓ (*)

$$\int_0^x x^3 J_0(x)\,dx = \int_0^x x^2\left[x J_0(x)\right] dx$$

$u = x^2$
$du = 2x\,dx$
$dv = x J_0(x)\,dx$
$\int dv = \underbrace{\int x J_0(x)\,dx}$
$v = x J_1(x)$

$$= \left[x^2\{x J_1(x)\}\right]_0^x - \int_0^x 2x\{x J_1(x)\}\,dx$$

$$= x^3 J_1(x) - 2\int_0^x x^2 J_1(x)\,dx$$

$$\int_0^x x^3 J_0(x)\,dx = x^3 J_1(x) - 2x^2 J_2(x) \quad \text{(using (*) with } p=2\text{)}$$