# Ordinary Differential Equations

Course Code: 21M03CC

UNIT - I

Dr. A. Tamilselvan
Professor and Chair
School of Mathematical Sciences

BHARATHIDASAN UNIVERSITY
Tiruchirappalli- 620024, Tamil Nadu, India

D. Tamuli 23/11/2020

Recall

$$f(m) = m(m-1) + m p_0 + q_0$$

$$a_0 f(m) = 0$$

$$a_1 f(m+1) + a_0 (m p_1 + q_1) = 0$$

$$a_2 f(m+2) + a_0 (m p_2 + q_2) + a_1 ((m+1) p_1 + q_1) = 0$$

$$\vdots$$

$$a_n f(m+n) + a_0 (m p_n + q_n) + \cdots + a_{n-1} [(m+n-1) p_1 + q_1] = 0$$

$$\vdots$$

$$a_0 \neq 0 \Rightarrow f(m) = 0 \Rightarrow \boxed{m(m-1) + m p_0 + q_0 = 0}$$

indicial equation

roots of indicial equation are called the exponents of the respective diff$^l$ eq$^n$.

—o—

$$y = x^m \sum_{n=0}^{\infty} a_n x^n$$

$m_1$ & $m_2$

$m = m_1 \rightarrow y_1 = x^{m_1} \sum_{n=0}^{\infty} a_n x^n$

$m = m_2 \rightarrow y_2 = x^{m_2} \sum_{n=0}^{\infty} a_n x^n$

when will this happen? $m_1 - m_2 \neq n$

$m_1 = m_2 + n$

$m_1 - m_2 = n$

$m_1 = m_2$ say ($m$)

$$y_1 = x^m \sum_{n=0}^{\infty} a_n x^n$$

what about the another (second) sol$^n$?

$$y_2 = v y_1$$

$$v = \int \frac{1}{y_1^2} e^{-\int p(x) dx} dx$$

Note The $a_n$ are therefore determined in terms of $a_0$ for each choice of $m$ except $f(m+n) = 0$ for some +ve integer $n$, in which case the process breaks off.

Thus if $m_1 = m_2 + n$ for some integer $n \geq 1$ the choice $m = m_1$ gives a formal solution but in general $m = m_2$ does not — since $f(m_2+n) = f(m_1) = 0$.

If $m_1 = m_2$, we also obtain only one formal solution. $(y_2 = J y_1)$

In all other cases where $m_1$ and $m_2$ are numbers, this procedure gives two independent formal solutions.

We are not discussing the case when $m_1$ and $m_2$ are complex conjugate numbers.

— o —

Example 1: Consider the Bessel's equation of order 1

$x^2 y'' + x y' + (x^2 - 1) y = 0$. Show that $m_1 - m_2 = 2$ and that the equation has only one Frobenius series solution. Then find it.

"order $p$"

$x^2 y'' + x y' + (x^2 - p^2) y = 0$

order $\frac{1}{2}$

$x^2 y'' + x y' + (x^2 - \frac{1}{4}) y = 0$

— o —

Given that

$$x^2 y'' + x y' + (x^2 - 1) y = 0 \quad \text{—} (1)$$

Std form:

$$y'' + \frac{1}{x} y' + \left(\frac{x^2 - 1}{x^2}\right) y = 0$$

$$P(x) = \frac{1}{x}, \quad Q(x) = \frac{x^2 - 1}{x^2}$$

$x = 0$ is a singular point.

$x P(x) = 1$, $x^2 Q(x) = x^2 - 1$ $\Rightarrow$ $x = 0$ is a r.s.p

Let $y = x^m \sum_{n=0}^{\infty} a_n x^n$, $(a_0 \neq 0)$ be a solution of (1).

i.e., $y = \sum_{n=0}^{\infty} a_n x^{n+m}$ , $y' = \sum_{n=0}^{\infty} (n+m) a_n x^{n+m-1}$

$$y'' = \sum_{n=0}^{\infty} (n+m)(n+m-1) a_n x^{n+m-2}$$

① becomes

$$x^2 \sum_{n=0}^{\infty} (n+m)(n+m-1) a_n x^{n+m-2} + x \sum_{n=0}^{\infty} (n+m) a_n x^{n+m-1}$$

$$+ x^2 \sum_{n=0}^{\infty} a_n x^{n+m} - \sum_{n=0}^{\infty} a_n x^{n+m} = 0$$

$$\sum_{n=0}^{\infty} (n+m)(n+m-1) a_n x^{n+m} + \sum_{n=0}^{\infty} (n+m) a_n x^{n+m}$$

$$+ \sum_{n=0}^{\infty} a_n x^{n+m+2} - \sum_{n=0}^{\infty} a_n x^{n+m} = 0$$

$$\sum_{n=0}^{\infty} (n+m)(n+m-1) a_n x^{n+m} + \sum_{n=0}^{\infty} (n+m) a_n x^{n+m} + \sum_{n=2}^{\infty} a_{n-2} x^{n+m}$$

$$- \sum_{n=0}^{\infty} a_n x^{n+m} = 0$$

$$m(m-1) a_0 x^m + m(m+1) a_1 x^{m+1} + \sum_{n=2}^{\infty} (n+m)(n+m-1) a_n x^{n+m}$$

$$+ m a_0 x^m + (m+1) a_1 x^{m+1} + \sum_{n=2}^{\infty} (n+m) a_n x^{n+m}$$

$$+ \sum_{n=2}^{\infty} a_{n-2} x^{n+m}$$

$$- a_0 x^m - a_1 x^{m+1} - \sum_{n=2}^{\infty} a_n x^{n+m} = 0$$

$$\left[m(m-1)+m-1\right]a_0 x^m + \left[m(m+1)+(m+1)-1\right]a_1 x^{m+1}$$
$$+\sum_{n=2}^{\infty}\left[(n+m)(n+m-1)a_n + (n+m)a_n + a_{n-2} - a_n\right]x^{n+m} = 0$$

$$\Rightarrow \left[m(m-1)+m-1\right]a_0 = 0 \quad -(i)$$
$$\left[m(m+1)+(m+1)-1\right]a_1 = 0 \quad -(ii)$$
and $$\left[(n+m)(m+n-1)+(n+m)-1\right]a_n + a_{n-2} = 0 \quad -(iii)$$

$$(iii) \Rightarrow \quad (m+n-1)(n+m+1)a_n + a_{n-2} = 0$$
$$\Rightarrow a_n = -\frac{a_{n-2}}{(m+n-1)(n+m+1)}, \quad \forall\, n \geq 2$$

$$(i) \Rightarrow m^2 - m + m - 1 = 0 \quad (\because a_0 \neq 0)$$
$$\Rightarrow m^2 - 1 = 0 \Rightarrow m = \pm 1 \qquad m_1 = 1,\ m_2 = -1$$
$$\boxed{m_1 - m_2 = 2} \checkmark$$

<u>case(i)</u> <u>$m = m_1 = 1$</u>

$$a_n = \frac{-a_{n-2}}{n(n+2)}, \quad \forall\, n \geq 2$$

<u>From (ii)</u>
$$\left[m(m+1)+(m+1)-1\right]a_1 = 0 \Rightarrow 3a_1 = 0 \Rightarrow a_1 = 0$$

<u>$n=2$</u> $$a_2 = -\frac{a_0}{2\cdot 4} = -\frac{a_0}{2^2\cdot 2!}$$

<u>$n=3$</u> $$a_3 = -\frac{a_1}{3\cdot 5} = 0$$

<u>$n=4$</u> $$a_4 = -\frac{a_2}{4\cdot 6} = \frac{a_0}{\underset{2\cdot 2^2\cdot 2^2\cdot 2\cdot 3}{2\cdot 4\cdot 4\cdot 6}} = \frac{a_0}{2^4\cdot 2!\, 3!}$$

$\underline{n=5}$ $\quad a_5 = -\dfrac{a_3}{5\cdot 7} = 0$

$\underline{n=6}$ $\quad a_6 = -\dfrac{a_4}{6\cdot 8} = -\dfrac{a_0}{\underset{2^6\cdot 3!4!}{2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8}}$

$\vdots$

$\therefore$ The solution corresponding to $m = m_1 = 1$ is

$$y = x^1\left\{a_0 - \frac{a_0}{2\cdot 4}x^2 + \frac{a_0}{2\cdot 4\cdot 4\cdot 6}x^4 - \frac{a_0}{2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8}x^6 + \cdots\right.$$

$$y = x\, a_0\left\{1 - \frac{x^2}{2^2\cdot 2!} + \frac{x^4}{2^4 2!\, 3!} - \frac{x^6}{2^6\cdot 3!\, 4!} + \cdots\right\}$$

$\underline{\text{case (ii)}}$ $\underline{m = m_2 = -1}$

A. Tamil
24/11/2020

$\therefore$ (11) $\Rightarrow \left[m(m+1) + (m+1) - 1\right] a_1 = 0$

$\underline{\text{when } m=-1}$ : $\quad -a_1 = 0 \Rightarrow a_1 = 0$

$$a_n = \frac{-a_{n-2}}{(m+n+1)(m+n-1)}, \quad \forall\, n \geq 2$$

$\underline{m=-1}$

$$\Rightarrow \quad a_n = \frac{-a_{n-2}}{n(n-2)}, \quad \forall\, n \geq 2$$

$\underline{n=2}$ $\quad a_2 = \dfrac{-a_0}{2\cdot 0}$ Can't find $a_2$

$\underline{n=3}$ $\quad a_3 = -\dfrac{a_1}{3\cdot 1} = 0$

$\underline{n=4}$ $\quad a_4 = \dfrac{-a_2}{4\cdot 2}$ can't find $a_4$

$\vdots$

The second solution $y_2$ can't be found.

$\therefore$ we have only one solution corresponding to $m=1$.

$$\boxed{\begin{array}{l} y_2 = x^{-1}\{a_0\} = \dfrac{a_0}{x} \\ x \to 0,\ y_2 \text{ unbdd} \end{array}}$$

$m_1 - m_2 = 2$

ie, $m_1 = m_2 + n$

Take $a_0 = 1$

$$y = x\left\{1 - \frac{x^2}{2!\,2^2} + \cdots\right\}$$

---

Example 2. Consider the Bessel's equation of order $p = \frac{1}{2}$ given by

$$x^2y'' + xy' + (x^2 - \tfrac{1}{4})y = 0.$$

Show that $m_1 - m_2 = 1$, but that nevertheless the equation has two independent Frobenius series solutions. Then find them.

---

Solution

Given that

$$x^2y'' + xy' + (x^2 - \tfrac{1}{4})y = 0 \quad \text{--- (1)}$$

Std form:

$$y'' + \frac{1}{x}y' + \frac{[x^2 - \frac{1}{4}]}{x^2}y = 0$$

$$P(x) = \frac{1}{x}, \quad Q(x) = \frac{[x^2 - \frac{1}{4}]}{x^2}$$

$x = 0$, a singular point.

Also $xP(x) = 1$, $x^2Q(x) = x^2 - \frac{1}{4}$ which are analytic at $x = 0$. $\therefore$ $x = 0$ is a r.s.p

Let $y = x^m \sum_{n=0}^{\infty} a_n x^n$, $(a_0 \neq 0)$ be a solution of ①.

ie, $y = \sum_{n=0}^{\infty} a_n x^{n+m}$, $y' = \sum_{n=0}^{\infty} (n+m) a_n x^{n+m-1}$,

$$y'' = \sum_{n=0}^{\infty} (n+m)(n+m-1) a_n x^{n+m-2}$$

Then ① becomes

$$x^2 \sum_{n=0}^{\infty} (n+m)(n+m-1) a_n x^{n+m-2} + x \sum_{n=0}^{\infty} (n+m) a_n x^{n+m-1}$$

$$+ x^2 \sum_{n=0}^{\infty} a_n x^{n+m} - \frac{1}{4} \sum_{n=0}^{\infty} a_n x^{n+m} = 0$$

ie,
$$\sum_{n=0}^{\infty} (n+m)(n+m-1) a_n x^{n+m} + \sum_{n=0}^{\infty} (n+m) a_n x^{n+m}$$

$$+ \sum_{n=0}^{\infty} a_n x^{n+m+2} - \frac{1}{4} \sum_{n=0}^{\infty} a_n x^{n+m} = 0$$

ie,
$$\sum_{n=0}^{\infty} (n+m)(n+m-1) a_n x^{n+m} + \sum_{n=0}^{\infty} (n+m) a_n x^{n+m}$$

$$+ \sum_{n=2}^{\infty} a_{n-2} x^{n+m} - \frac{1}{4} \sum_{n=0}^{\infty} a_n x^{n+m} = 0$$

$$m(m-1) a_0 x^m + m(m+1) a_1 x^{m+1} + \sum_{n=2}^{\infty} (n+m)(n+m-1) a_n x^{n+m}$$

$$+ m a_0 x^m + (m+1) a_1 x^{m+1} + \sum_{n=2}^{\infty} (n+m) a_n x^{n+m}$$

$$+ \sum_{n=2}^{\infty} a_{n-2} x^{n+m}$$

$$-\frac{1}{4}a_0 x^m - \frac{1}{4}a_1 x^{m+1} - \frac{1}{4}\sum_{n=2}^{\infty} a_n x^{n+m} = 0$$

$$\left[m(m-1) + m - \frac{1}{4}\right]a_0 = 0 \quad -(i)$$

$$\left[m(m+1) + (m+1) - \frac{1}{4}\right]a_1 = 0 \quad -(ii)$$

$$\left[(n+m)(n+m-1)a_n + (n+m)a_n + a_{n-2} - \frac{1}{4}a_n\right] = 0 \quad (iii)$$

From (i), $\quad m(m-1) + m - \frac{1}{4} = 0 \quad$ as $a_0 \neq 0$

$$m^2 - m + m - \frac{1}{4} = 0 \Rightarrow m^2 = \frac{1}{4} \Rightarrow m = \pm\frac{1}{2}$$

$$m_1 = \frac{1}{2}, \quad m_2 = -\frac{1}{2} \qquad \boxed{m_1 - m_2 = 1}$$

Case (i) $m = m_1 = \frac{1}{2}$

$$(ii) \Rightarrow \left[\frac{1}{2}\left(\frac{1}{2}+1\right) + \left(\frac{1}{2}+1\right) - \frac{1}{4}\right]a_1 = 0$$

$$\left[\frac{3}{4} + \frac{3}{2} - \frac{1}{4}\right]a_1 = 0 \Rightarrow 2a_1 = 0$$

$$a_1 = 0$$

$$(iii) \Rightarrow \left[(n+m)(n+m-1) + (n+m) - \frac{1}{4}\right]a_n + a_{n-2} = 0$$

$m = m_1 = \frac{1}{2}$

$$\left[(n+m)(n+m-1+1) - \frac{1}{4}\right]a_n + a_{n-2} = 0$$

$$(*) \leftarrow \left[(n+m)^2 - \frac{1}{4}\right]a_n + a_{n-2} = 0$$

$$\left[\left(n+\frac{1}{2}\right)^2 - \frac{1}{4}\right]a_n + a_{n-2} = 0$$

$$\left[n^2 + n + \frac{1}{4} - \frac{1}{4}\right]a_n + a_{n-2} = 0$$

$$n(n+1)a_n = -a_{n-2}$$

$$a_n = -\frac{a_{n-2}}{n(n+1)}, \quad \forall\, n \geq 2$$

$\underline{n=2}$ $\quad a_2 = -\frac{a_0}{2.3} = \frac{-a_0}{3!}$

$\underline{n=3}$ $\quad a_3 = -\frac{a_1}{3\cdot 4} = 0 \quad (\because a_1 = 0)$

$\underline{n=4}$ $\quad a_4 = \frac{-a_2}{4\cdot 5} = \frac{a_0}{2\cdot 3\cdot 4\cdot 5} = \frac{a_0}{5!}$

$\vdots$

The solution corresponding to $m = m_1 = \frac{1}{2}$ is

$$y_1 = x^{\frac{1}{2}}\left[a_0 - \frac{a_0}{3!}x^2 + \frac{a_0}{5!}x^4 - \cdots\right]$$

$$= \frac{x^{\frac{1}{2}}}{x}\cdot\left[a_0 x - \frac{a_0 x^3}{3!} + \frac{a_0}{5!}x^5 - \cdots\right]$$

$$\boxed{y_1 = x^{-\frac{1}{2}} a_0\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right]} \rightarrow \underline{y_1 = x^{-\frac{1}{2}} a_0 \sin x}$$

(I)

$\underline{\text{Case (ii)}}$ $m = m_2 = -\frac{1}{2}$

$\therefore$ (ii) $\rightarrow \left[m(m+1) + (m+1) - \frac{1}{4}\right]a_1 = 0$

$m = -\frac{1}{2}$ $\quad \left[-\frac{1}{2}\left(-\frac{1}{2}+1\right) + \left(-\frac{1}{2}+1\right) - \frac{1}{4}\right]a_1 = 0$

$\left[-\frac{1}{4} + \frac{1}{2} - \frac{1}{4}\right]a_1 = 0 \; ; \; \underline{\underline{0\cdot a_1 = 0}}$

$(*) \Rightarrow \left[(n+m)^2 - \frac{1}{4}\right]a_n + a_{n-2} = 0$

$\underline{m = -\frac{1}{2}}$ $\quad \left[\left(n - \frac{1}{2}\right)^2 - \frac{1}{4}\right]a_n + a_{n-2} = 0$

$\left[n^2 - n + \frac{1}{4} - \frac{1}{4}\right]a_n + a_{n-2} = 0$

$$n(n-1)a_n + a_{n-2} = 0$$

$$\Rightarrow \quad a_n = -\frac{a_{n-2}}{n(n-1)}, \quad \forall \; n \geq 2$$

<u>$n=2$</u> $\quad a_2 = -\frac{a_0}{2\cdot 1} = -\frac{a_0}{2!}$

<u>$n=3$</u> $\quad a_3 = -\frac{a_1}{3\cdot 2} = 0$

<u>$n=4$</u> $\quad a_4 = -\frac{a_2}{4\cdot 3} = \frac{a_0}{1\cdot 2\cdot 3\cdot 4} = \frac{a_0}{4!}$

<u>$n=5$</u> $\quad a_3 = -\frac{a_3}{5\cdot 4} = 0$

$\vdots$

$\therefore$ The solution corresponding to $m = m_2 = -\frac{1}{2}$ is

$$y_2 = x^{-\frac{1}{2}}\left[a_0 - \frac{a_0}{2!}x^2 + \frac{a_0}{4!}x^4 - \cdots\right]$$

$$\boxed{y_2 = x^{-\frac{1}{2}} a_0 \left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right]} \rightarrow y_2 = x^{-\frac{1}{2}} a_0 \cos x \quad \text{(II)}$$

Take $a_0 = 1$ in (I) and (II)

$\therefore$ The two independent solutions are given by

$$\left.\begin{aligned} y_1 &= x^{-1/2} \sin x \\ y_2 &= x^{-1/2} \cos x \end{aligned}\right\}$$

—— o ——

**Note : 1.**

Assume that $y'' + P(x)y' + Q(x)y = 0$ ① has a regular singular point at $x = 0$. Then method of Frobenius seeks a solution of the form

$$y = x^m \sum_{n=0}^{\infty} a_n x^n, \quad (a_0 \neq 0).$$

Substituting $y$, $y'$ and $y''$ into ①, we get the ✓indicial equation $[m(m-1) + m p_0 + q_0 = 0]$ which gives the roots $m_1$ and $m_2$ (called exponents of ①) and the recurrence relation ✓

**Step 1** Using the recurrence relation and one root $m_1$ leads to a solution

$$y_1(x) = x^{m_1} \sum_{n=0}^{\infty} a_n x^n.$$

**Step 2** A second linearly independent solution $y_2$ is obtained as follows :

**case (i)** If $m_1 - m_2$ is not an integer, then

$$y_2 = x^{m_2} \sum_{n=0}^{\infty} a_n x^n$$

**case (ii)** If $m_1 = m_2$, then [HINT : $y_2 = v y_1$]

$$y_2(x) = x^{m_1} \sum_{n=0}^{\infty} a_n x^n + [\log x]\, y_1(x)$$

**case (iii)** If $m_1 - m_2 = n$ is a +ve interger then

$$y_2(x) = x^{m_2} \sum_{n=0}^{\infty} a_n x^n + c[\log x]\, y_1(x)$$

Note-2 ordinary point

Let $x_0$ be an ordinary point of the diff$^{l}$ eq$^{n}$ $y'' + P(x)y' + Q(x)y = 0$ — ①, and let $a_0$ and $a_1$ be arbitrary constants. Then there exists a unique function $y(x)$ that is analytic at $x_0$ is a solution of ① in a certain neighbourhood of this point and satisfies the initial conditions $y(x_0) = a_0$ and $y'(x_0) = a_1$. Furthermore, if the power series expansions of $P(x)$ and $Q(x)$ are valid on an interval $|x - x_0| < R$, $R > 0$, then the power series expansion of this solution $[y(x)]$ is also valid on the same interval.

—o—

$x_0 = 0$ $x = 0$ is an ordinary pt

$P(x)$ and $Q(x)$ are analytic at $x = 0$

$|x - x_0| < R \Rightarrow |x| < R$

$\mathbb{R}$ $|x - x_0| < R \to (x_0 - R, x_0 + R)$

$|x| < R \to (-R, R)$

—o—