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Programme: M.Sc., Mathematics

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Binomial Theorem and its Proof

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Qn: Number of primes

Euclid Theorem

The number of primes is infinite.

That is, there is no end to the sequence of primes

$$2, 3, 5, 7, 11, 13, 17, \ldots$$

Proof: Let k be any +ve integer. Suppose that $p_1, p_2 \ldots p_k$ are the first k primes. Then let us define

$$n = p_1 p_2 \cdots p_k + 1 \qquad (1)$$

It is clear that n is not divisible by any of the primes $p_1, p_2, \ldots, p_k$

If $p \mid n$, then p is different from $p_1, p_2, \ldots, p_k$.

By the canonical factorization "any integer $n > 1$ can be factored into product of primes";

either n is prime or n has prime factor p,

$\Rightarrow$ there is a prime distinct from $p_1, p_2, \ldots, p_k$.

Thus we have that for any +ve int k the number of primes is not exactly k.

Hence the number of primes is infinite

Definiton : n is said to be composite if there exists an integer a s.t. $1 < a < n$ and $a | n$.

Theorem: There are large gaps between primes i.e) For any +ve int k, there are k consecutive composite numbers

Proof: Consider these numbers

$$(k+1)!+2, (k+1)!+3, \ldots, (k+1)!+(k+1)$$

Binomial Theorem:

Definition Let α be any real number and let k be a nonnegative integer. Then the binomial coefficient $\binom{\alpha}{k}$ is given by

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}$$

If n & k are both integers & $0 \leq k \leq n$ then

$$\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}$$

$$= \frac{n!}{k!\,(n-k)!}$$

(we know)

$$\boxed{n! = \prod_{k=1}^{n} k}$$

$0! = 1$ (conversionally).

<u>Theorem</u> (combinatorics)

Let P be a set containing exactly n elements. For any non-negative integer k the number of subsets of P containing k elements is $\binom{n}{k}$.

Idea : X - no. of subsets of P of k elements

$$X = \binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

$$\Rightarrow X \, k!\,(n-k)! = n! \quad //$$

P is set of n elements

$P = \{1, 2, \dots, n\}$

The numbers of P can be listed in various orders,

listing $\rightarrow$ permutation π.

$P = \{1, 2, \dots, n\}$

There are $n!$ permutations of P

each permutation contains n terms.

first term may be any one of n numbers

the second term any one of $n-1$ remaining numbers

and so on.

$$n(n-1)(n-2)\cdots 2\cdot 1 = n!$$

n element set has $n!$ permutations.

X – number of subsets of P having k elements

Let A be a sub set of K elements.

Then there are $K!$ permutations of the elements of A, each permutation contains K elements

III^{ly} there are $(n-K)!$ permutations of the $n-K$ elements not in A

$K!$ $\quad$ $n-K!$ $\quad$ $n!$

Take a permutation of elements of A & a permutation of elements not in A

we attach each permutation of elements not A to the right of each permutation of elements of A

$$\left(\overset{k \text{ elem}}{\ldots\ldots\ldots} \quad \overset{n-k}{\text{elem}} \right)$$ — permutation of n element

↓

permutation of P

This way we generate $k!(n-k)!$ permutations of P

For each subset of P having k elements, we generate $k!(n-k)!$ permutations of P

All the permutations are generated from some subset of k element.

$$x \cdot k!\,(n-k)! = n!$$

$$x = \frac{n!}{k!\,(n-k)!} = \binom{n}{k}$$

$\therefore$ The no. of subsets of P having precisely k elements is $\binom{n}{k}$

$$\Rightarrow \boxed{\binom{n}{k} \text{ is an integer.}}$$

Theorem: The product of any k consecutive integers is divisible by $k!$

Proof:

$$n(n-1)(n-2)\cdots(n-k+1)$$

case(1) $n \geq k$

then the product

$$n(n-1)(n-2)\cdots(n-k+1)$$

does not contain 0 as a factor.

By the defn of Binomial coeff.

$$n(n-1)\cdots(n-k+1) = \binom{n}{k} k!$$

By the previous result $\binom{n}{k}$ is an integer.

$$\therefore k! \mid \text{the product.}$$

the result holds

$n < k$

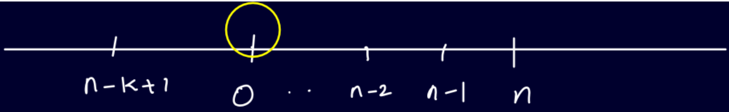
$n \leq k-1$

$n-k+1 \leq 0$

<u>case(1)</u> $0 \leq n < k$

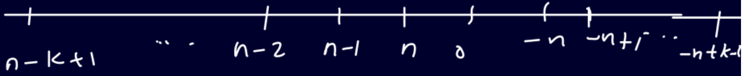
$$n(n-1)\cdots\cdots(n-k+1) = 0$$

because $n-k+1 \leq 0$ & 0 is a factor



$$k! \mid n(n-1)\cdots(n-k+1)$$

<u>case (3)</u> $n < 0 \leq k$



$$n(n-1)(n-2)\cdots(n-k+1)$$

$$= (-1)^n (-n)(-n+1)(-n+2)\cdots(-n+k-1)$$

$$= (-1)^n m(m-1)\cdots(m-k+1)$$

where $m = -n+k-1$

$$\boxed{\begin{aligned} m &= -n+k+1 \\ m-1 &= -n+k-2 \\ &\vdots \\ m-k+2 &= -n+1 \\ m-k+1 &= -n \end{aligned}}$$

Since $m > 0$, by case (i) & case (2),

$$k! \mid m(m-1)\cdots(m-k+1)$$

$$\Rightarrow k! \mid (-1)^n m(m-1)\cdots(m-k+1)$$

$$= k! \mid n(n-1)\cdots(n-k+1)$$

The Product of any K consecutive integers is divisible by $K!$

Binomial theorem:

For any integer $n \geq 1$ and any real numbers x and y

$$\boxed{(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.}$$

$$= x^n + \binom{n}{1} x^{n-1} y + \binom{n}{2} x^{n-2} y^2 + \cdots + \cdots + y^n.$$