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Tamil Nadu, India

Programme : M. Sc. Mathematics

Course Title : ALGEBRA - I

Course Code : 24S2M05CC

UNIT - II

ISOMORPHISMS AND DIRECT PRODUCT

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Group Homomorphisms

- A map ϕ from a group $(G, \star)$ into a group (G', Δ) is a homomorphism if

$$\phi(a \star b) = \phi(a)\Delta\phi(b) \text{ for all } a, b \in G.$$

Example

1. For any groups G and G', there is always at least one homomorphism: $\phi : G \longrightarrow G'$ defined by $\phi(g) = e'$ for all $g \in G$ where e' is the identity in G'. We call it the **trivial homomorphism or zero-homomorphism.**
2. Let G be a group. Then the identity map is a group homomorphism. This homomorphism is called the **identity homomorphism.**

Continue ...

- Let $r \in \mathbb{Z}$ and let $\phi_r : \mathbb{Z} \longrightarrow \mathbb{Z}$ be defined by $\phi_r(n) = rn$ for all $n \in \mathbb{Z}$. Then ϕ is a homomorphism.
- Let $\phi : \mathbb{Z}_2 \times \mathbb{Z}_4 \longrightarrow \mathbb{Z}_2$ be defined by $\phi(x, y) = x$ for all $x \in \mathbb{Z}_2, y \in \mathbb{Z}_4$. Then ϕ is a homomorphism.
- Let G be a group and $g \in G$. Then the map $\phi : \mathbb{Z} \to G$ defined by $\phi(n) = g^n$ for all $n \in \mathbb{Z}$ is a homomorphism.

Properties of Homomorphisms

- Let ϕ be a homomorphism of a group G into a group G'. Then
  1. If e is the identity element in G, then $\phi(e)$ is the identity element e' in G'.
  2. If $a \in G$, then $\phi(a^{-1}) = \phi(a)^{-1}$.
  3. If H is a subgroup of G, then $\phi(H)$ is a subgroup of G'.
  4. If K' is a subgroup of G', then $\phi^{-1}(K')$ is a subgroup of G.
- Let ϕ be a homomorphism of a group G into a group G'. Then the **kernel of** ϕ is defined by $Ker(\phi) = \{g \in G \mid \phi(g) = e'\}$.
- If $\phi : G \longrightarrow G'$ is a group homomorphism, then $Ker(\phi)$ is a normal subgroup of G.
- $im(f)$ is a subgroup of G'.
- A group homomorphism $\phi : G \longrightarrow G'$ is a one-to-one map if and only if $Ker(\phi) = \{e\}$

Isomorphisms of Groups

- A homomorphism $\phi : G \to G'$ is said to be an **isomorphism** if it is both one-to-one and onto. It is denoted by $G \cong G'$.
- **Fundamental Theorem of Homomorphism**
  Let $\phi : G \longrightarrow G'$ be a homomorphism. Then $\frac{G}{Ker\phi} \cong \phi(G)$.
- If $\phi : G \to G'$ is an isomorphism, then
  - the identity $e \in G, e' \in G'$, $\phi(e) = e'$.
  - $\phi(a^n) = (\phi(a))^n$ for all $a \in G, n \in \mathbb{Z}$.
  - for any $a, b \in G$, a, b commute $\Leftrightarrow \phi(a), \phi(b)$ commute.
  - $G =< a > \Leftrightarrow G' =< \phi(a) >$.
  - $|a| = |\phi(a)|$ for all $a \in G$.
  - If G is finite, then G, G' have exactly the same no. of elements of every order.

isomorphism ...

If $\phi : G \to G'$ is an isomorphism, then

- G is cyclic $\Leftrightarrow$ G' is cyclic.
- G is Abelian $\Leftrightarrow$ G' is Abelian.
- $\phi(Z(G)) = Z(G')$.
- If H, H' is a Subgroups of G, G' respectively. Then $\phi(H), \phi^{-1}(H')$ is a Subgroups of G', G respectively.
- Are the Homomorphisms:
  1. Let $r \in \mathbb{Z}$ and let $\phi_r : \mathbb{Z} \longrightarrow \mathbb{Z}$ be defined by $\phi_r(n) = rn$ for all $n \in \mathbb{Z}$. Then ϕ is a homomorphism.
  2. Let $\phi : \mathbb{Z}_2 \times \mathbb{Z}_4 \longrightarrow \mathbb{Z}_2$ be defined by $\phi(x, y) = x$ for all $x \in \mathbb{Z}_2, y \in \mathbb{Z}_4$. Then ϕ is a homomorphism.
- are they isomorphisms?

Continue ...

- An isomorphism from a group onto itself is said to be an **automorphism.**
- $Aut(G) = \{\phi : G \to G \mid \phi$ is an isomorphism $\}$ and $Inn(G) = \{\phi_a : G \to G \mid \phi_a(x) = axa^{-1}$ for all $x \in G$ and $a \in G\}$.

External Direct Products

- Let $(G_1, *_1), (G_2, *_2), \ldots, (G_n, *_n)$ be a finite collection of groups. Then the External direct product of $G_1, \ldots, G_n$ is $G = G_1 \oplus \cdots \oplus G_n = \{(g_1, \ldots, g_n) | g_i \in G_i\}$ is group under the operation defined by

$$(x_1, \ldots, x_n)(y_1, \ldots, y_n) = (x_1 *_1 y_1, \cdots, x_n *_n y_n)$$

  for all $(x_1, \ldots, x_n)(y_1, \ldots, y_n) \in G$.
- $\mathbb{Z}_2 \oplus \mathbb{Z}_3 = \{(0,0), (0,1), (0,2), (1,0), (1,1), (1,2)\}$.
- The order of an element $(g_1, \ldots, g_n) \in G$ is $lcm(o(g_1), \ldots, o(g_n))$.
- Let $m = n_1 n_2 \cdots n_k$. Then $\mathbb{Z}_m \cong \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_k} \Leftrightarrow n_i$ and n_j are relatively prime for $i \neq j$.

Internal Direct Product

- $\mathbb{Z}_2 \oplus \mathbb{Z}_{30} \cong \mathbb{Z}_6 \oplus \mathbb{Z}_{10}$. But $\mathbb{Z}_2 \oplus \mathbb{Z}_{30} \ncong \mathbb{Z}_{60}$.
- **Fundamental theorem of finite Abelian Groups**
  Every finite Abelian group is a direct product of cyclic groups of prime power order.
- Let $H_1, H_2, \ldots, H_n$ be the normal subgroups of a group G. G is said to be the **Internal direct Products of** $H_1, \ldots, \times H_n$ if every element g of G is written as $g = h_1 h_2 \cdots h_n$ in a unique way.
- G is the Internal direct product of H and K iff H, K are normal in G and $H \cap K = \{e\}$.

Internal Direct Product

- Suppose that $G = H_1 H_2 \cdots H_n$ where each H_i is a normal subgroup of G. Then the following conditions are equivalent
  1. G is the internal direct product of the H_i.
  2. $H_1 H_2 \cdots H_{i-1} \cap H_i = \{e\}$ for all $i = 1, 2, \cdots, n$
- $H_1 H_2 \ldots H_n \cong H_1 \oplus \cdots \oplus H_n$.

Finite Abelian Groups

* If G is abelian gp of order p^n, then G is a direct product of its cyclic subgps

$$o(H_i) = p^{n_i} \text{ for } i = 1, 2, \dots r$$

$$G = H_1 H_2 H_3 \cdots H_r$$

$$p^{n_1} p^{n_2} p^{n_3} \cdots p^{n_r} = p^{n_1+n_2+\cdots+n_r}$$

$\Rightarrow n = n_1 + n_2 + \cdots + n_r$

Each H_i is cyclic of order p^{n_i}

$\Rightarrow H_i \cong \mathbb{Z}_{p^{n_i}}$

$\Rightarrow G = H_1 H_2 \cdots H_r \cong \mathbb{Z}_{p^{n_1}} \times \mathbb{Z}_{p^{n_2}} \times \cdots \times \mathbb{Z}_{p^{n_r}}$

$\therefore G \cong \mathbb{Z}_{p^{n_1}} \mathbb{Z}_{p^{n_2}} \cdots \mathbb{Z}_{p^{n_r}}$

where $n_1, n_2 \dots n_r$ are partition of n

$\# G = 16$, 2^4

$$\begin{aligned} 4 &= 4 \\ &= 3+1 \\ &= 2+2 \\ &= 2+1+1 \\ &= 1+1+1+1 \end{aligned}$$

$G \cong \mathbb{Z}_{2^4}$ or $\mathbb{Z}_{2^3} \times \mathbb{Z}_{2^1}$

or $\mathbb{Z}_{2^2} \times \mathbb{Z}_{2^2}$ or $\mathbb{Z}_{2^2} \times \mathbb{Z}_{2^1} \times \mathbb{Z}_{2^1}$

or $\mathbb{Z}_{2^1} \times \mathbb{Z}_{2^1} \times \mathbb{Z}_{2^1} \times \mathbb{Z}_{2}$

$\Rightarrow$ G is isomorphic to any one of the above gps.

$\# G = p^n$ where G is an abelian gp

$\Rightarrow$ G is isomorphic to any one of $p(n)$, the partition of n, gps.

② Let G be an abelian gp of order $p^n q^m$ where p & q are primes & m, n are +ve integers.
Then G is isomorphic to any one of $p(n)p(m)$ abelian gps.

Eg: $\# G = 72 = 2^3 \times 3^2$

$3 = 3,\ 2+1,\ 1+1+1$; $2 = 2,\ 1+1$

G is isomorphic to any one of the following gps

$\mathbb{Z}_{2^3} \times \mathbb{Z}_{3^2}$, $\mathbb{Z}_{2^3} \times \mathbb{Z}_{3^1} \times \mathbb{Z}_{3^1}$

$\mathbb{Z}_{2^2} \times \mathbb{Z}_{2^1} \times \mathbb{Z}_{3^2}$, $\mathbb{Z}_{2^2} \times \mathbb{Z}_{2^1} \times \mathbb{Z}_3 \times \mathbb{Z}_3$

$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{3^2}$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$

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Note: ① Let G be an abelian gp of order m. If $o(m) = p^n$, then the no of non-isomorphic abelian of order is $p(n)$, the partition of n.

② If $o(n) = p^r q^s$ where p & q are primes, then the no of non-isomorphic abelian gp of order $p^r q^s$ is $p(r)p(s)$.

③ If $o(n) = p_1^{n_1} p_2^{n_2} \cdots p_r^{n_r}$ where p_i's are distinct primes then there are $p(n_1)p(n_2)\cdots p(n_r)$ non-isomorphic abelian groups of order $p_1^{n_1} p_2^{n_2} \cdots p_r^{n_r}$.

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