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UNIT - III

GROUP ACTIONS AND SYLOW'S THEOREMS

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- Let G be a finite group of order n. If p, a prime, divides n, then there is an element in of order p.
  This theorem is known as the **Cauchy's Theorem.**

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Cauchy's Theorem

$p \mid o(G) \Rightarrow \exists$ an elt $a \in G \ni o(a) = p$

Subgp H of G of order p
$\quad\quad = \langle a \rangle$

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p-group The order of each elt in G is a power of p

$a \in G \Rightarrow o(a) = p^i$ for some i

Corollary Let G be a finite gp. Then G is a p-group iff $o(G) = p^n$ for some +ve integer n.

Proof: Assume that G is a p-group

Suppose $o(G) = p^n \underline{m}$ where n & m are +ve integers and $p \nmid m$.

If $m > 1$, then there exists a prime q divides m

$\Rightarrow q \mid o(G)$

Since q is a prime

$\Rightarrow \exists$ an elt a of G of order q, $a \Rightarrow \Leftarrow$ The order of each elt in G is a power of p

$\therefore m > 1$ is impossible

$\Rightarrow m = 1$

$\therefore o(G) = p^n$

Conversely, we assume that $o(G) = p^n$

Let $a \in G$

$\Rightarrow a^{o(G)} = e$

$\Rightarrow o(a) \mid o(G) = p^n$

$\Rightarrow o(a) = p^i$ for $1 \le i \le n$

$\Rightarrow$ the order a is p^i for some i

$\therefore$ the order of every elt in G is a power of p.

——— x ———

Let $X = G$. Define an action

$$* : X \times G \longrightarrow G$$
$$\underline{(x, g)} \longmapsto g^{-1} x g$$

for all $x \in X$, $g \in G$.

We know that X is a G-set

$$xG = \{ xg \mid g \in G \}$$
$$xG = \{ g^{-1} x g \mid g \in G \}$$

WKT $\# xG = [G : G_x] = \# \left(\frac{G}{G_x} \right)$

$$G_x = \{ g \in G \mid xg = x \}$$
$$= \{ g \in G \mid g^{-1} x g = x \}$$

$\therefore G_x = \{ g \in G \mid gx = xg \}$

Since $X = G$

$\Rightarrow G_x$ is the normalizer of x, $N(x)$

$$\therefore \# xG = \# \left(\frac{G}{N(x)} \right)$$

$$X_G = \{ x \in X \mid xg = x \text{ for all } g \in G \}$$
$$= \{ x \in X \mid g x g^{-1} = x \text{ for all } g \in G \}$$
$$= \{ x \in X \mid gx = xg \text{ for all } g \in G \}$$

Since $X = G$

$\Rightarrow X_G = Z(G)$

WKT

$X = X_G \cup (\cup xG)$

Since G is a finite G-set and $X = G$

$\Rightarrow \# G = (\# Z(G)) + \sum \# \left(\frac{G}{G_x}\right)$ | $\# H = G$

$\therefore \# G = \# Z(G) + \sum_{x \notin Z(G)} \frac{\# G}{\# G_x}$

This is known as class equation

Note that $\#\left(\frac{G}{H}\right) \mid \# G$ and $Z(G)$ is a subgp of G

$\Rightarrow$ each term in the class equation has a divisor of $o(G)$.

① $\# G = 20$

$\Rightarrow$ class equation

$\# G = \underline{3} + \underline{4} + \underline{5} + \underline{8}$ | $20 \neq$ $\# G = 2+4+5+5+1$ not a class equation

$\therefore$ This is not a class equation for a gp of order 20 because 3 & 8 are not a divisor of 20.

② If G is an abelian

$\Rightarrow G = Z(G)$,

$\Rightarrow$ the equation

$\# G = 1 + 1 + \cdots + 1$

$\leftarrow \# \frac{Z(G)}{G} \rightarrow$

$N(a) = G$

$\# \frac{G}{N(a)} = 1$

Example Let $G = S_3$

$= \{e, (1,2), (1,3), (2,3), (1,2,3), (1,3,2)\}$

$cl(e) = \{e\}$

$cl((1,2)) = \{(1,3), (2,3), (1,2)\}$

$cl(1,2,3) = \{(1,2,3), (1,3,2)\}$

The class equation of S_3 is

$$\# S_3 = 1 + 3 + 2$$

—— x ——

Theorem Every group of order prime square is abelian

Proof: Let G be a group of order p^2 where p is a prime

Claim $Z(G) = G$

Suppose $Z(G) \subsetneq G$

$\Rightarrow \exists\, a \in G$ but $a \notin Z(G)$

Clearly $Z(G) \subseteq N(a)$ and $a \in N(a)$

$$\Rightarrow Z(G) \subsetneq N(a)$$

Since $Z(G)$ is a subgp of G

$$\Rightarrow \# Z(G) \mid \# G = p^2$$

$$\Rightarrow \# Z(G) = 1 \text{ or } p \text{ or } p^2$$

Since $\# G = p^2 \Rightarrow \# Z(G) \neq 1$

$$\therefore \# Z(G) = p \text{ or } p^2$$

Since $Z(G) \neq G$ and $\# G = p^2$

$$\Rightarrow \# Z(G) = p$$

Since $N(a)$ is a subgp of G $\Rightarrow o(N(a)) \mid o(G) = p^2$

$$\Rightarrow o(N(a)) = 1 \text{ or } p \text{ or } p^2$$

Since $Z(G) \subsetneq N(a)$ & $\# Z(G) = p$ $\Rightarrow o(N(a)) = p^2$

Since $N(a)$ is a subgp of G & $\# G = \# N(a)$ $\Rightarrow G = N(a)$

$\Rightarrow$ a commute with every elt of G $\Rightarrow a \in Z(G)$, a $\Rightarrow\Leftarrow$ to $a \notin Z(G)$

$\therefore Z(G) = G$ $\therefore$ G is abelian //

Lemma If $o(G) = p^n$, then $Z(G) \neq \{e\}$ i.e. $\# Z(G) > 1$

Proof. Suppose $Z(G) = \{e\}$

$\Longrightarrow$ The class equation

$$\# G = \# Z(G) + \sum_{a \notin Z(G)} \# \left(\frac{G}{N(a)}\right) \quad (1)$$

$a \notin Z(G) \Rightarrow N(a) \subsetneq G$

$$\Rightarrow \# \frac{G}{N(a)} \neq 1.$$

Since $\# \frac{G}{N(a)} \Big| \# G = p^n$ $\quad \# \frac{G}{N(a)} \neq 1$

$$\Longrightarrow \# \frac{G}{N(a)} = p^i \text{ for some } i > 1$$

$$\Longrightarrow p \Big| \# \frac{G}{N(a)}$$

$$\Rightarrow p \Big| \sum_a \# \frac{G}{N(a)}$$

$$\Longrightarrow p \Big| \# G - \sum_a \# \left(\frac{G}{N(a)}\right) = \# Z(G)$$

Since $p > 1$ a prime

$$\Longrightarrow \# Z(G) \geq p > 1$$

$$\therefore Z(G) \neq \{e\}$$

Note. (1) Let $G = \{\pm 1, \pm i, \pm j, \pm k\}$

with $\quad ij = -ji, \; jk = -kj, \; ki = -ik$

G is a non-abelian gp with 2^3 elts

(2) D_4 — dihedral gp of degree 4

$$\# D_4 = 8$$

is non-abelian gp.

Sylow's Theorem

Let G be a gp. Let X be the set of all subgps of G.

Define

$$* : X \times G \longrightarrow X$$
$$(H, g) \longmapsto g^{-1}Hg$$

for all $H \in X$ & $g \in G$

WKT if H is a subgp of G, then $g^{-1}Hg$ and gHg^{-1} are subgroups of G

Ex. Show that X is a G-set, ie, $*$ is an action of G on X

Let $H \in X$, then

$$G_H = \{g \in G \mid Hg = H\}$$
$$= \{g \in G \mid g^{-1}Hg = H\}$$

is a subgp of G (isotropy subgp)

X G-set
$x \in X$
$G_x = \{g \in G \mid xg = x\}$

Let $h \in H$

$$\Rightarrow h^{-1}Hh = Hh = H$$
$$\Rightarrow h \in G_H$$

Clearly $H \subseteq G_H$

Ex. $H \triangleleft G_H$.

Let $h \in H$, $g \in G_H$

$ghg^{-1} \in H$

$$g^{-1}Hg = H$$
$$\Rightarrow H = gHg^{-1}, \quad ghg^{-1} \in gHg^{-1} = H$$

G_H is the largest subgp of G having H as a normal subgp.

—— x ——

$$G_H = \{g \in G \mid g^{-1}Hg = H\} = \{g \in G \mid Hg = gH\}$$
$$= N(H)$$

normalizer of H

Lemma: Let H be a p-subgp of a finite gp G. Then

$$[N(H):H] \equiv [G:H] \pmod{p}$$

Proof: Let $X = \{Hg \mid g \in G\}$ be the set of all right cosets of H in G.

$$\Rightarrow \# X = [G:H]$$

Define

$$* : X \times H \longrightarrow X$$
$$(Hg, h) \longmapsto H(gh)$$

for all $Hg \in X$, $h \in H$

Clearly $*$ is an action of H on X (Ex.)

$\Rightarrow$ X is a H-set

$$X_H = \{x \in X \mid xh = x \ \forall h \in H\}$$
$$= \{Hg \in X \mid (Hg)h = Hg \ \forall h \in H\}$$
$$= \{Hg \in X \mid H(gh) = Hg \ \forall h \in H\}$$

$$X_H = \{Hg \in X \mid ghg^{-1} \in H \ \forall h \in H\}$$

Since $N(H) = \{g \in G \mid gHg^{-1} = H\}$

$X_H = \{ Hg \in X \mid g \in \underline{N(H)} \}$

$\Rightarrow$ X_H is the colln of cosets of H in $N(H)$

$\Rightarrow \boxed{\# X_H = [N(H) : H]}$

We know that if G is a p-gp, then $\# X \equiv \# X_G \pmod p$

Since H is a p-gp

$\Rightarrow \# X \equiv \# X_H \pmod p$

$\Rightarrow [G : H] \equiv [N(H) : H] \pmod p$

$\Rightarrow [N(H) : H] \equiv [G : H] \pmod p$

$\boxed{a \equiv b \pmod n \text{ iff } b \equiv a \pmod n}$

Sylow's Theorem

- **Sylows First Theorem**

  Let G be a finite group and let p be a prime. If $p^k|\#(G)$, then

  1. G has at least one subgroup H of order p^k.
  2. H has a normal subgroup K of order p^{k-1}.

- A subgroup of G of order p^k where $p^k|\#(G)$ but $p^{k+1} \nmid \#(G)$ is called a Sylow p-subgroup.

- Sylows Second theorem

  Any two Sylow p-groups are conjugate. That is, if H and K are two Sylow p-groups, then $H = gKg^{-1}$ for some $g \in G$.

- Sylow's Third Theorem

  Let $|G| = p^k m, p \nmid m$. The number of Sylow p-subgroups of G is equal to 1 modulo p and divides $\#(G)$.

- If p is a prime and $p^k|\#(G)$, then the number of subgroups of G

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Lemma: Let H be p-subgp of a finite gp G. Then

$[N(H):H] \equiv [G:H] \pmod p$

Note: If $p \mid [G:H]$, $\Rightarrow$ $p \mid [N(H):H]$

$\Rightarrow N(H) \neq H$, i.e. $H \subsetneq N(H)$

WKT $H \triangleleft N(H)$.

$N(H) = \{g \in G \mid g^{-1}Hg = H\}$

$Hg = gH$

$1, p, p^2, \ldots p^n$

First part of Sylow's Theorem

Statement: Let G be a finite gp of order $p^n m$ where $p \nmid m$

Then

① there exists a subgp H of order p^i for $1 \leq i \leq n$

② there exists a subgp K of order p^{i+1} such that H is a normal subgp of K.

Proof: ① We prove this by induction on i.

If $i=1$, then $p^1 \mid o(G)$, by Cauchy's theorem

$\Rightarrow \exists$ a subgp of H of order p

We assume that ~~this is true for $i<n$.~~

Claim $\exists$ a subgp K of order p^{i+1}

Let H be a subgp of G of order p^i

$\Rightarrow p \mid [G:H] = \dfrac{o(G)}{o(H)} = \dfrac{mp^n}{p^i} = mp^{n-i}$

$i<n$

$n-i>0$

$\Rightarrow p \mid [N(H):H]$ since $[G:H] \equiv [N(H):H] \pmod p$

Since $H \triangleleft N(H)$, $\Rightarrow \frac{N(H)}{H}$ is a group

Since $p \mid [N(H):H] \Rightarrow p \mid \#\left(\frac{N(H)}{H}\right)$

By Cauchy's theorem, there exists a subgp K'

of $\frac{N(H)}{H}$ with $\#(K') = p$

WKT. every subgp of $\frac{N(H)}{H}$ is $\frac{K}{H}$ where K is <u>some subgp of $N(H)$ contains H.</u>

$\Rightarrow$ Let us take $K' = \frac{K}{H}$

Since $\#K' = p \Rightarrow p = \#\left(\frac{K}{H}\right) = \frac{\#(K)}{\#H}$

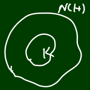
$$\Rightarrow \#K = p(\#H) = p \cdot p^i = p^{i+1}$$

$\therefore$ K is a <u>subgp of $N(H)$ and hence a subgp of G of order p^{i+1} & K containing H.</u>

$\therefore$ G has a subgp of order p^i for $1 \le i \le n$

(2) From the above proof,

$$H \le K \subseteq N(H)$$



Since $H \triangleleft N(H) \Rightarrow H \triangleleft K$

Thus we proved the subgp H of order p^i is a normal subgp of K of order p^{i+1} for $1 \le i < n$

—x—

If $o(G) = 24 = 2^3 \times 3$

$1, 2, 1, 2, 1, 2 \quad 1, 3$

$\Rightarrow$ G has subgps of orders $1, 2, 2^2 = 4, 2^3 = 8$ & 3

<u>Definition</u> Sylow <u>p-subgp</u>

— <u>maximal p-subgp</u>

A subgp H of G is said to be a Sylow p-subgroup if $o(H) = p^n$ & $p^n \mid o(G)$ but $p^{n+1} \nmid o(G)$

A maximal p-subgroup of a group is said to be a <u>Sylow p-subgp</u>.

$o(G) = 50 = 2 \times 25 = 2 \times 5^2$

There exist Sylow 2-subgp, Sylow 5-subgp

2   25

$\# G = 72 = 2^3 \times 3^2$

— Sylow 2-Subgp - order 2^3

Sylow 3-Subgp - order 3^2

H & K are conjugate subgps of G if $H = g^{-1} K g$

$K = g^{-1} H g$

V_4 — Klein 4-gp $= G = \langle a, b, c, e \rangle$

$G_1 = \langle e, a \rangle$ $a^2 = b^2 = c^2 = e$

$G_2 = \langle e, b \rangle$, $G_3 = \langle e, c \rangle$

2nd Part of Sylow's Theorem

Statement: Any two Sylow p-subgps are Conjugate

Proof: Let P_1 and P_2 be two Sylow p-subgps of a finite gp G

Let $X = \{P_1 g \mid g \in G\}$, then $\# X = [G : P_1]$

Define

$$* : X \times P_2 \to X$$

by $*(P_1 g, y) = P_1(gy) \in X$

for all $P_1 g \in X$ & $y \in P_2$

Clearly X is a P_2-set

WKT

$$\# X \equiv \# X_{P_2} \pmod p$$

$\# G = p^n m$

$p \nmid m$

WKT $\# X = [G : P_1] = \dfrac{\# G}{\# P_1}$

$X_H = \{x \in X \mid xg = x \ \forall g \in H\}$

$\Rightarrow p \nmid \# X$

$X_{P_2} = \{P_1 g \mid P_1 g y = P_1 g$

$\Rightarrow p \nmid \# X_{P_2}$

$\Rightarrow \exists$ an elt $P_1 g \in X_{P_2}$ for some $g \in G$

$\Rightarrow (P_1 g) y = P_1 g$ for all $y \in P_2$

$\Rightarrow P_1(gy) = P_1 g$ "

$\Rightarrow g^{-1} y g \in P_1$ for all $y \in P_2$

$\Rightarrow g^{-1} P_2 g \subseteq P_1$

$\Rightarrow P_1 = g^{-1}P_2 g$ since $\# P_1 = \# P_2 = \# g^{-1}P_2 g$

i.e. P_1 and P_2 are conjugate

$\therefore$ Any two Sylow p-subgps are conjugate

3rd Part of Sylow's Theorem

Statement: The no. of Sylow p-subgps of a finite gp is congruent to 1 modulo p and it divide the order the gp.

$r \equiv 1 \pmod p$

$r = kp+1$ for some k

Proof: Let X be the set of all Sylow p-subgps of a finite group G

Let P be a Sylow p-subgp of G.

Define an action

$* : X \times P \longrightarrow X$

by $*(T, y) = y^{-1} T y \in X$

for all $T \in X$ & $y \in P$.

clearly X is a P-set.

$\Rightarrow \# X \equiv \# X_p \pmod p$

Let $T \in X_p = \{H \in X \mid Hg = H\}$

$\Rightarrow Tg = T$ for all $g \in P$

$\Rightarrow g^{-1} T g = T$ $\forall g \in P$ by definition of action

clearly $a \in H \Rightarrow aH = H$, $Ha = H$

$P \in X_p = \{x \mid xp = x\}$

$Pg = P$

$g^{-1}Pg = P$ $\forall g \in P$

$\Rightarrow P \leq N(T)$ $\quad N(H) = \{g \in G \mid g^{-1}Hg = H\}$

Since $T \leq N(T)$ & P, T are Sylow p-subgp of G

$\Longrightarrow P$ & T are Sylow p-subgps of $N(T)$

By the 2nd part of Sylow's theorem

$\Rightarrow P = g^{-1}Tg$ for some $g \in N(T)$

Since $N(T) = \{g \in G \mid g^{-1}Tg = T\}$
and $g \in N(T)$

$\Longrightarrow g^{-1}Tg = T$

$\therefore P = g^{-1}Tg = T$

$\therefore T = P$

$\therefore X_p = \{P\}$ i.e. $\# X_p = 1$

$\therefore \# X \equiv 1 \pmod p$

$\therefore$ The no Sylow p-subgps of G is congruent to 1 modulo p.

— x —

Note. The no Sylow p-subgps of G is $\equiv 1 \pmod p$

$\Longrightarrow$ The no of Sylow p-subgps of $G = 1 + kp$ for some k

Define an action

$*: X \times G \longrightarrow X$

by $*(P, g) = g^{-1}Pg$ for all $P \in X, g \in G$

Clearly X is a G-set

$\Rightarrow \# X \equiv \# X_G \pmod p$ —— (1)

Consider

$X_G = \{ P \in X \mid Pg = P \ \forall g \in G \}$

$= \{ P \in X \mid \underline{g^{-1}Pg = P \ \forall g \in G} \}$

$= \{ g^{-1}Pg \mid g \in G \}$ = conjugate class of P

$X_G = \{ Pg \mid g \in G \}$ = orbit of P

i.e., X_G is the orbit of P

$\Rightarrow \# X_G = \#(\text{orbit of } P)$

$= \# \left(\frac{G}{G_P} \right)$

$\Rightarrow$ Since $\# \frac{G}{G_P} \mid \# G$

$\Rightarrow \# X_G \mid \# G$

Since $\# X \equiv \# X_G \pmod p$

$\Rightarrow \# X \mid \# G$

i.e., The no. of Sylow p-subgps divides $o(G)$.

$xG = \{ xg^{-1} \mid g \in G \}$

orbit of x

$\# xG = \frac{\# G}{\# G_x}$ $N(x)$

$\# \frac{G}{H} \mid \# G$

$r \, \# H = \# G$

(3) No. of Sylow p-subgp $= 1 + kp$

$1+kp \mid \# G$

(2) A & B are Sylow p-subgp $\Rightarrow A = g^{-1}Bg$ for any $g \in G$

(1) $\# G = p^n m$, $p \nmid m$

$\Rightarrow \exists$ a subgp of order p^i $1 \le i \le n$ | subgp of order of p^{i+1}, $1 \le i < n$

& the subgp H of order p^i is a normal subgp of ...

Example Let $G = S_3$

$\# G = 6 = 2 \times 3$

$\Rightarrow$ Sylow 2-subgp & Sylow 3-subgp exist

$\Rightarrow$ order 2 & order 3

$\#$ of Sylow 3-subgp $= 1+3k$

$1+3k \mid \# G = 6 = 3 \times 2$

$\left| \begin{array}{l} (j+3k, 3^i) = 1 \\ 1 \le j \le 2 \end{array} \right.$

Since $(1+3k, 3) = 1 \Rightarrow 1+3k \nmid 3$

$\Rightarrow 1+3k \mid 2$

1, 2   1   2

$1+3k = 1$ ✓ or $1+3k = 2$ ✗

$\therefore 1+3k = 1$ is possible

$\Rightarrow$ the no of Sylow 3-Subgp is only one

$H = \{e, (1,2,3), (1,3,2)\}$

$\#$ of Sylow 2-Subgps $= 1+2k$

$1+2k \mid \# G = 3 \times 2$

$\Rightarrow 1+2k \mid 3$

1   3

$\Rightarrow 1+2k = 1$ or 3

$\Rightarrow$ G has either one Sylow 2-subgp or 3 Sylow 2-Subgps

$S_3 = \{e, (1,2), (1,3), (2,3), (1,2,3), (1,3,2)\}$

$G_1 = \{e, (1,2)\}$, $G_2 = \{e, (1,3)\}$, $G_3 = \{e, (2,3)\}$

If G has only one Sylow p-subgp, then it is normal

Proof: Let H be the only Sylow p-subgp of G

Since $\# H = \#(g^{-1}Hg)$ for all $g \in G$

$\Rightarrow g^{-1}Hg$ is a Sylow p-subgp of G for all $g \in G$

Given that G has only one Sylow p-subgp H.

$\Rightarrow H = g^{-1}Hg$ for all $g \in G$

$\therefore$ H is a normal subgp of G

—— x ——

$\therefore$ The Sylow 3-subgp of S_3 is normal

—— ^ ——

A group G is said to be simple if it has no nontrivial normal proper subgp.

i.e., G has no normal subgp other than $\{e\}$ & G

Problem Every group of order 15 is not a simple group.

Proof: Let $\#G = 15$

$15 = 3 \times 5$

$\Rightarrow$ G has Sylow 3-subgp & Sylow 5-subgp of order 3 and 5 respectively

The # of Sylow 5-subgps $= 1+5k$

$1+5k \mid \#G = 5 \times 3$

$\Rightarrow 1+5k \mid 3$

$\Rightarrow k=0$ is the only possibilities

$\therefore 1+5k = 1$ i.e. G has only one Sylow 5-subgp

$\Rightarrow$ it is normal subgp and is non-trivial & proper

$\therefore$ G is not simple

# of Sylow 3-subgps $= 1+3k$

$1+3k \mid \#G = 5 \times 3$

$\Rightarrow 1+3k \mid 5$

$\Rightarrow k = 0$ is the only possibility

$\therefore$ G has only one Sylow 3-Subgp H and only one Sylow 5-Subgp K of order 3 & 5 respectively

ie. $o(H) = 3$ & $o(K) = 5$

$\Rightarrow$ H & K are cyclic and hence abelian

Let $x \in H \cap K$

$\Rightarrow x \in H$ & $x \in K$

$\Rightarrow o(x) \mid o(H)$ & $o(x) \mid o(K)$

ie $o(x) \mid 3$ & $o(x) \mid 5$

$1, 3$ $\quad$ $1, 5$

$\Rightarrow o(x) = 1$

ie, $x = e$

$\therefore H \cap K = \{e\}$

Let $x, y \in G$

$\Rightarrow$ ① $x, y \in H$ or K ② $x \in H$, $y \in K$

① Since H & K are abelian $\Rightarrow xy = yx$

$\Rightarrow$ ① proved

② Since H & K are normal

$\Rightarrow xyx^{-1} \in K$ since $x \in H \leq G$, $y \in K$

Since $y \in K \Rightarrow y^{-1} \in K$

$\therefore xyx^{-1}y^{-1} \in K$

Since $x \in H \Rightarrow x^{-1} \in H$

Since H is normal $\Rightarrow yx^{-1}y^{-1} \in H$ $\forall y \in K \leq G$

Since $x \in H$ $\Rightarrow x(yx^{-1}y^{-1}) \in H$

$\therefore xyx^{-1}y^{-1} \in H \cap K = \{e\}$

$\therefore xyx^{-1}y^{-1} = e$

$\Rightarrow xy = yx$

$\therefore$ G is an abelian gp

Since H & K are cyclic

$H = \langle a \rangle$, $K = \langle b \rangle$

Let $c = ab \in G$

Consider $c^{15} = (ab)^{15}$

$= a^{15} b^{15} = (a^3)^5 (b^5)^3 = e^5 e^3 = e$

i.e. $c^{15} = e$

$\Rightarrow o(c) \mid 15$ i.e. $o(c) \le 15$

Let $o(c) = r$

$\Rightarrow c^r = e$

$\Rightarrow (ab)^r = e \Rightarrow a^r b^r = e$

$\Rightarrow a^r = e$ & $b^r = e$ since $H \cap K = \{e\}$

$\Rightarrow o(a) \mid r$ & $o(b) \mid r$

i.e. $3 \mid r$ & $5 \mid r$

$\Rightarrow 15 \mid r$ i.e. $15 \le r = o(c)$

$o(c) = 15$ i.e. $G = \langle c \rangle$

$\therefore$ G is a cyclic group

Let $o(G) = 11^2 \cdot 13^2$ Prove that G is abelian

Sylow 11-Subgp — $1 + 11k \mid o(G) = 11^2 13^2$

" 13-Subgp — $1 + 13k \mid o(G) = 11^2 13^2$

$1 + 13k \mid 11^2$

$\Rightarrow k = 0$ is only possible

$\Rightarrow$ G has only one Sylow 13-Subgp H of order 13^2

Similarly . . . Sylow 11-Subgp K 11^2

$\Rightarrow$ H & K are abelian

① $H \cap K = \{e\}$

② $x, y \in G \quad xy = yx$

① $x, y \in H$ or K , $x \in H, y \in K$

Continue ...

- A group G is said to be a **simple group** if it has not nontrivial normal subgroups.
- Let G be a group of order pq where p and q are distinct primes. Then
  1. If $q \equiv 1 \pmod{p}$, then G has a normal Sylow p-subgroup.
  2. G is not simple.
  3. If $p \equiv 1 \pmod{q}$ and $q \equiv 1 \pmod{p}$, then G is a cyclic group.
- The only groups of order 255 is $\mathbb{Z}_{255}$.
- There are exactly 4 groups of order 66 namely, $\mathbb{Z}_{66}, D_{33}, D_{11} \oplus \mathbb{Z}_3$ and $D_3 \oplus \mathbb{Z}_{11}$.

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