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Programme : M.Sc. Mathematics
Course Title : Complex Analysis
Course code : 24S3M09CC

UNIT 3

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Complex Analysis, Sep 2, 2020.

Proof of chain rule

(i) $\gamma : [a,b] \to U$ diff

$f : U \to \mathbb{C}$ diff

$g = (f \circ \gamma) : [a,b] \to \mathbb{C}$

Let $t_0 \in (a,b)$

claim: $g'(t_0)$ exists & equal to

$$g'(t_0) = (f\circ\gamma)'(t_0) = f'(\gamma(t_0)) \cdot \gamma'(t_0)$$

Let $\gamma(t_0) = b$

$\because$ γ is diff at t_0

$\exists$ fn γ_1 on $[a,b]$ s.t

(i) $\gamma(t) = \gamma(t_0) + \gamma_1(t)(t - t_0)$, $t \in [a,b]$

(ii) γ_1 is conts at t_0 & $\gamma'(t_0) = \gamma_1(t_0)$

lly f is diff at b $\exists$ fn f_1 on U s.t

(i) $f(z) = f(b) + f_1(z)(z-b)$, $z \in U$

(ii) f_1 is conts at b & $f_1(b) = f'(b)$

$$\begin{aligned} g(t) &= (f\circ\gamma)(t) \\ &= f(\gamma(t)) \\ &= f(b) + f_1(\gamma(t))(\gamma(t) - b) \\ &= f(\gamma(t_0)) + f_1(\gamma(t))(\gamma(t_0) + \gamma_1(t)(t-t_0) - b) \\ &= f(\gamma(t_0)) + f_1(\gamma(t))\,\gamma_1(t)(t-t_0) \\ &= g(t_0) + (t-t_0)\,g_1(t) \end{aligned}$$

choose $g_1(t) = f_1(\gamma(t))\,\gamma_1(t)$

composition of conts is conts

$$\left\{ \lim_{t\to t_0} g_1(t) = f_1(\gamma(t_0))\,\gamma_1(t_0) = g_1(t_0) \right\}$$

$$g_1(t_0) = f_1(b)\cdot \gamma_1(t_0)$$
$$= f'(b)\cdot \gamma'(t_0)$$
$$= f'(\gamma(t))\cdot \gamma'(t_0) \checkmark$$

Part (ii)

$f : U \to V$ diff, $g : V \to W$ diff

$h = (g \circ f) : U \to W$ is diff on U

Ex

Every diff saly $f : U \to \mathbb{C}$ is conts on U

Let $f : U \to \mathbb{C}$ diff on U

Let $a \in U$

$\because f$ is diff at a, $\exists$ a fn $f_1 : U \to \mathbb{C}$ s.t

(i) $f(z) = f(a) + f_1(z)(z-a)$, $z \in U$

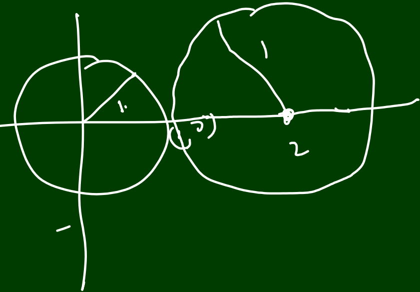
(ii) f_1 is Conts at a & $f_1(a) = f'(a)$

By Algebra of Conts fns f is conts.

1) U - open set
$f : U \to \mathbb{C}$ is dif on U & $\forall z \in U$, $f'(z) = 0$
Then f is constant on U

The above statement is false, Why?

$$f(z) = \begin{cases} 1 & \forall z \in B(0,1) = U_1 \\ 2 & \forall z \in B(2,1) = U_2 \end{cases}$$



$\forall z \in G = U_1 \cup U_2$

$f'(z) = 0$

f is locally constant is correct statement.

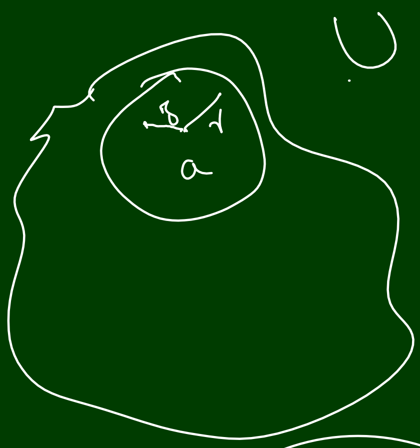
Let us prove this

Thm Let $U \overset{\text{open}}{\subseteq} \mathbb{C}$, $f \in H(U)$ and $\forall z \in U$, $\boxed{f'(z) = 0}$

Then f is locally const

i.e $\forall a \in U \big(\exists r > 0$ s.t $B(a,r) \subset U$

$\exists$ const $M \in \mathbb{C} \big(\forall z \in B(a,r) \, (f(z) = M) \big) \big)$.



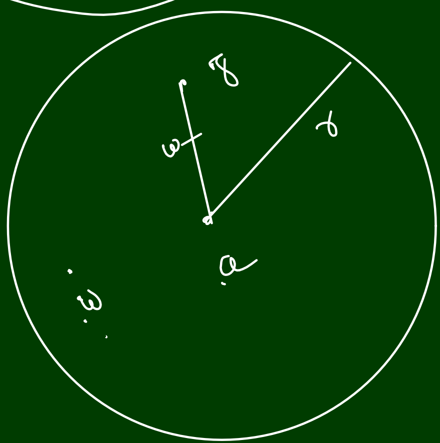
Proof:

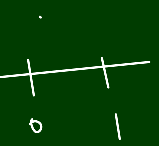
Hint: Use chain rule.

Let $a \in U$

$\exists r > 0$ s.t $B(a,r) \subset U$ $[\because U$ is open$]$

Let $z \in B(a,r)$





$w = (1-t)a + tz, \quad t \in [0,1]$

$\gamma : [0,1] \to U$

$\gamma(t) = (1-t)a + tz$

Note ~~that~~ $\gamma(t) \in B(a,r)$ $\forall t \in [0,1]$

Take $g = f \circ \gamma : [0,1] \to \mathbb{C}$

$\therefore$ By chain rule g is diff on $[0,1]$

& $g'(t) = f'(\gamma(t)) \cdot \gamma'(t)$

$= 0 \cdot \gamma'(t) = 0$

$g : [0,1] \to U$

$\therefore \forall t \in [0,1], \quad g'(t) = 0$

Claim g is constant

g_1 is const

g_2 is const

$g = \operatorname{Re} g + i \operatorname{Im} g$

$= g_1 + i g_2$

$g_1 : [0,1] \to \mathbb{R}$

$g_2 : [0,1] \to$

$g'(t) = g_1'(t) + i g_2'(t)$

$\therefore \; g(0) = g(1)$

i.e $f(a) = f(z)$

$\therefore \; \forall z \in B(a,r), \quad f(z) = f(a)$

$\therefore$ f is locally constant.

Cor $f : U \to \mathbb{C}$, U - open, connected &

$f' = 0$ on U then f is constant.

By P.T f is locally constant fn

Let $a \in U$

$$G = \{ z \in U : f(z) = f(a) \}.$$

$G \subseteq U$ connected

EST (i) $G \neq \phi$

(ii) G is clopen

$a \in G \; \therefore G \neq \phi$ let $f(a) = b$

claim 2 G is closed

$G = f^{-1}(\{b\})$ ($\because$ f is cont)

claim 2 G is open

Let $w \in G$

$\because$ f is locally constant $\exists \; r > 0$

$B(w,r) \subset U$

& $\forall z \in B(w,r), \; f(z) = f(w) = f(a)$

$\therefore B(w,r) \subset G$

$\therefore$ G is open $\therefore G = U$

We say $G \subseteq \mathbb{C}$ is region if G is open & connected

Define Convex set

$L \subseteq \mathbb{C}$ is convex iff

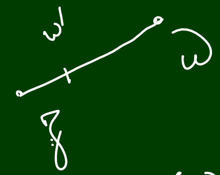
$\forall z, w \in L, \forall t \in [0,1], \quad (1-t)z + tw \in L$

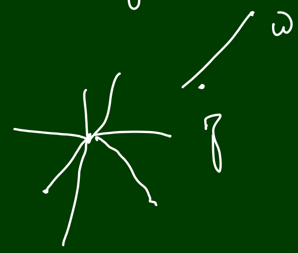
$[z,w] := \{(1-t)z + tw : t \in [0,1]\} \subset L$

Star shaped set

$A \subseteq \mathbb{C}$ is star shaped

at $z \in A$, $\forall w \in A$, $[z, w] \subset A$





z is called star centre

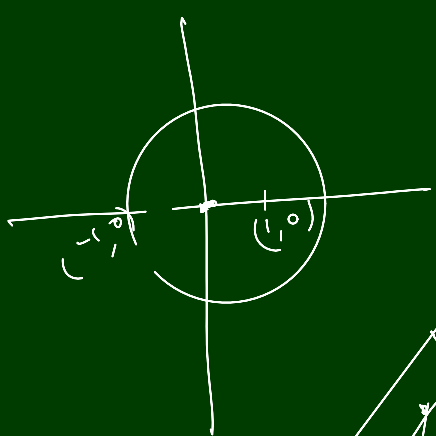
A is star shaped $\exists$ exists a star centre

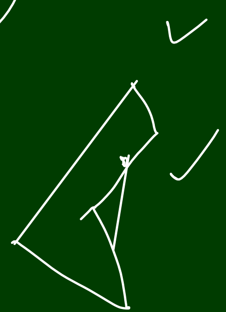
Every convex set is star shaped

Let $a \in C$. $\forall w \in C$, $[w, a] \subset C$

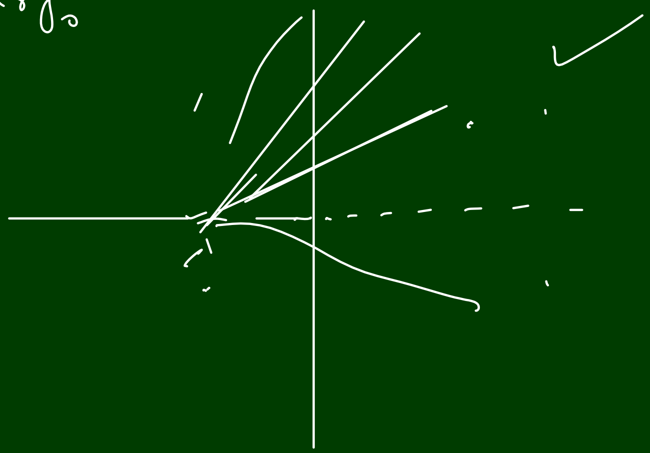
$B(0,1) \setminus \{0\}$

Give an example of a star shaped set but not convex





arg₀



Every star shaped is path connected
& hence connected.

Complex Analysis, Sep 3, 2020

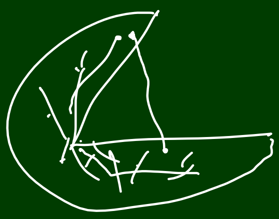
Recall: $A \subseteq \mathbb{C}$ is star shaped if $\exists$ a star center

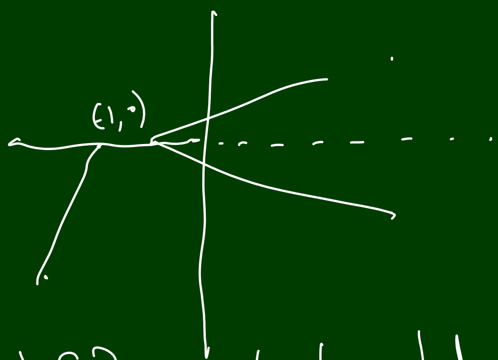
$a \in A$ is said star center of A if

$\forall z \in A$, $[z, a] \subseteq A$

$\mathbb{C} \setminus L_0$

Example





Exercise: $B'(0,1) = B(0,1) \setminus \{0\}$ is it star shaped?

Differentiability of Inverse fn.

Let U & V open sets in $\mathbb{C}$. Let $f : U \to V$ be a bijection & $g = f^{-1} : V \to U$

Let $a \in U$, $b = f(a)$. Assume that

(i) f is diff at $a \in U$ & $f'(a) \neq 0$

(ii) g is conts at b

Then g is diff at b & $g'(b) = (f'(a))^{-1} = \dfrac{1}{f'(a)}$

Proof

$\exists f_1 : U \to \mathbb{C}$ s.t

$f(z) = f(a) + (z-a) f_1(z)$, $\forall z \in U$ ——— (1)

& f_1 is conts at a & $f_1(a) = f'(a)$

Let $w = f(z)$

$$\frac{1}{f_1} = \begin{cases} \dfrac{z-a}{f(z)-f(a)} & z \neq a \\ \dfrac{1}{f'(a)} & z = a \end{cases}$$

Find $g_1 : V \to \mathbb{C}$ s.t

$g(w) - g(b) = (w-b) g_1(w)$

$g(w) = g(f(z)) = z$

$g(b) = g(f(a)) = a$

Put in eqn (1)

$$w - b = \big(g(w) - g(b)\big) f_1(z)$$

$$\therefore g(w) - g(b) = (w-b) \cdot \frac{1}{f_1(z)}$$

$$= (w-b) \cdot \frac{1}{f_1(g(w))}$$

$$= (w-b) \cdot \frac{1}{(f_1 \circ g)(w)}$$

$$= (w-b) g_1(w) \quad \text{where } g_1(w) = \frac{1}{(f_1 \circ g)(w)}$$

g is conts at b & f_1 is conts at $g(b) = a$

$\therefore (f_1 \circ g)$ is conts at b ✓

& $(f_1 \circ g)(b) = f_1(g(b)) = f_1(a) = f'(a) \neq 0$

$g : V \to U$

$f_1 : U \to \mathbb{C}$

$(f_1 \circ g) : V \to \mathbb{C}$

$\underbrace{f_1(g(w)) \neq 0}_{}$ $\forall\, w \in V$ because of defn of f_1

$$g_1(b) = \frac{1}{f_1 \circ g(b)} = \frac{1}{f_1(a)} = \frac{1}{f'(a)}$$

$y = \log x$

$y' = \frac{1}{x}$

Application:

Cor: $\log_\alpha$ is holomorphic on $\mathbb{C} \setminus L_\alpha$

& $\log_\alpha'(z) = \frac{1}{z}$ on $\mathbb{C} \setminus L_\alpha$

Let $f(z) = \exp(z) = e^{\operatorname{Re} z} \cdot e^{i \operatorname{Im} z}$

st $\operatorname{Im} z \in (\alpha, \alpha + 2\pi)$ Take $g = \log_\alpha$

Let $w = f(z)$ $g(w) = \log_\alpha f(z)$

$$\therefore g'(w) \overset{\text{Cor}}{=} \frac{1}{f'(z)} = \frac{1}{\exp(z)} = \frac{1}{w}$$

✓

[exp maps $\{z \in \mathbb{C} : \alpha < \operatorname{Im} z < \alpha + 2\pi\}$ to $\mathbb{C} \setminus L_\alpha$

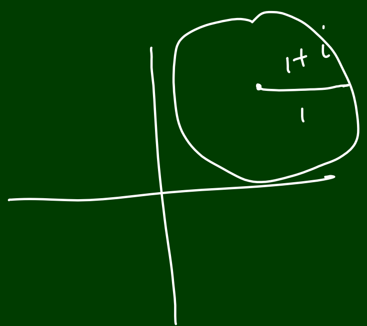
& $\log_\alpha : \mathbb{C} \setminus L_\alpha \to \mathbb{C}$ is its inverse]

Defn We say a conts fn $g : U \to \mathbb{C}$ is logarithm on U

in $\exp(g(z)) = z \quad \forall z \in U$

Ex If g is logarithm on U

Then S.T $\forall z \in U$ $g'(z) = \frac{1}{z}$



$U = B(1+i, 1)$

$f : U \to \mathbb{C}$

f is diff at z

$f(z) = \operatorname{Re} f(z) + i \operatorname{Im} f(z)$

$= u(z) + i v(z)$

$u, v : U \to \mathbb{R}$

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} \qquad u(x,y) \qquad z = x + iy$$

Case(i) $h \in \mathbb{R}$ & $h \to 0$

$$f'(z) = \lim_{\substack{h \to 0 \\ h \in \mathbb{R}}} \frac{u(x+h,y) - u(x,y) + i\left(v(x+h,y) - v(x,y)\right)}{h}$$

$u : \mathbb{R}^2 \to \mathbb{R}$

$$= \lim_{\substack{h \to 0 \\ h \in \mathbb{R}}} \frac{u(x+h,y) - u(x,y)}{h} + i \lim_{\substack{h \to 0 \\ h \in \mathbb{R}}} \left(\qquad \right)$$

$$f'(z) = u_x + i v_x$$

Case(ii) give incr in ih & $h \in \mathbb{R}$ $h \to 0$

$$f'(z) = \lim_{h \to 0} \frac{u(x,y+h) - u(x,y)}{ih} + i \left(\qquad \right)$$

$$f'(z) = -i\left(u_y + i v_y\right)$$

$$= v_y - i u_y$$

$$\therefore \boxed{\begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned}} \rightsquigarrow \text{Cauchy - Remainn equations}$$

$x = \frac{z + \bar{z}}{2}$

$y = \frac{z - \bar{z}}{2i}$

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial z} + i \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial z} = \frac{1}{2}\left(f_x - i f_y\right)$$

$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial \bar{z}} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial \bar{z}} = \frac{1}{2}\left(f_x - \frac{1}{i} f_y\right)$$

$$= \frac{1}{2}\left(f_x + i f_y\right)$$

$$df = \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \bar z} d\bar z$$

$$f'(z) = \frac{\partial f}{\partial z} + \frac{\partial f}{\partial \bar z} = f_x = \frac{\partial f}{\partial x}$$

$f'(z) = u_x + i v_x$

$f'(z) = v_y - i u_y$

$$\boxed{f'(z) = \frac{\partial f}{\partial x}}$$

$$f'(z) = -i\frac{\partial f}{\partial y} \quad \text{(C.R equat)}$$

$$\frac{\partial f}{\partial \bar z} = 0 \quad \text{(C-R eq).}$$

U open

$f : U \to \mathbb{C}$ f is diff on U then

$$\boxed{u_x = v_y \quad \& \quad u_y = -v_x}$$

are the C-R equations

Let U be a region (U is open & connected)

(i) If f is diff on U & $f' = 0$ on U then f is constant

$f'(z) = u_x + i v_x = 0$

$u, v : U$

(ii) If f is holom on U & $\text{Re} f$ (Im f) is Const on U then f is constant on U

$f = \text{Re} f + i \,\text{Im} f$

$f'(z) = u_x + i v_y$ $\qquad u_x = 0$

$= 0$ $\qquad v_y = 0$

(iii) If f is holom on U & $\bar f$ assumes only real values then f is const on U

$g(z) = i f(z)$

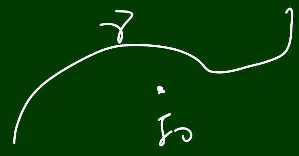
$\text{Re}\, g = 0$

Bye - Bye

Complex Analysis, Sep 7, 2020

$\gamma : [a,b] \to \mathbb{C}$ $\qquad \gamma \in C^1([a,b], \mathbb{C})$

$\forall t \in [a,b]$, $\gamma(t) \neq z_0$



$$\boxed{g(t) = \int_a^t \frac{\gamma'(s)}{\gamma(s) - z_0}\, ds}$$

$h = \operatorname{Re} h + i \operatorname{Im} h$

$\operatorname{Re} h, \operatorname{Im} h : [a,b] \to \mathbb{R}$

$$h(t) = e^{-g(t)}[\gamma(t) - z_0]$$

S.T h is const $\qquad$ Not $h : [a,b] \to \mathbb{C}$

EST $\underline{\forall t \in [a,b]}$, $h'(t) = 0$

$$h'(t) = e^{-g(t)}[\gamma'(t)] - (\gamma(t) - z_0)\, e^{-g(t)} \cdot \underline{g'(t)} \quad (*)$$

$w = \gamma(s) - z_0$ $\qquad$ change of variable

$dw = \gamma'(s)\, ds$

$$g(t) = \int_{\gamma(a) - z_0}^{\gamma(t) - z_0} \frac{dw}{w} = \ln w \Big|_{\gamma(a) - z_0}^{\gamma(t) - z_0}$$

$$= \ln\left(\frac{\gamma(t) - z_0}{\gamma(a) - z_0}\right)$$

$\ln f(t))$

$\frac{1}{f(t)}$

$$g'(t) = \frac{(\gamma(a) - z_0)}{\gamma(t) - z_0} \cdot \frac{\gamma'(t)}{(\gamma(\cancel{a}) - z_0)}$$

$$h'(t) = e^{-g(t)}\gamma'(t) - (\gamma(t) - z_0) \cdot e^{-g(t)} \cdot \frac{\gamma'(t)}{\cancel{\gamma(t)} - z_0}$$

$$= 0$$

$\forall t \in [a,b]$ $\quad h(t) = e^{-g(t)} \cdot (\gamma(t) - z_0) = c$

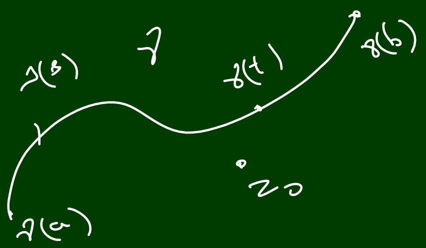
$c \exp g(t) = \gamma(t) - z_0$

Note that at $t=a$, $g(t)=0$

$\therefore \; C = \gamma(a) - z_0$

$\gamma,\ t \in [a,b]$

$\therefore \; \exp g(t) = \dfrac{\gamma(t) - z_0}{\gamma(a) - z_0}$

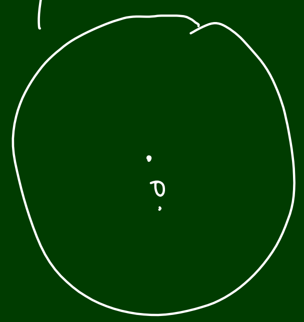


$\exp g(b) = \dfrac{\gamma(b) - z_0}{\gamma(a) - z_0} = \dfrac{\gamma(a) - z_0}{\gamma(a) - z_0} = 1$

$[a,b]$ If $\gamma(a) = \gamma(b)$, γ is a called closed path

$\gamma : [0, 2\pi] \to \mathbb{C}$, $\gamma(t) = e^{it}$

$\gamma(0) = \gamma(2\pi)$



$e^{g(b)} = 1 = \exp^{i2n\pi}$

$g(b) = 2n\pi i \quad n \in \mathbb{Z}$

$$g(t) := \int_a^t \frac{\gamma'(s)}{\gamma(s) - z_0}\, ds$$

$\dfrac{1}{2\pi i} g(t) \in \mathbb{Z}$

$g(b) = 2n\pi i$

$$\boxed{\frac{1}{2\pi i}\int_a^b \frac{\gamma'(s)}{\gamma(s) - z_0}\, ds} = n$$

Example $\gamma : [0, 2\pi] \to \mathbb{C}$

$\gamma(t) = e^{it}$

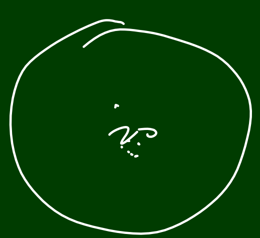
$\rho : [0, 4\pi] \to \mathbb{C}$

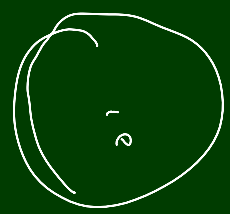
$\gamma(t) = e^{it}$

$$\frac{1}{2\pi i}\int_0^{2\pi} \frac{\gamma'(s)}{\gamma(s) - z_0}\, dz = \frac{1}{2\pi i}\int_0^{2\pi} \frac{ie^{is}}{e^{is} - 0}\, ds$$

$$= \frac{1}{2\pi i}\int_0^{2\pi} i\, ds$$

$= 1$





'Winding No.'

Look for some more examples (like z_0 is outside)

Ex 1: Integ by parts $f, g : [a,b] \to \mathbb{C}$ and $f, g \in C^1[a,b]$

$$\int_a^b f(t)\, g'(t)\, dt = \Big[f(t)\, g(t) \Big]_a^b - \int_a^b g(t) \cdot f'(t)\, dt$$

Proof Diff $(fg)(t)$

Ex 2 Let $h : [c,d] \to \mathbb{R}$ be contsly dif & $f : [a,b] \to \mathbb{R}$ be cont

$$h([c,d]) \subseteq [a,b]$$

$$\int_c^d (f \circ h)(s) \cdot h'(s)\, ds = \int_{h(c)}^{h(d)} f(t)\, dt$$

Thm Let $f : [a,b] \to \mathbb{C}$ be conts.

Then $$\left| \int_a^b f(t)\, dt \right| \le \int_a^b |f(t)|\, dt.$$

Proof $z = |z| e^{i\theta}$ $e^{i\theta}$ – trick

Take $z = \int_a^b f(t)\, dt$

$$|z| = z\, e^{-i\theta}$$

$$\left| \int_a^b f(t)\, dt \right| = e^{-i\theta} \int_a^b f(t)\, dt$$

$g(t) = e^{-i\theta} f(t)$

$g(t) = \mathrm{Re}\, g + i\, \mathrm{img}$

$\int_a^b g(t)\,dt$ is real.

$\therefore \int_a^b \mathrm{Img}\, g(t)\,dt = 0$

$\mathrm{Re}\, z \leq |z|$

$$= \int_a^b \left(e^{-i\theta} f(t)\right) dt \qquad \mathrm{Re}\left(\int\right) = \int_a^b \mathrm{Re} + i \int_a^b \mathrm{Im}$$

$$= \int_a^b \mathrm{Re}\left(e^{-i\theta} f(t)\right) dt$$

$$\leq \int_a^b \left|e^{-i\theta} f(t)\right| dt$$

$$= \int_a^b \left|e^{-i\theta}\right| |f(t)|\, dt$$

$$= \int_a^b |f(t)|\, dt$$

$$\therefore \left|\int_a^b f(t)\,dt\right| \leq \int_a^b |f(t)|\, dt$$

Proposition

Let $f_n : [a,b] \to \mathbb{C}$ cont, $f : [a,b] \to \mathbb{C}$

$f_n \rightrightarrows f$ on $[a,b]$ then (ii) $\int_a^b f_n(t)\,dt \to \int_a^b f(t)\,dt$.

(i) f is Cont.

Let $\epsilon > 0$

$t_0 \in [a,b]$

$|t - t_0| < \delta$ (This δ is from Cont f_N at t_0),

$$|f(t) - f(t_0)| \leq |f(t) - f_N(t)| + |f_N(t) - f_N(t_0)| + |f_N(t_0) - f(t_0)|$$

$$\leq \|f - f_N\|_\infty + \qquad + \|f - f_N\|_\infty$$

$$< \epsilon/3 + \epsilon/3 + \epsilon/3$$

$n \geq N$

$$\left| \int_a^b (f_n(t) - f(t))\, dt \right| \leq \int_a^b |f_n(t) - f(t)|\, dt$$

$$\leq \|f_n - f\|_\infty \int_a^b dt$$

$$< (b-a) \frac{\epsilon}{b-a} = \epsilon$$

$f_n \rightrightarrows f$

Given $\epsilon > 0$ $\exists N \in \mathbb{N}$

$\forall n \geq N$ $\|f_n - f\|_\infty < \frac{\epsilon}{b-a}$

$|f_n(t) - f(t)| \leq \|f_n - f\|_\infty$

$g_n(t) \leq C_n$

Cauchy theory for $\mathbb{C}$

Parseval's identity.

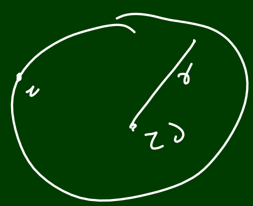
st $\quad f(z) = \sum_n a_n (z - z_0)$, $\quad z \in B(z_0, R)$

$z - z_0 = re^{it}$

$z = z_0 + re^{it}$

Then for $0 \leq r < R$, $\quad 0 \leq t \leq 2\pi$

(i) $$\frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + re^{it})|^2\, dt = \sum_{n=0}^{\infty} |a_n|^2 r^{2n} \quad \text{(1)}$$



(ii) Let $|f(z)| \leq M(r)$ for $|z - z_0| = r$

then $$\sum_{n=0}^{\infty} |a_n|^2 r^{2n} \leq (M(r))^2$$

Proof: I.P.S.

$(C[0, 2\pi], \| \ \|_2)$

$$\langle f, g \rangle_{\text{int}} = \int_a^b f(t)\, \overline{g}(t)\, dt$$

$f_n(t) = e^{int}$

$f_m(t) = e^{imt}$

$n \neq m$, $$\langle f_n, f_m \rangle_{\text{int}} = \int_0^{2\pi} e^{int}\, e^{-imt}\, dt$$

$$= \int_0^{2\pi} e^{i(n-m)t}\, dt$$

$$= \begin{cases} 0 & n \neq m \\ 2\pi & n = m \end{cases}$$

$\{f_n : n \in \mathbb{N}\}$ is orthogonal.

$$\left\{ \frac{e^{int}}{2\pi} : n \in \mathbb{Z} \right\} \text{ is an o.n. set}$$

$$\|f_1 + \cdots f_n\|^2 = \langle f_1 + \cdots f_n, f_1 + \cdots f_n \rangle$$
$$= \sum_{i=1}^{n} \langle f_i, f_1 + \cdots f_n \rangle$$
$$= \sum_{i=1}^{n} \langle f_i, f_i \rangle = \sum_{i=1}^{n} \|f_i\|^2 .$$

$$z - z_0 = r e^{it}$$

$$f(z) = \sum_n a_n (z - z_0)^n$$

$$f(z_0 + re^{it}) = \sum_n a_n r^n e^{int}$$

$$\langle a_k r^k e^{ikt}, a_k r^k e^{ikt} \rangle$$

$$f_n(t) = \sum_{k=0}^{n} a_k r^k e^{ikt}$$

$$\int_0^{2\pi} |f_n(t)|^2 dt = \langle f_n(t), f_n(t) \rangle = \left\langle \sum_{k=0}^{n} a_k r^k e^{ikt}, \sum_{k=0}^{n} a_k r^k e^{ikt} \right\rangle$$

$$= \sum_{k=0}^{n} \left\langle a_k r^k e^{ikt}, \sum_{k=0}^{n} a_k r^k e^{ikt} \right\rangle$$

$$= 2\pi \sum_{k=0}^{n} |a_k|^2 r^{2k} . \checkmark$$

$$g_n(t) \rightrightarrows g(t)$$

$$\int_0^{2\pi} |f_n(t)|^2 dt = 2\pi \sum_{k=0}^{n} |a_k|^2 r^{2k} \longrightarrow 2\pi \sum_{k=0}^{\infty} |a_k|^2 r^{2k}$$

$$f_n(t) \rightrightarrows f(z_0 + re^{it})$$

$f_n \rightrightarrows f$

$g_n \rightrightarrows g$?

$f_n g_n \rightrightarrows fg$.

$$\int_0^{2\pi} |f(z_0 + re^{it})|^2 dt$$

$$\frac{1}{2\pi} \int_0^{2\pi} \left| f(z_0 + re^{it}) \right|^2 dt = \sum_{k=0}^{\infty} |a_k|^2 r^{2k}$$

Note that $|f_n(t)|^2 \rightrightarrows |f(z_0 + re^{it})|^2$?.

$\therefore$ By P.T done.

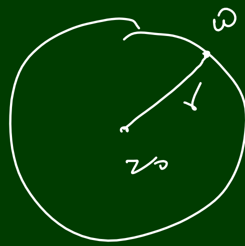
(ii) $z - z_0 = re^{it}$

$|z - z_0| = r$

$$\frac{1}{2\pi}\int_0^{2\pi} |f(z_0 + re^{it})|^2 dt = \sum_{n=0}^{\infty} |a_n|^2 r^{2n}$$

Give the z s.t. $|z - z_0| = r$, $|f(z)| \leq M(r)$

i.e. $|f(z_0 + re^{it})|^2 \leq (M(r))^2$



$$\therefore \sum_{n=0}^{\infty} |a_n|^2 r^{2n} = \frac{1}{2\pi}\int_0^{2\pi} |f(z_0 + re^{it})|^2 dt$$

$$\leq \frac{1}{2\pi}\int_0^{2\pi} (M(r))^2 dt \quad \text{(By monot of integral)}$$

$$\leq \frac{1}{\cancel{2\pi}} (M(r))^2 \cancel{2\pi}$$

— x —

Complex Analysis, Sep 14, 2020

Set of Handouts

1. Basic calculus in $\mathbb{C}$ → Lim, cont, Diff, Integration

Intro 1. Basic top in $\mathbb{C}$ → open, closed, compact } connected

2. Exponential & logarithms

2. Power Series (Radius of conv, Diff, Analyticity).

* 3. C-R-equations

4. Path integrals.

Recall Let γ be any path on $\mathbb{C}$

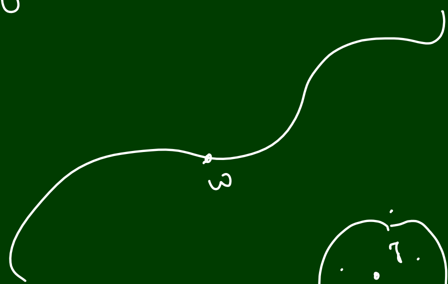
$$z \to \int_\gamma \frac{dw}{w-z} \text{ is Conts fn on } \mathbb{C}\setminus[\gamma]$$

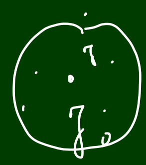
U_γ

$$\phi : \mathbb{C}\setminus[\gamma] \to \mathbb{C}$$

$$\phi(z) = \int_\gamma \frac{dw}{w-z}$$

Fix $z_0 \in \mathbb{C}\setminus[\gamma]$





$$\phi(z) - \phi(z_0) = \int_\gamma \frac{dw}{w-z} - \int_\gamma \frac{dw}{w-z_0}$$

$$= \int_\gamma \frac{(z-z_0)\,dw}{(w-z)(w-z_0)}$$

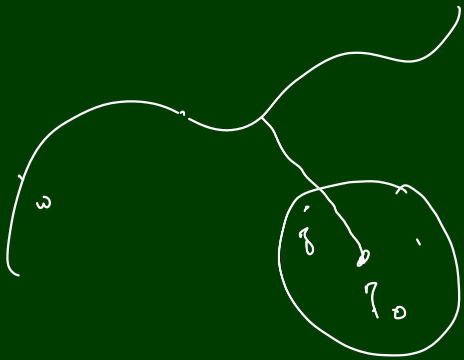
By M.L formula

$$|\phi(z) - \phi(z_0)| \leq \text{upper bound} \left\{ \frac{|z-z_0|}{|w-z||w-z_0|} : w \in [\gamma] \right\} \times L(\gamma)$$

$[\gamma] = \gamma([a,b]) \subseteq \mathbb{C}$

$\rightsquigarrow$ compact

& hence closed



$U = \mathbb{C} \setminus [\gamma]$

$d = d(\gamma_0, [\gamma])$

Then $d > 0$ $[\gamma]$

Ass $\delta := \frac{d}{2}$

For any $w \in [\gamma]$, $|w - \gamma_0| \geq d$

$|\gamma - \gamma_0| \leq \frac{d}{2}$

$-|\gamma - \gamma_0| \geq -\frac{d}{2}$

$$d \leq |w - \gamma_0| \leq |w - \gamma + \gamma - \gamma_0|$$

$$d \leq |w - \gamma| + |\gamma - \gamma_0|$$

$$d - \frac{d}{2} \leq |w - \gamma|$$

$$|w - \gamma| \geq \frac{d}{2}$$

$$\left| \frac{1}{(w-\gamma)(w-\gamma_0)} \right| \leq \frac{2}{d} \cdot \frac{1}{d}$$

$$\therefore \left| \phi(\gamma) - \phi(\gamma_0) \right| \leq \frac{2 |\gamma - \gamma_0|}{d^2} L(\gamma)$$

$$< \frac{2}{d^2} \delta L(\gamma) < \epsilon$$

$$\delta \leq \frac{\epsilon d^2}{2 L(\gamma)}$$

✓

$$\delta = \min \left\{ \frac{d}{2}, \frac{\epsilon d^2}{2 L(\gamma)} \right\}$$

Proposition: Let $f : U \to \mathbb{C}$ be conts ($U \subseteq \mathbb{C}$ open)

Let γ be any closed path in U.

f has a primitive in $U \iff \int_\gamma f = 0$ $\quad F' = f$

$\Rightarrow$ Already done. Just we recall

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\cdot\gamma'(t)\,dt$$
$$= \int_a^b F'(\gamma(t))\cdot\gamma'(t)\,dt$$
$$= \int_a^b (F\circ\gamma)'(t)\,dt$$
$$= F(\gamma(b)) - F(\gamma(a))$$
$$= 0 \qquad \gamma(a) = \gamma(b)$$

$\Leftarrow$ Hypothesis: For any closed path γ in U, $\int_\gamma f = 0$

Claim: f has a primitive in U

Find $F : U \to \mathbb{C}$ s.t. $F' = f$

Case (i) U is connected

Fix $z_0 \in U$

$F(z) = \int_\gamma f(z)\,dz$ where

γ is any path from z_0 to z

We need to show that integral is well def

If γ_1 & γ_2 are paths from z_0 to z: $\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz$

$$\gamma = \gamma_1 + \tilde{\gamma}_2$$

Then γ is a closed path

$\therefore$ By hypo, $\int_\gamma f = 0$

$$\int_{\gamma_1 + \tilde{\gamma}_2} f = 0$$

$\gamma_1 : [0, 2\pi]$

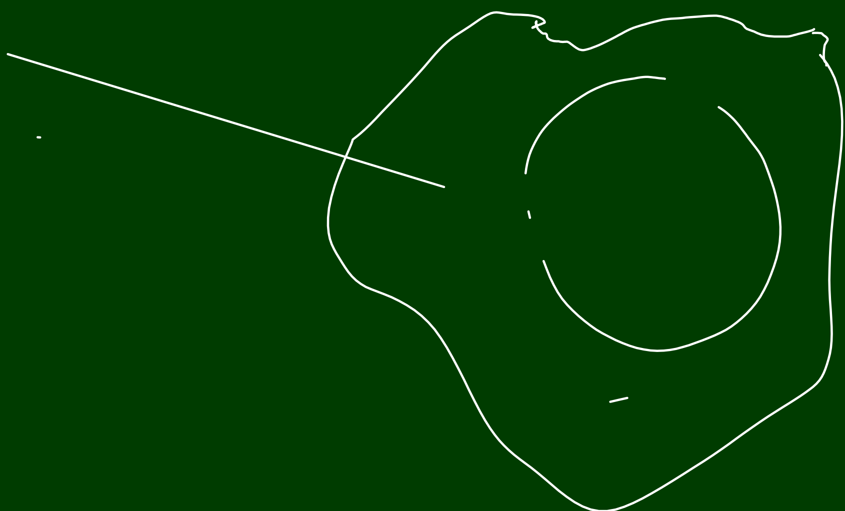
(prove this)

$$\int_{\gamma_1} f + \int_{\tilde{\gamma}_2} f = 0$$

$$\int_{\gamma_1} f - \int_{\gamma_2} f = 0$$

$$\int_{\gamma_1} f = \int_{\gamma_2} f$$

$F(z) = \underbrace{\int_\gamma f(w)\,dw}$, where γ is a any path from γ_0 to z



$$F(z+h) - F(z) = \int_{[z_0, z+h]} f - \int_{[z_0, z]} f$$

$$= \int_{[z_0, z+h]} f + \int_{[z, z_0]} f$$

$$= \int_{[z, z+h]} f$$

$\gamma = \gamma_1 + \gamma_2$

$$\int_\gamma f = \int_{\gamma_1} f + \int_{\gamma_2} f$$

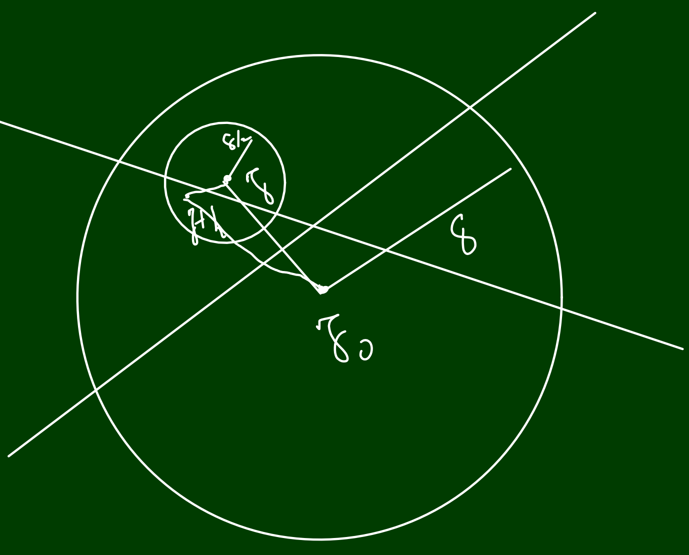
$$\left| \frac{F(z+h) - F(z)}{h} - f(z) \right| = \left| \frac{1}{h} \int_{[z, z+h]} f(w)\,dw - \frac{1}{h} \int_{[z, z+h]} f(z)\,dw \right|$$

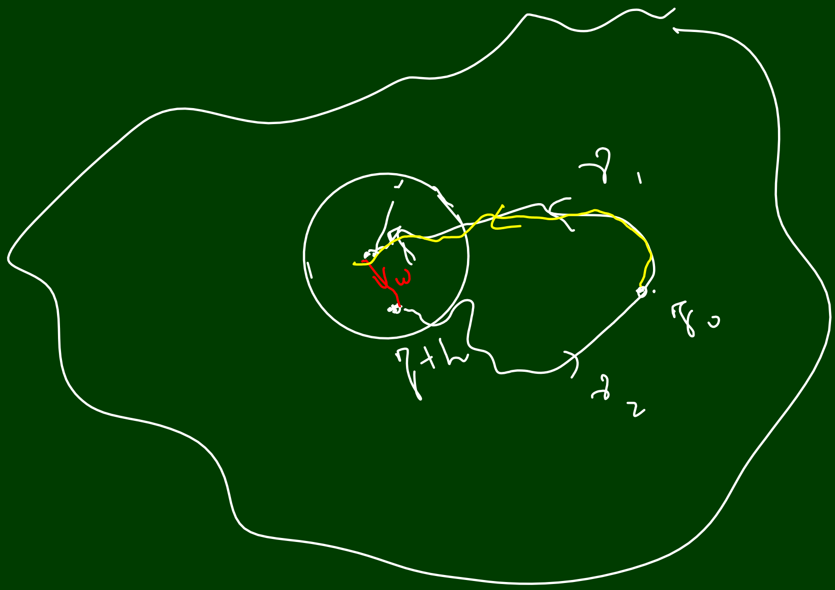
$$= \left| \frac{1}{h} \int_{[z, z+h]} \big(f(w) - f(z)\big)\,dw \right|.$$

$z_0 \in U$

$\therefore \exists\, \delta > 0$ s.t.

$B[z_0, \delta] \subset$





U

Let $z_0 \in U$

$z \in U$

$B[z, \delta] \subseteq U$

$z+h \in B(z, \delta)$

$|h| < \delta$

$$F(z+h) = \int_{\gamma_2} f(w)\,dw$$

$$F(z) = \int_{\gamma_1} f(w)\,dw$$

$$\int_{\gamma_1} + \int_{[z, z+h]} + \int_{\tilde{\gamma}_2} = 0$$

$$\int_{[z, z+h]} = \int_{\gamma_2} - \int_{\gamma_1}$$

$$\int_{[z, z+h]} f(w)\,dw = F(z+h) - F(z)$$

$$\left|\frac{1}{h}F\left(z+h)-F(z)\right)-f(z)\right| = \left|\frac{1}{h}\int_{[\eta,\eta+h]}\left(f(\omega)-f(z)\right)d\omega\right|$$

$\because$ f is conts at z, Given $\epsilon > 0$, $\exists\ \delta > 0$ s.t

if $|\omega-\eta| < \delta$ then $|f(\omega)-f(z)| < \epsilon$

Now use m-l formula. $< \frac{1}{|h|}\epsilon\,|h|$ ✓

✓

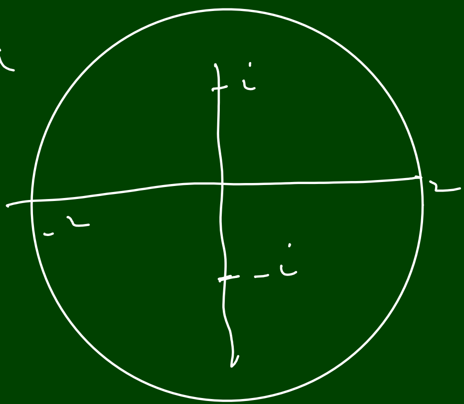
— x —

Complex Analysis, Sep 25 2020.

Let $\gamma(t) = 2e^{it}$, $0 \le t \le 2\pi$

$$\frac{1}{2\pi i}\int_\gamma \frac{e^{az}}{z^2+1}\,dz = \sin a$$

$$z^2+1 = (z+i)(z-i)$$



$$\frac{1}{(z+i)(z-i)} = \frac{A}{z+i} + \frac{B}{z-i}$$

$$= \frac{A(z-i)+B(z+i)}{(z+i)(z-i)}$$

$-Ai + Bi = 1$

$A + B = 0$

$(-A+B)i = 1$

$2Bi = 1$, $\quad B = \frac{1}{2i}$

$A = \frac{-1}{2i}$

$$\frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}\,dz$$

$$= f(z)\cdot\frac{1}{2\pi i}\int \frac{dz}{w-z}$$

$$= f(z)\cdot 1$$

$$I = A\int_\gamma \frac{e^{az}}{z+i}\,dz + B\int_\gamma \frac{e^{az}}{z-i}\,dz$$

$$= A\int_\gamma \frac{f(z)}{z-(-i)}\,dz + B\int_\gamma \frac{f(z)}{z-i}\,dz$$

$$= A\,2\pi i f(-i) + B\,2\pi i f(i)$$

$$= \frac{-1}{\cancel{2i}}\cancel{2\pi i}\,e^{-ia} + \frac{1}{\cancel{2i}}\cancel{2\pi i}\,e^{ia}$$

$$= \pi\left[e^{ia} - e^{-ia}\right]$$

$$= \pi\left[\cancel{\cos a} + i\sin a - \cancel{\cos a} + i\sin a\right] = 2\pi i \sin a$$

$$\frac{1}{2\pi i}\int_\gamma \frac{z e^{az}}{z^2+1}\,dz = \cos a$$

$$\underline{I} = \frac{1}{2\pi i}\int_\gamma \frac{z e^{az}}{z^2+1}\,dz \qquad g = z e^{az}$$

$$2\pi i\,\underline{I} = A\,2\pi i\, g(-i) + B\,2\pi i\, g(i)$$
$$= -\pi\left(-i e^{-ia}\right) + \pi i e^{ia}$$
$$= \pi i\left(e^{ia} + e^{-ia}\right)$$
$$= \pi i\,(2\cos a)$$

$$\underline{I} = \cos a$$

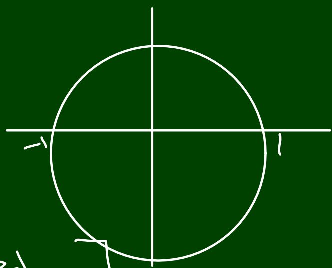
2) $\gamma(t) = e^{it}$, $0 \le t \le 2\pi$,

S.T. $\displaystyle\int_\gamma \frac{e^z}{z} = 2\pi i$

& hence deduce that

$$\int_0^{2\pi} e^{k\cos\theta}\cos(k\sin\theta)\,d\theta = 2\pi, \quad k \in \mathbb{Z}$$

$$\int_\gamma \frac{e^z}{z}\,dz = \int_\gamma \frac{f(z)}{z}\,dz$$



$$= 2\pi i\left[\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-0}\,dz\right]$$

$$= 2\pi i\left[f(0)\right]$$

$$\int_\gamma \frac{e^z}{z}\,dz = 2\pi i \checkmark$$

$$\int_0^{2\pi} \frac{e^{e^{it}}}{\cancel{e^{it}}}\, i\cancel{e^{it}}\,dt = 2\pi i$$

$e^{it} = \cos t + i\sin t$

$$\int_0^{2\pi} e^{\cos t} \cdot e^{i\sin t}\, dt = 2\pi$$

$$\int_0^{2\pi} e^{\cos t}\left[\left(\cos(\sin t) + i\sin(\sin t)\right)\right] dt = 2\pi$$

$$\int_0^{2\pi} e^{\cos t} \cos(\sin t)\, dt = 2\pi$$

Take $\gamma(t) = e^{ikt}$, $k \in \mathbb{N}$

W.k.T $\displaystyle\int_\gamma \frac{e^z}{z}\, dz = 2\pi i$

$$\int_0^{2\pi} \frac{e^{e^{ikt}}}{\cancel{e^{ikt}}} \cdot \cancel{i} k \cancel{e^{ikt}}\, dt = 2\pi \cancel{i} K$$

$$\int_0^{2\pi} \cancel{K} e^{\cos kt} \cdot e^{i\sin kt}\, dt = 2\pi \cancel{K}$$

$$\int_0^{2\pi} e^{\cos kt} \cos(\sin kt)\, dt = 2\pi$$

Ex (special case of argument principle)

Let p be a polyn s.t none of the zeros lie on the path $\gamma_R(t) = Re^{it}$, $0 \le t \le 2\pi$

$$\frac{1}{2\pi i}\int_{\gamma_R} \frac{p'}{p} = \text{No of zeros of } p \text{ in } B(0,R)$$

count with mult

$$\frac{P'(z)}{P(z)} = \sum_j \frac{1}{z-\alpha_j}$$

where α_j are roots of P

$$P(z) = (z-1)(z-2)$$
$$P'(z) = (z-1)+(z-2)$$
$$\frac{P'(z)}{P(z)} = \frac{1}{z-2} + \frac{1}{z-1}$$

$$\int_\gamma \frac{P'(z)}{P(z)} = |\{j : \alpha_j \text{ is root of } P \text{ in } B(0,R)\}|$$

$= \text{No of } \sim$

$$p(z) = \alpha(z-\alpha_1)\cdots(z-\alpha_k)$$
$$\frac{p'(z)}{p(z)} = \sum_{j=1}^{k} \frac{1}{z-\alpha_j}$$

$$p(z) = (z-\alpha)^2$$
$$p'(z) = 2(z-\alpha)$$
$$\frac{p'}{p} = \frac{2}{z-\alpha}$$
$= 2 \cdot 1$

2) Eva $\int_\gamma \frac{z^3}{z^4-1}\,dz = ?$

$\gamma(t) = 3e^{it}$

$$P(z) = z^4 - 1$$
$$P'(z) = 4z^3$$
$$\frac{P'(z)}{P(z)} = \frac{4z^3}{z^4-1}$$

$$\int \frac{P'(z)}{P(z)} = \text{No of zeroes of } z^4-1 \text{ in } B(0,3)$$

$z^4 = 1$

$z = 1, \omega, \omega^2, \omega^3$

$$\int \frac{P'(z)}{P(z)} = 4$$
$$\int \frac{z^3}{z^4-1} = \frac{1}{4}\cdot 4 = 1$$

2) $\displaystyle\int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \frac{2\pi}{\sqrt{3}}$

$\bar{z} = e^{-i\theta} = \frac{1}{z}$

$z = e^{i\theta}$

$dz = i e^{i\theta} d\theta$ $\qquad d\theta = \frac{1}{ie^{i\theta}} dz$

$= \frac{1}{iz} dz$

$\cos\theta = \frac{z+\bar{z}}{2}$

$\frac{1}{2+\cos\theta} = \frac{1}{2+\frac{(z+\bar{z})}{2}} = \frac{2}{4+z+\bar{z}}$

$= \frac{2}{4+z+\frac{1}{z}}$

$\frac{1}{2+\cos\theta} = \frac{2z}{4z+z^2+1}$

$\displaystyle\int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \frac{1}{i}\int \frac{2\, dz}{z^2+4z+1}$ $\qquad p(z) = z^2+4z+1$

$p'(z_1) = 2z+4$

$= -i \displaystyle\int_\gamma \frac{2\,dz}{z^2+4z+1}$

$\frac{1}{z^2+4z+1} = \frac{1}{(z-\omega_1)(z-\omega_2)} = \frac{1}{z-\omega_1}$,

$z = \frac{-4 \pm \sqrt{16-4}}{2}$

$= \frac{-4\pm 2\sqrt{3}}{2} = -2\pm\sqrt{3}$

$\omega = -2+\sqrt{3}$

$\sqrt{3} = 1.732$

$-2+1.7 = -0.3$

$-2-\sqrt{3} < -1$

$-<$ —

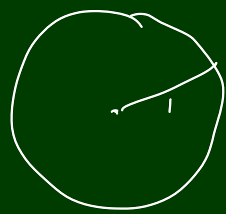
Complex Analysis, Sep 10, 2020

Recall

$$\int_\gamma f(z)\,dz := \int_a^b f(\gamma(t))\cdot\gamma'(t)\,dt$$ ✓

$0\le t\le 2\pi$

$\gamma_1(t) = e^{it}$, $\gamma_2(t) = e^{-it}$, $\gamma_3(t) = e^{2it}$



$f : \mathbb{C}^* \to \mathbb{C}$

$f(z) = \frac{1}{z}$

$$\int_{\gamma_1} f(z)\,dz = \int_0^{2\pi} f(\gamma(t))\,\gamma'(t)\,dt$$

$$= \int_0^{2\pi} \frac{1}{\cancel{e^{it}}}\, i\cancel{e^{it}}\,dt$$

$$= 2\pi i$$

$$\int_{\gamma_2} f(z)\,dz = \int_0^{2\pi} f(\gamma_2(t))\cdot\gamma_2'(t)\,dt$$

$$= \int_0^{2\pi} \frac{1}{e^{-it}}\, -ie^{-it}\,dt$$

$$= -2\pi i$$

$$\int_{\gamma_3} f(z)\,dz = \int_0^{2\pi} f(\gamma_3(t))\cdot\gamma_3'(t)\,dt$$

$$= \int_0^{2\pi} \frac{1}{\cancel{e^{2it}}}\, 2i\cancel{e^{2it}}\,dt$$

$$= 4\pi i$$

The integral depends on path not on trace

2) Let $\gamma_k(t) = e^{ikt}$, $k \in \mathbb{Z}$ $\quad 0 \le t \le 2\pi$

$$f_n(z) = z^n$$

$$\int_{\gamma_k} f_n(z)\,dz = \int_0^{2\pi} f_n(\gamma_k(t))\cdot \gamma_k'(t)\,dt$$

$$= \int_0^{2\pi} e^{iknt} \cdot (ik)\, e^{ikt}\,dt$$

$$= ik \int_0^{2\pi} e^{ik(n+1)t}\,dt$$

$$= \begin{cases} 2\pi i k & n = -1 \\ 0 & n \neq -1 \end{cases}$$

$l \in \mathbb{Z}$

$$\int_0^{2\pi} e^{ilt}\,dt = \begin{cases} 2\pi & l = 0 \\ 0 & l \neq 0 \end{cases}$$

$$\left.\frac{e^{il}}{il}\right|_0^{2\pi} = \frac{1}{il}(1-1) = 0$$

$C([\gamma], \mathbb{C})$ v.s

<u>Ex:</u> & $f \mapsto \int f$ is linear

<u>Def</u> Let $f : U \to \mathbb{C}$ be a fn. We say f has primitive in U $\exists\, F : U \to \mathbb{C}$ s.t $\boxed{F' = f}$

<u>Theorem:</u> Let $f : U \to \mathbb{C}$ be continuous & f has a primitive in U

Let $\gamma : [a,b] \to U$ be a path

Then
$$\int_\gamma f(z)\,dz = F(\gamma(b)) - F(\gamma(a))$$

In particular, if γ is a closed path then
$$\int_\gamma f(z)\,dz = 0$$

<u>Proof:</u>

<u>Case(1)</u> Let γ be <u>smooth</u>

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\cdot \gamma'(t)\,dt$$

$f =$

$$= \int_a^b F'(\gamma(t)) \cdot \gamma'(t)\, dt \qquad (F\circ\gamma)'(t) = F'(\gamma(t))\cdot\gamma'(t)$$

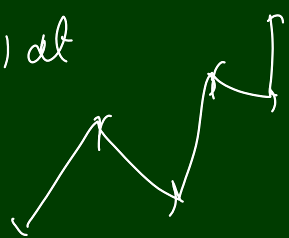
$$= \int_a^b (F\circ\gamma)'(t)\, dt \quad [\text{By chain rule}]$$

$$= (F\circ\gamma)(b) - (F\circ\gamma)(a) \quad [\text{By F.T.C (ii)}]$$

$$= F(\gamma(b)) - F(\gamma(a))$$

Case (ii) γ is piecewise smooth

$$\int_\gamma f(z)\, dt = \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} f(\gamma(t))\cdot\gamma'(t)\, dt$$



$$= \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} F'(\gamma(t))\cdot\gamma'(t)\, dt$$

$$= \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} (F\circ\gamma)'(t)\, dt$$

$$= \sum_{j=1}^{n} \left[(F\circ\gamma)(t_j) - (F\circ\gamma)(t_{j-1})\right]$$

$$= F(\gamma(\overset{b}{t_n})) - F(\gamma(\overset{a}{t_0}))$$

1) If F & G are primitives of f in U a region

Then $F - G = ?$

$F' = f, \quad G' = f$

$F' - G' = 0$

$(F - G)' = 0$ on U

$F - G$ is constant

2) Let $f(z) = \sum a_n z^n$ in $B(0,R)$

$$\int_\gamma f(z)\,dz = \underline{0}$$ where γ is any closed path in $B(0,R)$

$$F(z) = \sum_n \frac{a_n}{n+1} z^{n+1} \text{ in } B(0,R)$$

$$F'(z) = \sum \frac{a_n \cancel{(n+1)} z^n}{\cancel{(n+1)}} = f(z)$$

$\mathbb{C} \setminus L_0$

3) $$\int_\gamma \frac{dz}{z} = 0 \text{ for which } \gamma$$

$F(z) = \log z = \ln|z| + i \arg_0 z$

For γ lies in $\mathbb{C}\setminus L_0$

Vth $F'(z) = \frac{1}{z}$

3) $$\int_\gamma (z-a)^n\,dz$$

$\gamma(t) = a + re^{it}, \quad 0 \le t \le 2\pi$

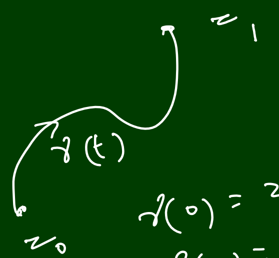
For $n \neq -1$
choose $F(z) = \dfrac{(z-a)^{n+1}}{n+1}$
Then $F' = f$

$$\int_\gamma (z-a)^n\,dz = \begin{cases} 0 & \text{if } n \neq -1 \\ 2\pi i & \text{if } n = 1 \end{cases}$$

$$\int_\gamma \frac{dz}{z-a} = \int_0^{2\pi} \frac{i\cancel{re^{it}}}{\cancel{re^{it}}}\,dt$$

$= 2\pi i$

$\gamma : [0,1] \to \mathbb{C}$



$\gamma(0) = z_0$
$\gamma(1) = z_1$

$\tilde{\gamma}(t) = \gamma(1-t)$

Defn

Let $\gamma : [a,b] \to \mathbb{C}$ be a path

$\tilde{\gamma} : [a,b] \to \mathbb{C}$ by

$$\tilde{\gamma}(t) = \gamma(a+b-t)$$

Then $\tilde{\gamma}$ is called reverse or opposite path

2. $\gamma(t) = a + re^{it}$ $\quad 0 \le t \le 2\pi$

Then $\tilde{\gamma}(t) = a + re^{-it}$

Prop Let $\gamma : [a,b] \to \mathbb{C}$ be a path. Let $f : [\gamma] \to \mathbb{C}$ be conts.

Then $$\int_{\tilde{\gamma}} f(z)\,dz = -\int_{\gamma} f(z)\,dz$$

Proof: As γ is smooth

$$\int_{\tilde{\gamma}} f(z)\,dz = \int_a^b f(\tilde{\gamma}(t)) \cdot \tilde{\gamma}'(t)\,dt$$

$\tilde{\gamma}'(t) = -\gamma'(a+b-t)$

$\tilde{\gamma}(t) = \gamma(a+b-t)$

$s = a+b-t$

$$= \int_b^a f(\gamma(s))\left(-\gamma'(s)\right)(-ds)$$

$$= \int_b^a f(\gamma(s)) \cdot \gamma'(s)\,ds$$

$$= -\int_a^b f(\gamma(s))\,\gamma'(s)\,ds$$

$$= -\int_{\gamma} f(z)\,dz$$

Thm Let F be a primitive of $f : U \to \mathbb{C}$ & let γ_1, γ_2 be two paths having same initial & end points

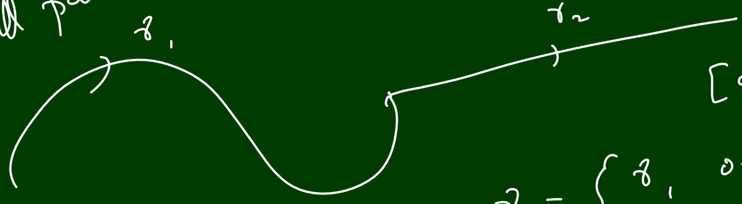
Then $$\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz$$

$\int_{\gamma_1} f = F(\gamma_1(b)) - F(\gamma_1(a))$

$\int_{\gamma_2} f = F(\gamma_2(b)) - F(\gamma_2(a))$

Thus integral value only depends on end points

The full path is γ



$[a,b] \cup [b,c]$

$$\gamma = \begin{cases} \gamma_1 & \text{on } [a,b] \\ \gamma_2 & \text{on } [b,c] \end{cases}$$

$$\gamma = \gamma_1 + \gamma_2$$

$$\gamma = \gamma_1 * \gamma_2$$

$$\int_\gamma f(z)\,dz = \int_{\gamma_1} f + \int_{\gamma_2} f \quad \left(\text{Proof ?}\right)$$

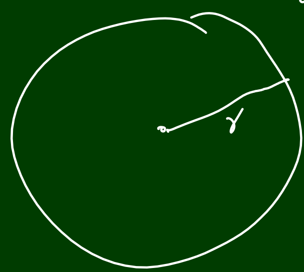
Easy do it.

Defn: Let $\gamma : [a,b] \to \mathbb{C}$ be a smooth path

$$L(\gamma) = \int_0^{2\pi} |\gamma'(t)|\,dt$$

$\gamma(t) = re^{it}$

$0 \le t \le 2\pi$



Complex Analysis, Sep 16, 2020

Let U be a connected subset of $\mathbb{C}$ & $f : U \to \mathbb{C}$ be cont. Then f has a primitive in U iff $\int_\gamma f = 0$ for every closed path γ lying entirely in U.

Proof: $\Rightarrow$ Use chain rule & F.T.I

$\Leftarrow$

$F : U \to \mathbb{C}$

$F' = f$

Note that U is path connected

Fix $z_0 \in U$,

For $z \in U$, $F(z) := \int_\gamma f(w)\,dw$ where γ is any path from z_0 to z

Suppose γ_1 & γ_2 are any two path from z_0 to z

To P.T $\int_{\gamma_1} f = \int_{\gamma_2} f$

$\gamma = \gamma_1 + \tilde{\gamma}_2$

Then γ is a closed path lying entirely in U

By hypo, $\int_\gamma f = 0$

ie $\int_{\gamma_1 + \tilde{\gamma}_2} f = 0$, i.e $\int_{\gamma_1} f + \int_{\tilde{\gamma}_2} f = 0$

$\int_{\gamma_1} f - \int_{\gamma_2} f = 0$

Fix $z_0 \in U$

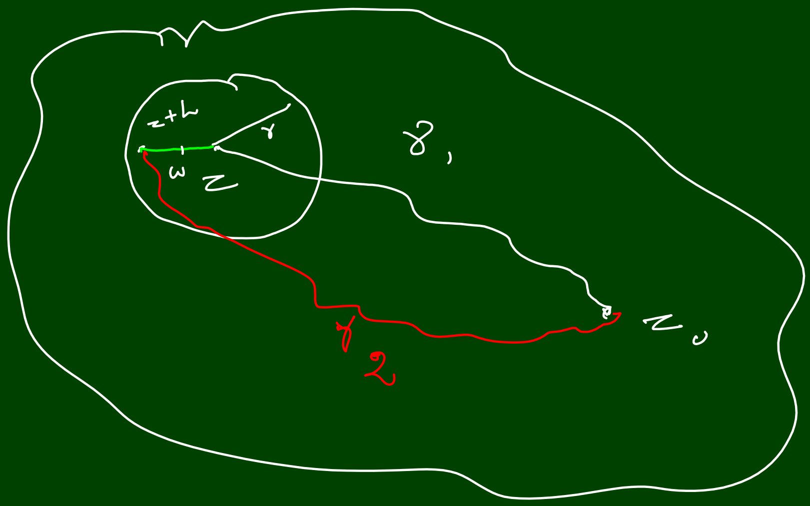
$F : U \to \mathbb{C}$

Let $z \in U$

$$F(z) = \int_\gamma f(w)\,dw$$ where γ is any path from z_0 to z

claim

$$\lim_{h \to 0} \frac{F(z+h) - F(z)}{h} = f(z)$$



Let $z \in U$

$\because U$ is open, $\exists\ r > 0$ s.t $B[z, r] \subseteq U$

Let $h \in \mathbb{C}$ s.t $|h| < R$ where $R = \min\{r, \delta\}$

Then $z+h \in B(z, r)$

choose $\gamma := \gamma_1 + [z, z+h] + \tilde{\gamma}_2$

Then γ is a closed path in U

$$\int_\gamma f = 0$$

i.e. $$\int_{\gamma_1} f + \int_{[z, z+h]} f + \int_{\tilde{\gamma}_2} f = 0$$

$$\int_{[z, z+h]} f = \int_{\gamma_2} f - \int_{\gamma_1} f$$

$$= F(z+h) - F(z)$$

Let $\epsilon > 0$ $\because$ f is conts at z, $\exists\ \delta > 0$ s.t. $|w - z| < \delta$
$|f(w) - f(z)| < \epsilon$

$$\left| \frac{1}{h}\left(F(z+h) - F(z)\right) - f(z) \right| = \left| \frac{1}{h}\int_{[z, z+h]} (f(w) - f(z))\, dw \right|$$

Use ML inequ

$$< \frac{1}{|h|} |h| \times \epsilon$$

Proposition Let $\gamma : [a,b] \to \mathbb{C}$ be a path.
Let $f_n : [\gamma] \to \mathbb{C}$ & $f : [\gamma] \to \mathbb{C}$ be conts
Assume $f_n \rightrightarrows f$ on $[\gamma]$. Then $\int_\gamma f_n \to \int_\gamma f$

Proof: $\because$ Each f_n's conts on $[\gamma]$, $f_n \rightrightarrows f$. Then f is conts

$\therefore \int_\gamma f(w)\,dw$ is defined.

$$\left| \int_\gamma f_n(w)\,dw - \int_\gamma f(w)\,dw \right| = \left| \int_\gamma (f_n - f)(w)\,dw \right| \quad \text{(by linearity of int)}$$

$\le \|f_n - f\|_\infty L(\gamma)$ (By ML form)

$f_n \rightrightarrows f$ uniformly on $[\gamma]$

For any $w \in [\gamma]$
$|f_n(w) - f(w)| \le \|f_n - f\|_\infty$

Cor Let $(f_n)_{n \in \mathbb{N}}$ be conts on $[\gamma]$, $f : [\gamma] \to \mathbb{C}$ be fn ~~$\left|\int_\gamma f\right| \le \int_\gamma |f|$~~

st $\sum f_n \rightrightarrows f$ on $[\gamma]$. Then

$$\boxed{\int_\gamma f = \sum_n \int_\gamma f_n}$$

$f = \sum f_n$

For $w \in [\gamma]$ $f(w) = \sum_{n=0}^{\infty} f_n(w)$.

W.P.T $\int_\gamma f(\omega)\,d\omega = \sum_{n=0}^{\infty} \int_\gamma f_n(\omega)\,d\omega$

Let $g_n = \sum_{k=0}^{n} f_k(\omega)\,d\omega$

$$\int f_1 + f_2 = \int f_1 + \int f_2$$

W.K.T $g_n \rightrightarrows f$

By P.T. $\int g_n \to \int f$

$$\int \sum_{k=0}^{n} f_k \to \int f$$

$$\int_\gamma f(\omega)\,d\omega = \lim_{n\to\infty} \int_\gamma \sum_{k=0}^{n} f_k(\omega)\,d\omega$$

$$= \lim_{n\to\infty} \sum_{k=0}^{n} \int_\gamma f_k(\omega)\,d\omega$$ (By linearity of the int.)

Ex. Let $f(z) = \sum_n a_n(z-a)^n$, $z \in B(a,R)$

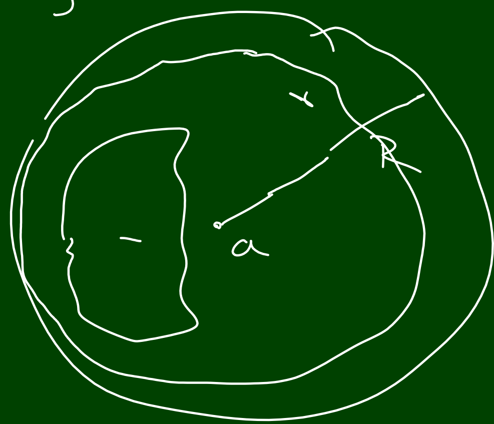
Let γ be any closed path in $B(a,R)$

Then $\int_\gamma f(z)\,dz = \int_\gamma \sum_{n=0}^{\infty} a_n(z-a)^n\,dz$

Choose $r > 0$

$\therefore$ $[\gamma] \subseteq B[a,r]$

Then power ser unif conq in $B[a,r]$



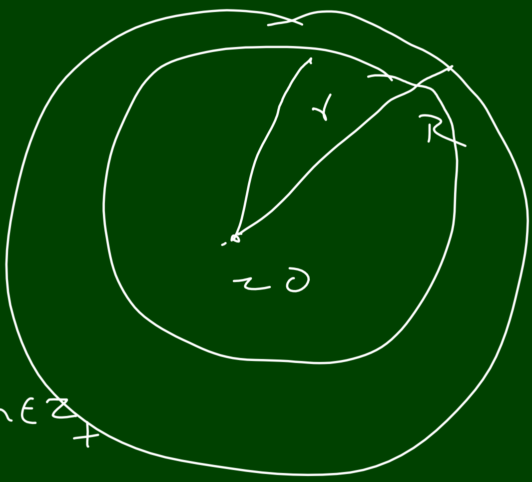
We can use thm

$$= \sum_{n=0}^{\infty} \int_\gamma a_n(z-a)^n\,dz$$

$$= 0$$

3) Let $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \quad z \in B(z_0, R)$

Let $\gamma = z_0 + re^{it}$ where $0 < r < R$



$$\frac{f^{(n)}(z_0)}{n!} = a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(z)\,dz}{(z-z_0)^{n+1}} \text{ where } n \in \mathbb{Z}_+$$

$$f(z) = \sum_n a_n (z-z_0)^n$$
$$= a_0 + a_1(z-z_0) + \cdots + a_n(z-z_0)^n + a_{n+1}(z-z_0)^{n+1} + \cdots$$

$$\frac{f(z)}{(z-z_0)^{n+1}} = \frac{a_0}{(z-z_0)^{n+1}} + \frac{a_1}{(z-z_0)^n} + \cdots + \frac{a_n}{(z-z_0)} + a_{n+1} + a_{n+2}(z-z_0) + \cdots$$
$= R.H.S$

$$\int_\gamma (z-z_0)^k dz = \begin{cases} 0 & \text{if } k \neq -1 \\ 2\pi i & \text{if } k = -1 \end{cases}$$

$$\frac{1}{2\pi i} \int_\gamma \frac{f(z)\,dz}{(z-z_0)^{n+1}} = \int_\gamma R.H.S$$
$$= \int_\gamma \frac{a_0}{(z-z_0)^{n+1}} dz + \int \cdots +$$
$$= 2\pi i\, a_n$$

$$\therefore \boxed{a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z-z_0)^{n+1}} dz}$$

$$\frac{|f(z)|}{|z-z_0|^{n+1}} \le \frac{M(r)\, r}{r^{n+1}}$$

Also $a_n = \dfrac{f^{(n)}(z_0)}{n!}$

3) Derive cauchy ineq

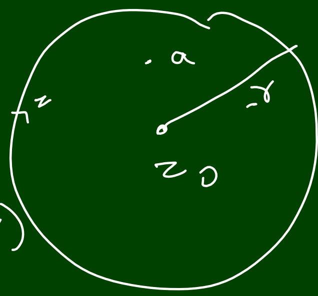
$$|f^{(n)}(z_0)| = n!\,|a_n| \le n! \frac{M(r)}{r^n}$$

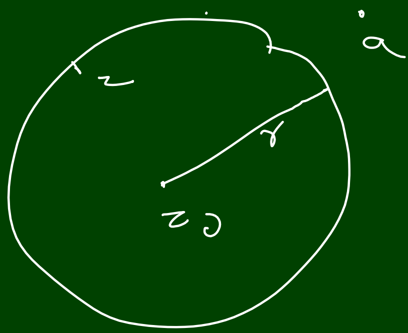
– x –

3) Very Important

$$\gamma(t) = z_0 + r e^{it}$$

$$\int_\gamma \frac{dz}{z-a} = \begin{cases} 0 & \text{if } a \notin \overline{B}[z_0; r] \\ 2\pi i & \text{if } a \in B(z_0; r) \end{cases}$$





$$\frac{1}{z-a} = \frac{1}{z-z_0+z_0-a}$$

Case (i) $a \in B(z_0, r)$

$$\therefore |a-z_0| < r = |z-z_0|$$

$$\frac{|a-z_0|}{|z-z_0|} < 1$$

$$\int_\gamma (z-z_0)^k dz = \begin{cases} 2\pi i & \text{if } k=-1 \\ 0 & \text{if } k \neq -1 \end{cases}$$

$$(1-w)^{-1}$$

$$= \frac{1}{(z-z_0)\left(1 - \frac{a-z_0}{z-z_0}\right)}$$

$$= \frac{1}{z-z_0} \cdot \frac{1}{1-w}$$

where $w = \frac{a-z_0}{z-z_0}$

$|w| < 1$

$$= \frac{1}{z-z_0} (1-w)^{-1}$$

$$= \frac{1}{z-z_0} \sum_{n=0}^{\infty} w^n$$

$$= \frac{1}{z-z_0} \sum_{n=0}^{\infty} \left(\frac{a-z_0}{z-z_0}\right)^n$$

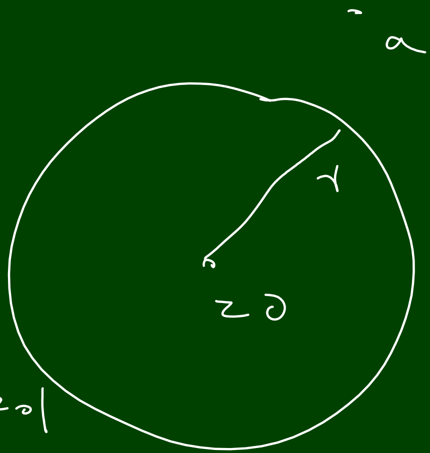
$$\frac{1}{z-a} = \sum_{n=0}^{\infty} \frac{(a-z_0)^n}{(z-z_0)^{n+1}}$$

$$\int_{\gamma} \frac{dz}{z-a} = \sum_{n=0}^{\infty} (a-z_0)^n \cdot \int_{\gamma} \frac{dz}{(z-z_0)^{n+1}}$$

$$= \sum_{n=0} + \underbrace{\sum_{n=1} ''}_{0}$$

$$= 2\pi i$$

Case (ii) $a \notin B[z_0, r]$



$$\frac{1}{z-a} = \frac{1}{z-z_0+z_0-a}$$

$$|a-z_0| > r = |z-z_0|$$

$$\frac{|z-z_0|}{|a-z_0|} < 1$$

$$= \frac{-1}{(a-z_0)\left(1-\frac{z-z_0}{a-z_0}\right)}$$

$$= \frac{-1}{a-z_0} \sum_{n=0}^{\infty} w^n \qquad \text{when } w = \frac{z-z_0}{a-z_0}$$

$$= \frac{-1}{a-z_0} \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{(a-z_0)^n}$$

$$= -\sum_{n=0}^{\infty} \frac{(z-z_0)^n}{(a-z_0)^{n+1}}$$

$$\int_{\gamma} (z-z_0)^n \, dz = 0$$

$$\int_{\gamma} \frac{dz}{z-a} = 0$$

— x —

Complex Analysis, Sep 17, 2020

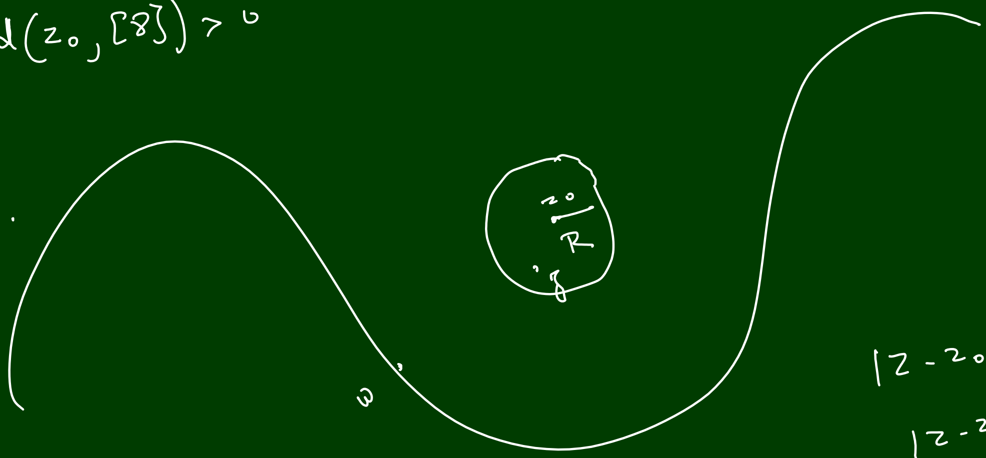
Thm Let $\gamma : [a,b] \to \mathbb{C}$. Let $f : [\gamma] \to \mathbb{C}$ be continuous

$$F : \mathbb{C} \setminus [\gamma] \to \mathbb{C}$$

$$F(z) = \int_\gamma \frac{f(w)}{w - z}\, dw$$

Then F is diff on $\mathbb{C} \setminus [\gamma]$. In fact F is analytic in $\mathbb{C} \setminus [\gamma]$

Proof $d = d(z_0, [\gamma]) > 0$



$$|z - z_0| < R < |w - z_0|$$

$$\frac{|z - z_0|}{|w - z_0|} < 1$$

Let $z_0 \in \mathbb{C} \setminus [\gamma]$

TST $F(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ for some $R > 0$ s.t. $z \in B(z_0, R)$

$$\frac{1}{w - z} = \frac{1}{w - z_0 + z_0 - z}$$

$$= \frac{1}{w - z_0} \cdot \left(\frac{1}{1 + \frac{z_0 - z}{w - z_0}} \right) \quad \text{where } w^* = \frac{z - z_0}{w - z_0}$$

$$= \frac{1}{w - z_0} \frac{1}{1 - w^*} \qquad |w^*| < 1$$

$$= \frac{1}{w - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{w - z_0} \right)^n$$

$$= \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(w - z_0)^{n+1}}$$

$$F(z) = \int_\gamma f(w) \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(w - z_0)^{n+1}}\, dw$$

$$= \sum_{n=0}^{\infty} \int_\gamma \frac{f(w)}{(w-z_0)^{n+1}} (z-z_0)^n \, dw$$

$$= \sum_{n=0}^{\infty} \left[\int_\gamma \frac{f(w)}{(w-z_0)^{n+1}} dw \right] (z-z_0)^n$$

$$= \sum_{n=0}^{\infty} a_n (z-z_0)^n \quad \text{where } a_n = \int_\gamma \frac{f(w)\, dw}{(w-z_0)^{n+1}}$$

How to settle for R.

$$d := d(z_0, [\gamma])$$

$$R := \frac{d}{2}$$

$$|z-z_0| < \frac{d}{2}, \qquad |w-z_0| \geq d$$

$$\frac{1}{|w-z_0|} \leq \frac{1}{d}$$

$$\therefore \frac{|z-z_0|}{|w-z_0|} < \frac{1}{2} < 1$$

$\therefore$ we have proved

$\forall z_0 \in \mathbb{C} \setminus [\gamma]$, $\exists R > 0$ & (a_n) in $\mathbb{C}$ s.t

$$F(z) = \sum_n a_n (z-z_0)^n, \quad z \in B(z_0, R).$$

— x —

Cauchy's Theorem for star shaped Domain.

Intro cum Motivation.

1) U - open set, $\gamma : [a,b] \to \mathbb{C}$ s.t γ lies in U & γ is a closed path, $f : U \to \mathbb{C}$ is conts & f has a primitive in U

then $\int_\gamma f = 0$

[if $\int_\gamma f = 0$ for any closed path lying in U then f has a primitive]

2) Converse (i) is true if U is open & Connected ✓

$z_0 \in U$

$F(z) := \int_\gamma f(w)\,dw$ $\quad$ γ is any path from z_0 to z

$F' = f$?

3) Same Converse relax the open & Connected

i.e through away Connectedness

Compensation on (U, γ, f)

(i) U is star shaped at $a \in U$

(ii) $\int_\gamma f = 0$ for some spec. γ

3 is Cauchy Thm

(i.e) [Let U be open set star shaped at $a \in U$

Let $f : U \to \mathbb{C}$ be Conts with $\int_{\partial T} f = 0$ $\forall\, T \subset U$]

Then f has a primitive in U

Cauchy - Goursat Thm

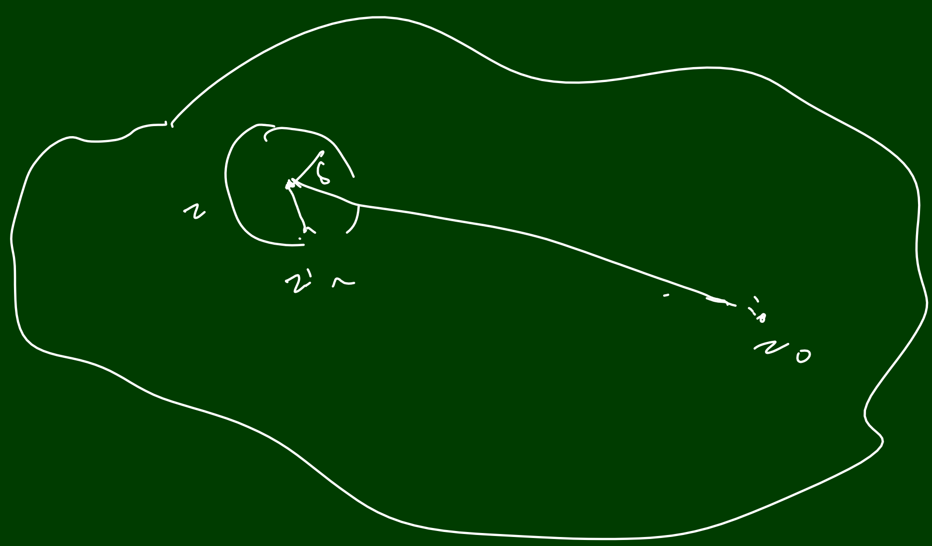
[Let U be an open set. Let $f \in H(U)$

Then $\int_{\partial \Delta} f = 0$

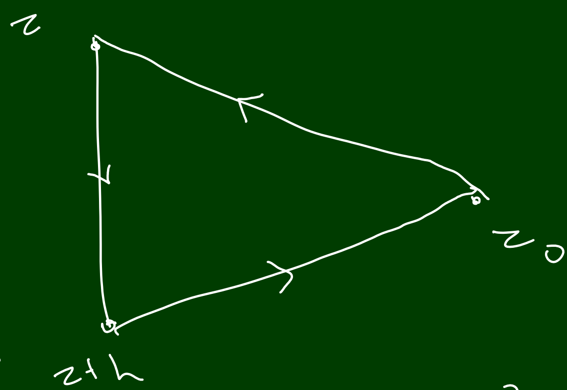
//

$[a, z] \subset U$

U



$F(z) := \int_{[z_0, z]} f(z)\,dz$ if U is starshaped at z_0.



Suppose F serves as a primitive

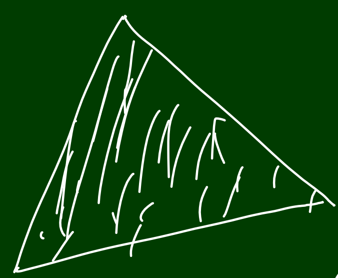
Then $\int_{[z_0, z]} + \int_{[z, z+h]} + \int_{[z+h, z_0]} = 0$

$T =$ triangle

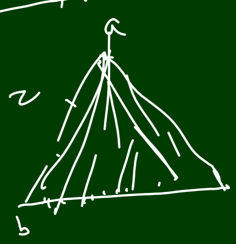
∂T

$\therefore$ Thus rough idea $\int_{\partial T} f = 0$

— Definition Let a, b, c



$\Delta(a,b,c) \equiv [a,b,c]$
$= [b,c,a]$
$= [c,a,b]$



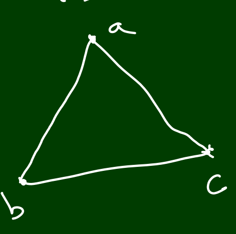
$\therefore \Delta(a,b,c) = \{ z \in [a,w] : w \in [b,c] \}$

Ex? $\doteq \{ w \in \mathbb{C} : w = t_1 a + t_2 b + t_3 c$ where $0 \le t_i \le 1$, $\sum_{i=1}^{3} t_i = 1 \}$

✓

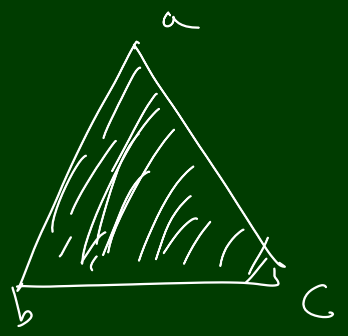
$[a,b] + [b,c] + [c,a]$

$[a,b] = \{ (1-t)a + tb : 0 \le t \le 1 \}$

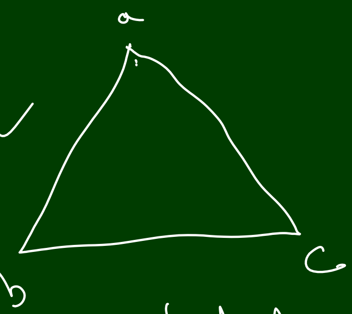


$\partial \Delta$, ∂T, $T(a,b,c)$

$\Delta(a,b,c) \longrightarrow$



$\partial \Delta = $ ✓



$[a,b]+[b,c]+[c,a]$

Thm Let U be an open set star shaped at $z_0 \in U$. Let $f: U \to \mathbb{C}$ be conts fn. Let $\int_T f = 0$ where T is a bdry of any triangle Δ. Then f has a primitive in U.

Proof: Define
$$F(z) := \int_{[z_0, z]} f(w)\, dw$$

Estimate
$$\left| \frac{F(z+h) - F(z)}{h} - f(z) \right|$$

U

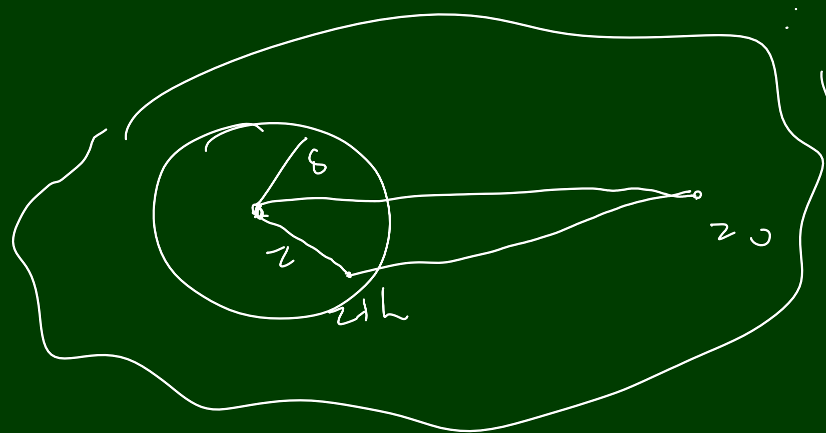
$z \in U$

$\exists r>0$ $B(z,r) \subset U$

$\therefore B[z, r/2] \subset U$

$|h| < R$

$R = \min\{\delta, r/2\}$



$$F(z+h) - F(z) = \int_{[z_0, z+h]} f(w)\, dw - \int_{[z_0, z]} f(w)\, dw$$

$$= \int_{[z, z+h]} f(w)\, dw$$

$$f(z) = \frac{1}{h} \int_{[z, z+h]} f(z)\, dw$$

$$\left| \frac{1}{h}\Big(F(z+h) - F(z)\Big) - f(z) \right| = \left| \frac{1}{h} \int_{[z, z+h]} \big(f(w) - f(z)\big)\, dw \right|$$

$$< \frac{1}{|h|}\, \epsilon\, L\big([z, z+h]\big) \quad \text{by ML est.}$$

$$< \frac{1}{\cancel{|h|}}\, \epsilon \cdot \cancel{|h|}$$

$\because f$ is conts at z, $\exists\, \delta > 0$ s.t. $B[z, \delta] \subseteq U$

$\forall\, w \in B(z, \delta)$, $|f(w) - f(z)| < \epsilon$

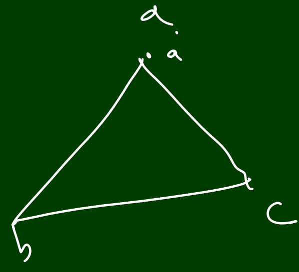
choose $h \in \mathbb{C}$, $|h| \leq \delta$

— x —

Complex Analysis, Sep 18, 2020.

Thm Let U be a open star shaped set at $z_0 \in U$. Let $f: U \to \mathbb{C}$ be cont. Let $\int_T f = 0$ where T is bdry of any Δ lying entirely in U. Then f has a primitive in U.

Given $[a,b,c]$ a triangle



Then 1) $\operatorname{diam}([a,b,c]) = \max\{|a-b|, |b-c|, |c-a|\}$

2) length $\partial[a,b,c] = |a-b| + |b-c| + |c-a|$

$E \subseteq X$ m.s

$\operatorname{diam} E := \operatorname{lub}\{d(x,y) : x, y \in E\}$

$E = [a,b,c]$

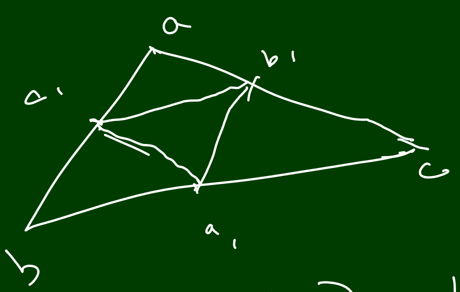
$E \subseteq \mathbb{C}$

E is a compact subset of $\mathbb{C}$

$E = B(a, r)$

$\operatorname{diam} E = 2r$

3)

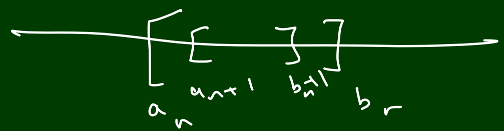


Δ_1

$T_1 = \partial[a_1, b_1, c_1]$

$T = \partial[a, b, c]$

$$L(T_1) = |a_1 - b_1| + |b_1 - c_1| + |c_1 - a_1|$$
$$= \frac{|a-b|}{2} + \frac{|b-c|}{2} + \frac{|c-a|}{2}$$
$$= \frac{L(T)}{2}$$



Nested interval thm gen $\quad \operatorname{diam}(\Delta_1) = \frac{\operatorname{diam}(\Delta)}{2}$

✓ **Cantor Intersection Thm** Let X be metric space. Let (K_n) be a sequence of non-empty cpt sets st $K_{n+1} \subseteq K_n$. Then $\bigcap_n K_n \neq \phi$.

If $\operatorname{diam}(K_n) \to 0$ then $\bigcap_n K_n$ is singleton.

Thm (Cauchy–Goursat Theorem)

Let U be a open set. Let $f \in H(U)$.

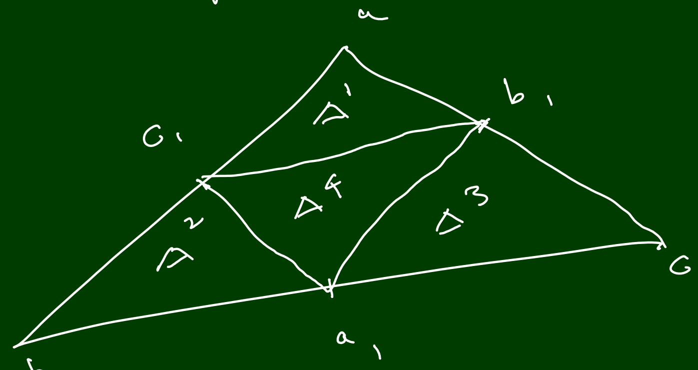
Let $\Delta = [a,b,c] \subseteq U$. Then $\int_T f = 0$ where $T = \partial\Delta$

Proof. Let $\Delta = [a,b,c] \subseteq U$, $T = \partial\Delta$.

Let $\epsilon > 0$

W.P.T $\left|\int_T f\right| < C\epsilon$ for some constant C depends on T.

$T_i = \partial\Delta^i$



$$\int_{T_1} f = \int_{[a,c_1]+[c_1,b_1]+[b_1,a]} f = \int_{[a,c_1]} f + \int_{[c_1,b_1]} f + \int_{[b_1,a]} f$$

$= ac_1 + c_1b_1 + b_1a$ (say for notation)

$$\int_{T_2} f = c_1b + ba_1 + a_1c_1$$

$$\int_{T_3} f = a_1c + cb_1 + b_1a_1$$

$$\int_{T_4} f = a_1b_1 + b_1c_1 + c_1a_1$$

$$I = \sum_{j=1}^{4} \int_{T_j} f$$

$$\sum_{j=1}^{4} \int_{T_j} f = ac_1 + c_1b_1 + b_1a + c_1b + ba_1 + a_1c_1 + a_1c + cb_1 + b_1a_1 + a_1b_1 + b_1c_1 + c_1a_1$$

$= ac_1 + c_1b + ba_1 + a_1c + cb_1 + b_1a$

$= ab + bc + ca$

$$= \int_{[a,b]+[b,c]+[c,a]} f$$

$$= \int_T f$$

$\because \sum_{j=1}^{4} \int_{T_j} f = \int_T f$ $\qquad I = \left| \int_T f \right|$

Let $I = \left| \sum_{j=1}^{4} \int_{T_j} f \right|$

$\leq \sum_{j=1}^{4} \left| \int_{T_j} f \right|$ ——— $(*)$

$\therefore \exists\, j \in \{1, \dots, 4\}$ s.t $\left| \int_{T_j} f \right| \geq \frac{1}{4} I$

If not, $\forall j \in \{1, 4\}$ $\left| \int_{T_j} f \right| < \frac{1}{4} I$

$\therefore \left| \sum_{j=1}^{4} \int_{T_j} f \right| \leq \sum_{j=1}^{4} \left| \int_{T_j} f \right| < \frac{1}{4} I + \dots + \frac{1}{4} I = I$

call this triangle as Δ_1

Repeat the above process

We get (Δ_n) s.t

The vertex of Δ_{n+1} are mid points of vert of Δ_n $\quad \big\}\ \Delta_0 = \Delta$

so that

$\forall n$

(i) $\text{diam}(\Delta_n) = \frac{\text{diam}(\Delta_{n-1})}{2} = \frac{\text{diam}(\Delta)}{2^n}$

(ii) $L(T_n) = \frac{L(T_{n-1})}{2} \dots = \frac{L(T)}{2^n}$

(iii) $\left| \int_{T_n} f \right| \geq \frac{I}{4^n}$ $\quad \Big\{ \boxed{\dots} \leq 4^n \left| \int_{T_n} f \right| \leq I.$

Also $\text{diam}(\Delta_n) \to 0$

By Cantor Int thm $\bigcap_{n \in \mathbb{N}} \Delta_n$ is singleton

Let $z_0 \in \bigcap_{n \in \mathbb{N}} \Delta_n$

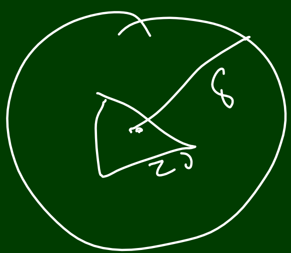
f is diff at z_0, given $\epsilon > 0$ $\exists$ $\delta > 0$ s.t

$0 < |z - z_0| < \delta$, $\left| f(z) - f(z_0) - (z - z_0) f'(z_0) \right| \leq \epsilon |z - z_0|$

$\because$ diam$(\Delta_n) \to 0$ $\exists N \in \mathbb{N}$, $\forall n \geq N$, diam $\Delta_n < \delta$

In part diam $\Delta_N < \delta$

$z_0 \in \Delta_N$



$\therefore \Delta_N \subseteq B(z_0, \delta)$

$$\left| \int_{T_N} f \right| = \left| \int_{T_N} \left(f(z) - f(z_0) - (z - z_0) f'(z_0) \right) + f(z_0) + (z - z_0) f'(z_0) \, dz \right|$$

Note that $z \to (z - z_0) f'(z_0)$ has primitive $\int$

$$= \left| \int_{T_N} \left[f(z) - f(z_0) - (z - z_0) f'(z_0) \right] dz \right|$$

$z_0, z \in \Delta_N$

$\leq \epsilon |z - z_0| \, L(T_N)$ (By ML inequality)

$\leq \epsilon \text{ diam } \Delta_N \, L(T_N)$

$= \epsilon \dfrac{\text{diam } \Delta}{2^N} \dfrac{L(T)}{2^N}$ ✓

$\therefore$

$4^N \left| \int_{T_N} f \right| \leq \epsilon \text{ diam}(\Delta) \cdot L(T)$

$$I \leq 4^N \left| \int_T f \right| \leq C\epsilon \qquad C = \text{diam}(\Delta) \cdot L(T)$$

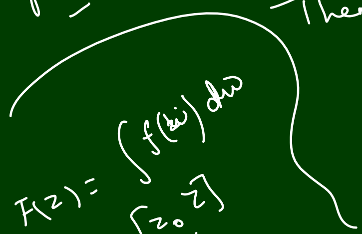
Thm Let U be star shaped and open. Let $f \in H(U)$. Then f has a primitive in U & hence $\int_\gamma f = 0$ for every cl. path $\gamma \subseteq U$

Proof By P.T for any $\Delta \subset U$ $\int_T f = 0$

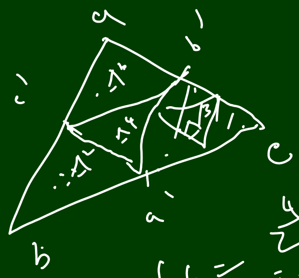
— x —

Complex Analysis, 21 Sep 2020

Recall Cauchy-Goursat Integral thm: Let U be open subset of $\mathbb{C}$ & $f \in H(U)$. Let T be bdry of any $\triangle \subseteq U$. Then $\int_T f = 0$.



$F(z) = \int_{[z_0, z]} f(w)\, dw$



$\int_T f = \sum_{j=1}^{4} \int_{T^j} f$

$I = \left| \int_T f \right|$

$I \le \left|\int_{T^1}\right| + \left|\int_{T^2}\right| + \left|\int_{T^3}\right| + \left|\int_{T^4}\right|$

$\exists j \in \{1, \dots, 4\}$

$\frac{I}{4} \le \left| \int_{T^j} \right|$

Cor: (Cauchy's thm)

Let U be open set & star shaped.

Let $f \in H(U)$. Then f has a primitive in U & hence for any closed path $\gamma \subseteq U$, $\int_\gamma f = 0$.

Cor: Let $U \overset{\text{open}}{\subseteq} \mathbb{C}$, $f \in H(U)$. Then f has local primitives in U.

i.e. $\forall a \in U$ ($\exists r > 0$ & $B(a,r) \subseteq U$ ($\exists F : B(a,r) \to \mathbb{C}$ s.t. $\forall z \in B(a,r)$, $F'(z) = f(z)$.))

Proof: Let $a \in U$

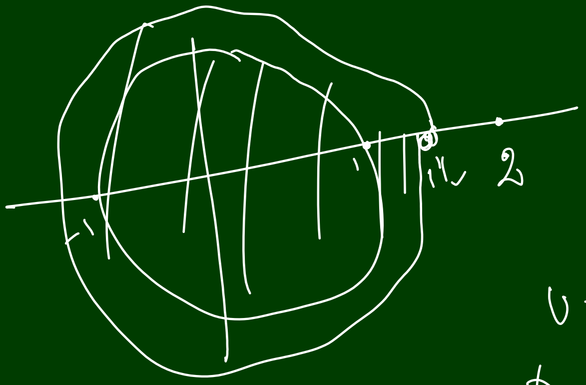
Note $f \in H(B(a,r))$ for some $r > 0$ s.t. $B(a,r) \subseteq U$

Apply previous Cor. we get $F : B(a,r) \to U$, $F' = f$ in $B(a,r)$

Applications of Cauchy thm

$\gamma(t) = e^{it}$, $0 \le t \le 2\pi$. Find $\int_\gamma f$

(i) $f(z) = \frac{e^z}{z-2}$, $f : \mathbb{C} \setminus \{2\} \to \mathbb{C}$



$U = B(0, 3/2)$

$D \subseteq U$

$\int_\gamma f = 0$ ✓

$f \in H(U)$?

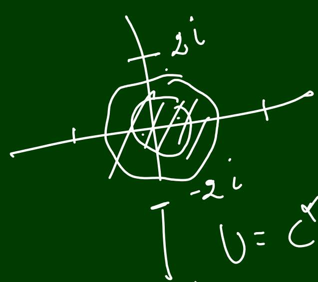
$f(z) = g_1(z)\, g_2(z)$

$g_1(z) = e^z$

$g_2(z) = \frac{1}{z-2}$.

2) $f(z) = \frac{z^3}{z^2+4}$

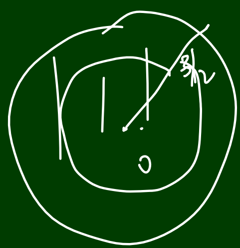
$= \frac{z^3}{(z+2i)(z-2i)}$



$U = \mathbb{C} \setminus \{\pm 2i\}$

$f \in H(U)$

$U = B(0, 3/2)$

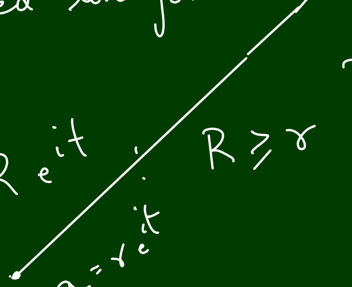


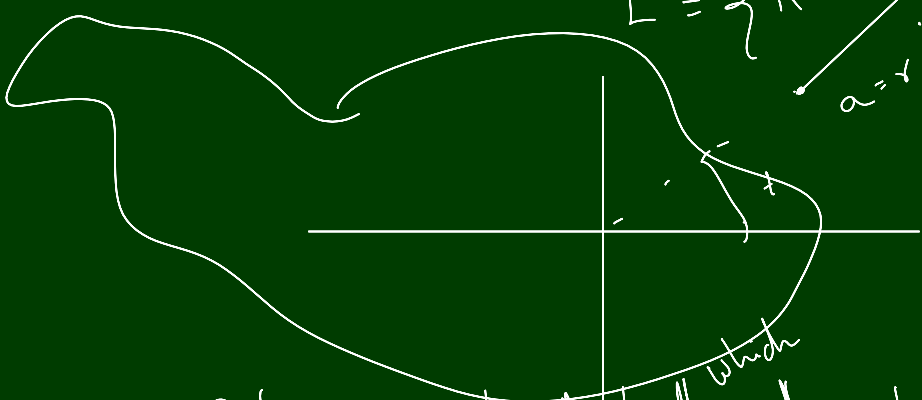
$\int_\gamma f = 0$ ✓✓

3) Let $a \in \mathbb{C}$ & L be a closed line join a & ∞

$a \neq 0$, $a = re^{it}$

$L = \{Re^{it} : R \geq r\}$





Let γ be any closed path which do not meet L

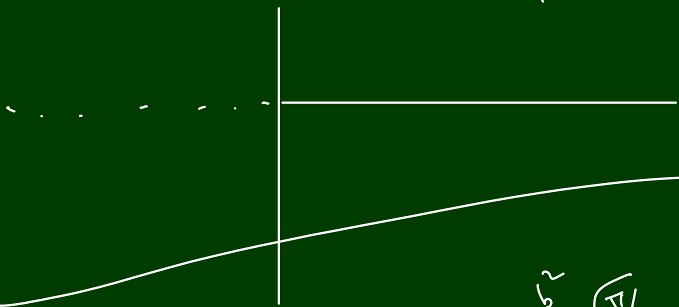
$$\int_\gamma \frac{dz}{z-a} := ?$$

$f : U \to \mathbb{C}$ $\quad U = \mathbb{C} \setminus L$

$f(z) = \dfrac{1}{z-a}$ $\quad \gamma \subseteq U$

$f'(z) = \dfrac{-1}{(z-a)^2}$

$a = 0$ $\quad \mathbb{C} \setminus (-\infty, 0]$ $\quad \log(z) = \displaystyle\int_1^z \frac{dw}{w}$



$$\boxed{\int_0^\infty e^{-x^2} \cos 2bx \, dx = e^{-b^2} \sqrt{\pi}/2}$$

$f(z) = e^{-z^2}$

More exercises (4 atleast) on the way towards Cauchy theorem

Thm An Extension of Cauchy theorem

Let U be an open set & $p \in U$

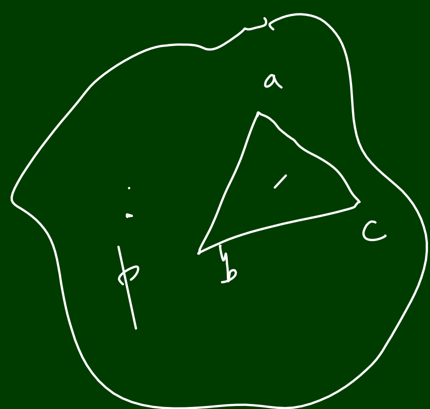
Let $f \in C(U)$ & $f \in H(U \setminus \{p\})$. Let $\Delta \subseteq U$ & $T = \partial \Delta$

Then $\displaystyle\int_T f = 0$

Proof

Case(i) $p \notin \Delta$ $\quad G := U \setminus \{p\} = U \cap \mathbb{C} \setminus \{p\}$

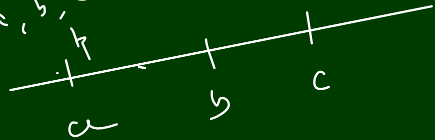
$f \in H(G)$



Case (ii) $p \in \Delta = [a, b, c]$

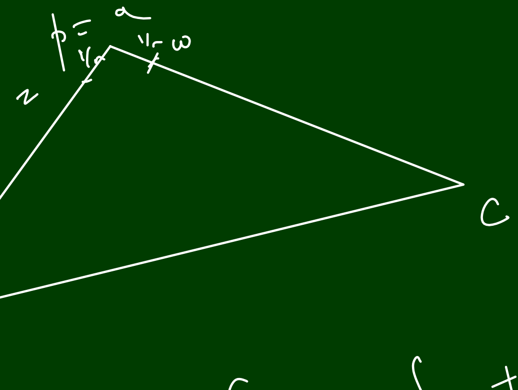
If p is one of the vertices.

Supp. all a, b, c are collinear



$$\int_{[a,b]+[b,c]+[c,a]} f \overset{?}{=} 0$$

Assume a, b, c are collinear



$$\int_T = \int_{[a,z]} + \int_{[z,b]} + \int_{[b,c]} + \int_{[c,w]} + \int_{[w,a]}$$

$$\text{—} \quad + \quad \text{—}$$

Complex Analysis, Sep 22, 2020.

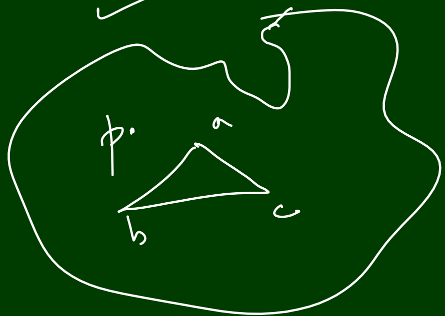
Recall Extension of Cauchy-Goursat's thm.

Let U be open subset of $\mathbb{C}$ & $p \in U$. Let $f \in C(U)$ & $f \in H(U \setminus \{p\})$. Let $\Delta \subseteq U$ & T be its bdrs

$$\int_T f = 0.$$

Let $\Delta = [a, b, c] \subseteq U$

Case (i) $p \notin \Delta$ ✓



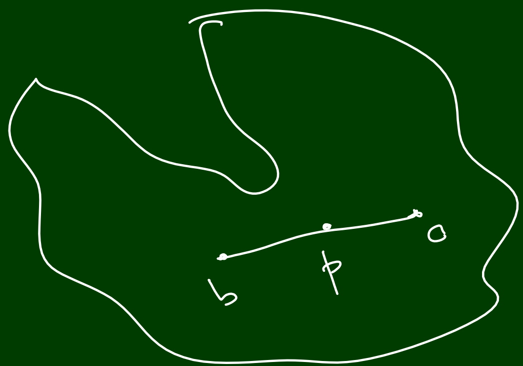
$G = U \setminus \{p\}$

G is open subset in $\mathbb{C}$

$\Delta \subseteq G$ By Cauchy-Goursat thm

$\therefore$ $f \in H(G)$, $\int_T f = 0$

Case (ii)

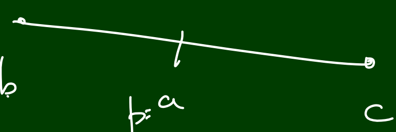


$p = a$

If p is one of the vertices & a, b, c are collinear

W.L.G. $p = a$

✓

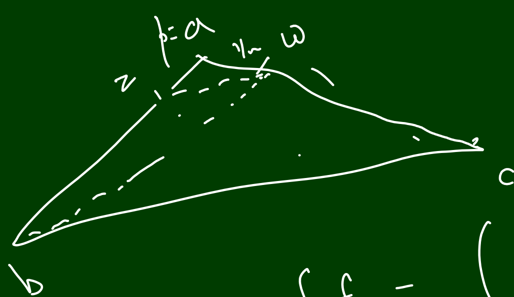


$$\int_{[a,b,c]} f = \int_{[a,b]} + \int_{[b,c]} + \int_{[c,a]}$$

$$= \cancel{-\int_{[b,a]}} + \cancel{\int_{[b,a]}} + \cancel{\int_{[a,c]}} + \cancel{\int_{[c,a]}}$$

$$= 0$$

Case (iii) a, b, c are not collinear



$$\int_T f = \int_{[a,z,w]} f + \int_{[z,b,w]} f + \int_{[b,c,w]} f \quad (?)$$

$$R.H.S = [a,z] + [z,\cancel{w}] + [w,a] + [z,b] + [b,\cancel{w}] + [w,\cancel{z}]$$

$$+ [b,c] + [c,w] + [w,\cancel{b}]$$

$$= [a,z] + [z,b] + [b,c] + [c,w] + [w,a]$$

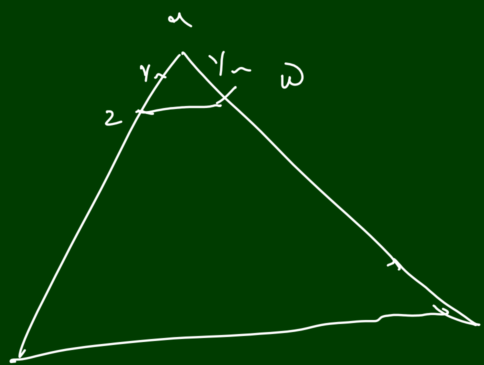
$$= [a,b] + [b,c] + [c,a] = L.H.S$$

$$\int_T f = \int_{[a,z,w]} f + \int_{(z,b,w)} f + \int_{[b,c,w]}$$

$p = a \notin [z,b,w]$

$p = a \notin [b,c,w]$

$$\therefore \int_T f = \int_{[a,z,w]} f$$



$$\left|\int_T f\right| \le \left|\int_{[a,z]} f\right| + \left|\int_{[z,w]} f\right| + \left|\int_{[w,a]} f\right|$$

$$\le M\left[|z-a| + |w-z| + |a-w|\right]$$

$$\le M\left[\frac{1}{n} + \frac{1}{n} + |w-z|\right]$$

$$\le \frac{4M}{n} < \epsilon$$

$$|w-z| = |w-a+a-z|$$

$$\le 2/n$$

$\exists\, z, w$ s.t $z \in [a,b]$, $w \in [a,c]$

& $|z-a| = 1/n$ & $|w-a| = 1/n$

choose $n \in \mathbb{N}$ s.t $\frac{\mathcal{F}(\Delta)}{n} < \epsilon$

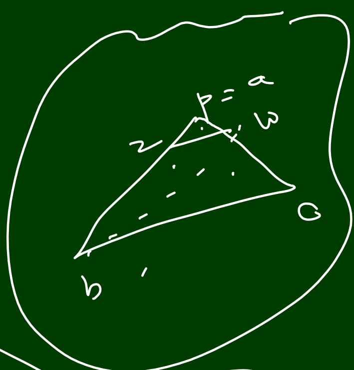
$z = (1-t)a + tb$

$z - a = t(b-a)$

$|z-a| = \frac{1}{n}$

$t|b-a| = \frac{1}{n}$

— x —

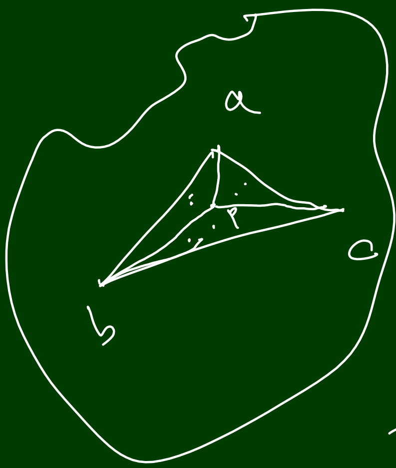


$G = U \setminus \{p\}$

$$\int_{[a,b,c]} f = \int_{[a,z,w]} f$$

The proof is over in all the cases.

$p \in \mathring{\Delta}$



$[a,b,c] = [a,b,p] + [b,c,p] + [c,a,p]$

By Previous Case $\int_{[a,b,p]} = 0$ ||ly other two

— x —

Extended Cauchy Theorem

(See below).

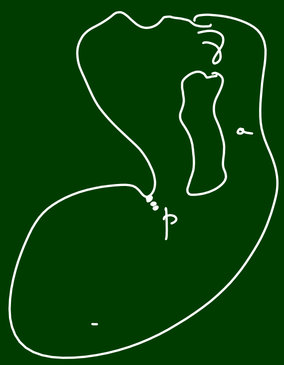
Let U be open & star shaped & $p \in U$. Let $f \in C(U)$ & $f \in H(U \setminus \{p\})$. Then (i) f has a primitive in U (ii) $\forall$ closed path γ in U

$$\int_\gamma f = 0$$

Cauchy - Integral formula.

Thm Let U be a open star shaped at $p \in U$. Let γ be closed in U. Let $f \in H(U)$

Then for any $a \in U \setminus [\gamma]$



$$\frac{1}{2\pi i} \int_\gamma \frac{f(w)\, dw}{w-a} = f(a) \left[\frac{1}{2\pi i} \int_\gamma \frac{dw}{w-a} \right]$$

$$= f(a)\, n_\gamma$$

Find g s.th $\int_\gamma g = 0$

$\therefore$ we get this result.

— x —

Complex Analysis, Sep 11, 2020

Recall Let $\gamma : [a,b] \to U$ be a smooth path, $U \subsetneq \mathbb{C}$

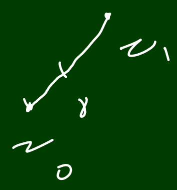
$$L(\gamma) := \int_a^b |\gamma'(t)|\, dt$$

If γ is piecewise smooth

$$L(\gamma) := \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} |\gamma'(t)|\, dt$$

Note

1. $L(\gamma)$ is diff from $[\gamma]$



2. If γ is line segment from z_0 to z_1

W.K.T $L(\gamma) = |z_1 - z_0|$

Verify with defn.

$$\gamma(t) = (1-t) z_0 + t z_1$$

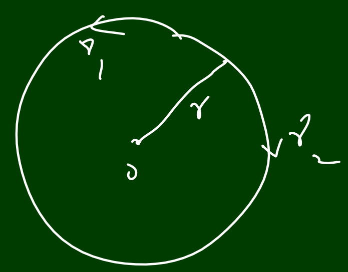
$$\gamma'(t) = -z_0 + z_1$$

$$L(\gamma) = \int_0^1 |z_1 - z_0|\, dt$$

$$= |z_1 - z_0|$$

3) $\gamma_1 = a + r e^{it}$ $\quad 0 \leq t \leq 2\pi$

$\gamma_2 = a + r e^{-it}$



$$L(\gamma_1) = \int_0^{2\pi} |r i e^{it}|\, dt$$

$\gamma_1 \quad \gamma_2$

$$= 2\pi r$$

$$L(\gamma_2) = 2\pi r$$

$L(\gamma)$ remains inv under reparametrization?

Idea γ_2 is reparametrization of γ_1 if $\exists\, h : [c,d] \to [a,b]$

$\gamma_1 : [a,b] \to \mathbb{C}$ $\quad h' \in C([c,d], \mathbb{R})$

$\gamma_2 : [c,d] \to \mathbb{C}$ $\quad \gamma_2(t) = \gamma_1(h(t))$.

$$[c,d] \xrightarrow{h} [a,b] \xrightarrow{\gamma_1} \mathbb{C}$$

Prove the result $L(\gamma)$ is remains inv under reparametrisation

γ_2 is rep γ_1

Then $L(\gamma_1) = L(\gamma_2)$

$\boxed{f : [a,b] \to \mathbb{C} \quad \checkmark \left|\int_a^b f\right| \le \int_a^b |f|}$

II Derive an u.b for $\left|\int_\gamma f\right|$

$$\cancel{\left|\int_\gamma f\right| \le \int_\gamma |f|} \qquad \text{why?}$$

wrong

$f : \mathbb{C} \to \mathbb{C}$

Let $f(z) = \frac{1}{z}$ Let γ be unit circle

$$\int_\gamma f(z)\,dz = \int_0^{2\pi} f(\gamma(t)) \cdot \gamma'(t)\,dt$$

$$= \int_0^{2\pi} \frac{i\,\cancel{e^{it}}}{\cancel{e^{it}}}\,dt$$

$$= 2\pi i$$

$g = |f|$

$$\int_\gamma g(z)\,dz = \int_0^{2\pi} g(\gamma(t)) \cdot \gamma'(t)\,dt$$

$$= \int_0^{2\pi} |f(\gamma(t))|$$

$2\pi \le 0$

$$\left|\int_\gamma f(z)\,dz\right| = 2\pi$$

$$\int_\gamma |f|\,dz = \int_0^{2\pi} \frac{1}{|\cancel{e^{it}}|}\, i e^{it}\,dt$$

$$= 0\,!$$

$$\left|\int_\gamma f(z)\,dz\right| = \left|\int_a^b f(\gamma(t))\cdot\gamma'(t)\,dt\right|$$

$$\leq \int_a^b |f(\gamma(t))|\,|\gamma'(t)|\,dt$$

$$\leq M \int_a^b |\gamma'(t)|\,dt \qquad |f(\gamma(t))| \leq \text{lub}\{|f(\gamma(t))| : t\in[a,b]\}$$

We have proved the foll

$$= M\,L(\gamma)$$

Thm (ML - formula) Inequality

Let $\gamma:[a,b]\to\mathbb{C}$ be a path. Let $f:[\gamma]\to\mathbb{C}$ be continuous fn. Let M be s.t $|f(\gamma(t))| \leq M$ $\forall t\in[a,b]$

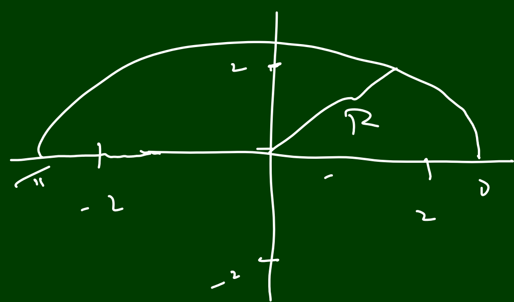
Then

$$\left|\int_\gamma f(z)\,dz\right| \leq M\,L(\gamma)$$

Prob

1) $\gamma(t) = Re^{it}$, $0\leq t\leq \pi$, $R>2$

$$\left|\int_\gamma \frac{dz}{z^2+4}\right| \leq \ ?$$



$f(z) = \dfrac{1}{z^2+4}$, $f:[\gamma]\to\mathbb{C}$

$$L(\gamma) = \int_0^\pi |iRe^{it}|\,dt \qquad |z| = R$$

$$= \pi R \qquad |z^2+4| \geq |z|^2 - 4$$

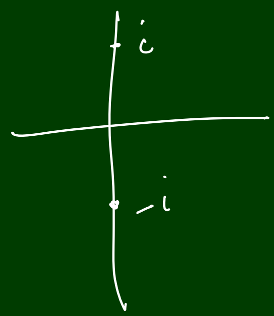
$$\geq R^2 - 4$$

$$\left|\frac{1}{z^2+4}\right| \leq \frac{1}{R^2-4}$$

$$\left|\int_\gamma f(z)\,dz\right| \le M\,L(\gamma)$$
$$\le \frac{1}{R^2-4}\,\pi R$$

2) $$\left|\int_\gamma e^{-z}\,dz\right| \le \, ?$$

γ is a line segment from $-i$ to i



By P.E

$$L(\gamma) = |i-(-i)| = 2$$

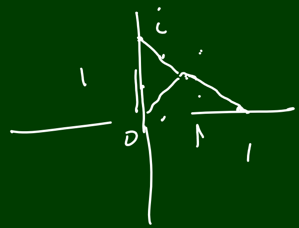
$$|e^{-z}| = \frac{1}{|e^{z}|} \qquad e^{z} = 1+z+\frac{z^2}{2!}+\cdots$$

Give a proof $|e^{z}| \ge 1$?

$\therefore |e^{-z}| \le 1 \qquad |e^{z}| = 1$

$$\left|\int_\gamma f(z)\,dz\right| \le 1\cdot 2 \quad ✓ \qquad |e^{z}| \ge 1$$

3) S.T $$\left|\int_\gamma \frac{dz}{z^4}\right| \le 4\sqrt{2} \qquad \gamma := [i,1]$$



$$L(\gamma) = |1-i| = \sqrt{1+1} = \sqrt{2}$$

~~T.P.T~~ $\cancel{f(z) = \frac{1}{z^4}}$ $\qquad \cancel{z \in [i,1]}$ $\qquad z = (1-t)i + t = t + i(1-t)$

$4t-2=0$, $t = 1/2$

$|z_0|^2 = \frac{1}{2}$

$f(z) = \frac{1}{z} \quad \cancel{= \frac{1}{t+i(1-t)}} \qquad z = \frac{1}{2} + i\left(1-\frac{1}{2}\right)$

$$= \frac{t-i(1-t)}{t^2-(1-t)^2}$$

$|z| = (1-t)i + t$

$|z|^2 = (1-t)^2 + t^2 = f(t)$

$f'(t) = 2t + 2(1-t)(-1)$

$= 2t + 2t - 2$

$= 4t-2$

$$f(z) = \frac{1}{z^4} \qquad = \left(\frac{t-i(1-t)}{2t-1}\right)^4$$

$$\cancel{|f(z)| = \left|\frac{1}{z^4}\right| = \left|\frac{t - i(1-t)}{2t-1}\right|^4}$$

$$\cancel{= \left(\frac{t^2}{(2t-1)^2} + \frac{(1-t)^2}{(2t-1)^2}\right)^2}$$

$|z|^2 \geq \frac{1}{4}$ ✓

γ is unit circle

5.1

4) a) $\left|\int_\gamma \frac{e^z}{z}\, dz\right| \le 2\pi e$

b) $\left|\int_\gamma \frac{dz}{4+3z}\right| \le 2\pi$

Improve this

$\left|\int_\gamma \frac{dz}{4+3z}\right| \le \frac{6}{5}\pi$

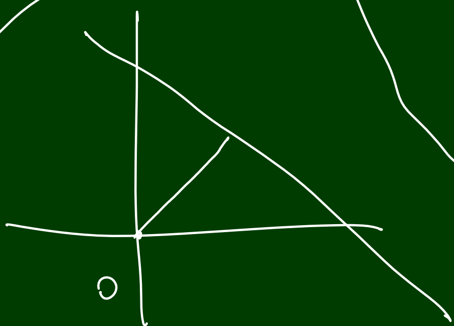
Go back to ~~the problem~~.

$|z|^2 =$

$z = (1-t)i + t$

$|z_0|^2 = \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2$

$= \frac{1}{2}$



$|z|^2 \geq |z_0|^2 = \frac{1}{2}$

$|z|^4 \geq \frac{1}{4}$

$\frac{1}{|z|^4} \leq 4$

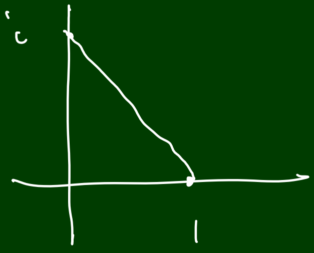
— x —

Complex Analysis, Sep 14, 2020

~~Appli~~ 'M - L' Estimates

1) $\gamma = [i, 1]$

$$\left| \int_\gamma \frac{dz}{z^4} \right| \le 4\sqrt{2}$$

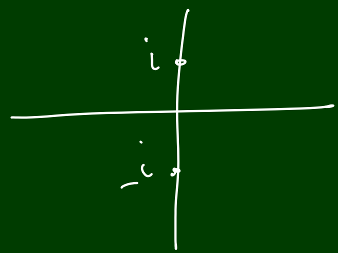


$$L(\gamma) = |i - 1|$$
$$= \sqrt{1+1} = \sqrt{2}$$

$$z = (1-t)i + t$$
$$|z|^2 = (1-t)^2 + t^2 = f(t)$$
$$f'(t) = 2(1-t)(-1) + 2t = 2t - 2 + 2t = 4t - 2$$
$$f''(t) = 4$$
$t = 1/2$ is point of min

$$|z|^2 \geq |z_0|^2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$
$$|z|^4 \geq |z_0|^4 = \frac{1}{4} \quad \text{on } \gamma \quad \frac{1}{|z|^4} \le 4$$
$$\left| \int_\gamma \frac{dz}{z^4} \right| \le 4 L(\gamma) = 4 \cdot \sqrt{2}$$

2) $$\left| \int_\gamma e^{-z} \, dz \right| \le 2 \qquad \gamma = [-i, i]$$



$$|e^z| = |e^{x+iy}|$$
$$= |e^x|$$
$$= e^x \qquad z \in [-i, i]$$
$$= e^0 = 1$$
$$\therefore \left| \frac{1}{e^z} \right| \le 1$$

$$\left| \int_\gamma e^{-z} \, dz \right| \le 1 \quad L(\gamma) = 1 \cdot 2 = 2$$

3) $\left| \int_\gamma \frac{e^z}{z}\, dz \right| \leq 2\pi e$

where γ is unit circle

$\gamma(t) = e^{it}, \quad 0 \leq t \leq 2\pi$

$$\left| \frac{e^z}{z} \right| = \frac{|e^z|}{|z|} = |e^z| = e^{\operatorname{Re} z} \leq e^{|z|} = e$$

for $z \in \gamma$

$x^2 \leq x^2 + y^2 \leq |z|^2$

$x \leq |z|$

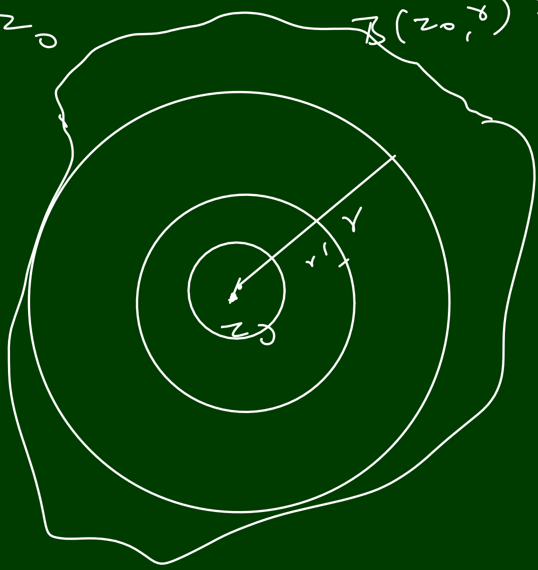
$\operatorname{Re} z \leq |z|$

2) Let $f : B(z_0, r) \to \mathbb{C}$ be continuous.

Let $\gamma_r(t) = z_0 + r e^{it}, \quad 0 \leq t \leq 2\pi$

S.T $\lim_{r \to 0} \frac{1}{2\pi i} \int_{\gamma_r} \frac{f(z)}{z - z_0}\, dz = f(z_0)$

$B(z_0, r) \subseteq U$



to p.t. $\lim_{r \to 0} (\qquad) = f(z_0)$

i.e Given $\epsilon > 0$ $(\exists \delta > 0 \; (|r| < \delta \; (|(\quad) - f(z_0)| < \epsilon)))$

We have to estimate

$$\left| \frac{1}{2\pi i} \int_{\gamma_r} \frac{f(z)}{z - z_0}\, dz - f(z_0) \right|$$

$$\left| \frac{1}{2\pi i} \int_{\gamma_r} \frac{f(z) - f(z_0)}{z - z_0}\, dz \right|$$

$$\int_{\gamma_r} \frac{f(z_0)}{z - z_0}\, dz = \int_0^{2\pi} \frac{f(z_0)\, i \cancel{re^{it}}}{\cancel{re^{it}}}\, dt$$

$$= 2\pi i f(z_0)$$

$$f(z_0) = \frac{1}{2\pi i} \int_{\gamma_r} \frac{f(z_0)}{z - z_0}\, dz$$

We have to estimate this

$$\therefore \left| \frac{1}{2\pi i} \int_{\gamma_r} \frac{f(z)}{z - z_0}\, dz - \frac{1}{2\pi i} \int_{\gamma_r} \frac{f(z_0)}{z - z_0}\, dz \right|$$

$$= \frac{1}{2\pi} \left| \int_{\gamma_r} \frac{f(z) - f(z_0)}{z - z_0}\, dz \right|$$

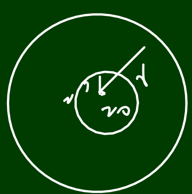
$\gamma_r = z_0 + re^{it}$, $0 \le t \le 2\pi$

$\int_\gamma f(z)\, dz := \int_a^b f(\gamma(t)) \cdot \gamma'(t)\, dt$

$$\int_{\gamma_r} \frac{f(z) - f(z_0)}{z - z_0}\, dz = \int_0^{2\pi} \frac{f(z_0 + re^{it}) - f(z_0)}{\cancel{re^{it}}}\, i\cancel{re^{it}}\, dt$$

$$\left| \int_{\gamma_r} \frac{f(z) - f(z_0)}{z - z_0}\, dz \right| = \left| i \int_0^{2\pi} \left[f(z_0 + re^{it}) - f(z_0) \right] dt \right|$$

$$= \left| \int_0^{2\pi} \left[f(z_0 + re^{it}) - f(z_0) \right] dt \right|$$



$$\le \int_0^{2\pi} \left| f(z_0 + re^{it}) - f(z_0) \right| dt$$

Use continuity of f at z_0, given $\epsilon' > 0$ ($\exists \delta' > 0$ ($|z - z_0| < \delta'$
$(|f(z) - f(z_0)| < \epsilon')$

Given $\epsilon > 0$
choose $\delta = \delta'$

$$|z - z_0| = r < \delta' \leq \delta$$

$$\leq \epsilon' \int_0^{2\pi} dt$$

$$\therefore \left| \frac{1}{2\pi i} \int \frac{f(z)}{z - z_0} dz - f(z_0) \right| \leq \frac{1}{\cancel{2\pi}} \cdot \epsilon' \cancel{2\pi}$$
$$< \epsilon$$

$$\boxed{\epsilon' = \epsilon}$$

Proof (2)

By M-L formula

$$\left| \frac{1}{2\pi i} \int_{\gamma_r} \frac{f(z) - f(z_0)}{z - z_0} dz \right| = \frac{1}{2\pi} \left| \int_{\gamma_r} \frac{f(z) - f(z_0)}{z - z_0} dz \right|$$

$\exists \delta > 0$

$\forall z \in I[z_0, \frac{\delta}{2}]$

$$|f(z) - f(z_0)| \leq C |z - z_0|$$

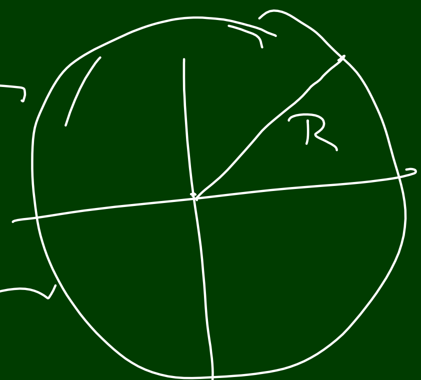
Continue the proof (2).

2) $f : \mathbb{C} \to \mathbb{C}$ be conts & bdd.

$$\gamma_R(t) = R e^{it}, \quad 0 \leq t \leq 2\pi$$

$\forall z_0 \in \mathbb{C}$

$$\lim_{R \to \infty} \frac{1}{2\pi i} \int_{\gamma_R} \frac{f(z)}{(z - z_0)^2} dz =$$



$$\int_{\gamma_R} \frac{f(z)}{(z - z_0)^2} dz = \int_0^{2\pi} \frac{f(z_0 + R e^{it})}{R^2 e^{i2t}} R i e^{it} dt$$

$$-\frac{i}{R}\int_0^{2\pi} f(z_0+Re^{it})\cdot e^{-it}\,dt$$

$$\left|\int_{\gamma_R}\frac{f(z)}{(z-z_0)^2}\,dz\right| = \frac{1}{R}\left|\int_0^{2\pi} f(z_0+Re^{it})\cdot e^{-it}\,dt\right|$$

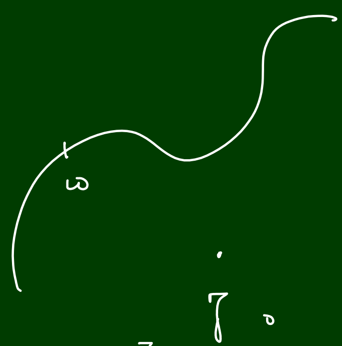
$$\le \frac{c}{R}\,2\pi$$

$$\lim_{R\to\infty}\frac{1}{2\pi i}\cdot\int_{\gamma_R}\frac{f(z)}{(z-z_0)^2}\,dz = 0$$

3) Let $\gamma:[a,b]\to\mathbb{C}$ be a path s.t. the fn $z\mapsto\int_\gamma\frac{dw}{w-z}$ is continuous on $\mathbb{C}\setminus[\gamma]$.

Let $\phi:\mathbb{C}\setminus[\gamma]\to\mathbb{C}$

$$\phi(z) = \int_\gamma\frac{dw}{w-z}$$



We prove ϕ is conti at $z_0\in\mathbb{C}\setminus[\gamma]$

$$\phi(z_0) = \int_\gamma\frac{dw}{w-z_0}$$

$w\in[\gamma]$

$z\in\mathbb{C}\setminus[\gamma]$

$\therefore z\neq w$ $\therefore \frac{1}{w-z}$ is well def

$$|\phi(z)-\phi(z_0)| = \left|\int_\gamma\frac{dw}{w-z}-\int_\gamma\frac{dw}{w-z_0}\right|$$

$|z-z_0|<\delta$

$$= \left|\int_\gamma\frac{(w-z_0)-(w-z)}{(w-z)(w-z_0)}\,dw\right|$$

$$= \left| \int_\gamma \frac{(z - z_0)}{(w-z)(w-z_0)}\, dw \right|$$

Exercise cont.

Just use M-L form

$\le$

Bye Bye

Complex Analysis, sep 28, 2020

$$I = \int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \frac{2\pi}{3}$$

$$z = e^{i\theta}$$
$$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

$$2+\cos\theta = 2 + \frac{1}{2}\left(z+\frac{1}{z}\right)$$
$$= 2 + \frac{1}{2}\left(\frac{z^2+1}{z}\right)$$
$$= \frac{4z+z^2+1}{2z}$$

$$z = e^{i\theta}$$
$$\frac{dz}{d\theta} = ie^{i\theta}$$
$$dz = ie^{i\theta}\,d\theta$$
$$\frac{-i}{z}\,dz = d\theta$$

$$\therefore I = \int_\gamma \frac{2\cancel{z}}{4z+z^2+1}\left(\frac{-i}{\cancel{z}}\,dz\right)$$
$$= -i \int_\gamma \frac{2\,dz}{4z+z^2+1}$$

$$\frac{2}{4z+z^2+1} = 2\left(\frac{A}{z-w_1} + \frac{B}{z-w_2}\right)$$

$$z = -2 \pm \sqrt{3}$$
$$w_1 = -2+\sqrt{3}$$
$$w_2 = -2-\sqrt{3}$$

$$A = -B$$
$$B(w_1 - w_2) = 1$$
$$B = \frac{1}{2\sqrt{3}}$$

$$A+B = 0$$
$$-(Aw_1 + Bw_2) = 1$$

$$A = -\frac{1}{2\sqrt{3}}$$

$$\int_\gamma \frac{2}{4z+z^2+1}\,dz = -\cancel{2}\cdot\frac{1}{\cancel{2}\sqrt{3}}\int_\gamma \frac{dz}{z-w_1} + 0$$

$$-i\int_\gamma \frac{2\,dz}{4z+z^2+1} = -\frac{1}{\sqrt{3}}\cdot 2\pi \cancel{i}$$

$$\int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \frac{2\pi}{\sqrt{3}}$$

$$p(z) = 4z+z^2-1$$

$$p'(z) = 4+2z$$

$$\int_\gamma \frac{p'(z)}{p(z)}\,dz = \int_\gamma \frac{4+2z}{4z+z^2-1}\,dz$$

Avoid it.

CIF for derivatives

CIF - D

Let $f \in H(U)$, $B(z_0,R) \subset U$

Let $0<r<R$, $\gamma_r(t) = z_0 + re^{it}$, $0 \le t \le 2\pi$

$$f^{(n)}(z) = \frac{n!}{2\pi i}\underbrace{\int_{\gamma_r} \frac{f(w)\,dw}{(w-z)^{n+1}}}$$

Proof

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n, \quad z \in B(z_0,R)$$

where

$$a_n = \frac{1}{2\pi i}\int_{\gamma_r} \frac{f(w)}{(w-z)^{n+1}}\,dw$$

Also W.K.T

$$a_n = \frac{f^{(n)}(z)}{n!}$$

$$\therefore f^{(n)}(z) = n!\, a_n$$

Def $f \in H(U)$ & $B(a,R) \subset U$

Then the taylor's expansion of f around a is given by $f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(z-a)^n$

Cor: The taylor ser of f converges to f absolutely & unif on cpt subsets of $B(a,R)$ $\quad K \subset B[a,r] \subset B(a,R)$

Remark Real Analogue is false

Next Thm (Weierstrass)

Let $f_n \in H(U)$, Assume $f_n \rightrightarrows f$ on compact subsets of U. Then $f \in H(U)$. Further, $\forall k \in \mathbb{N}$, $f_n^{(k)} \rightrightarrows f^{(k)}$ on cpt subsets of U.

Proof:

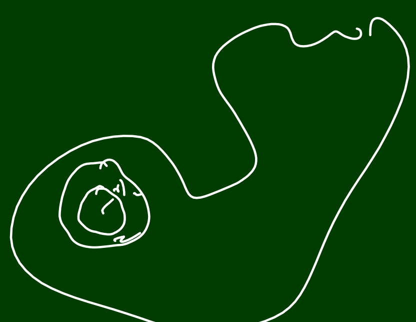
Note that f is conts on U $\quad B[z, r/2] \subset U$

$f_n \rightrightarrows f$

Let $z \in U$

Claim f is conts at z.

$B[z, r/2] \subseteq U$
& cpt
& $f_n \rightrightarrows f$ in $B[z, r/2]$
& $f_n \in C(B[z, r/2])$
$\therefore f \in C(B[z, r/2])$
hence f

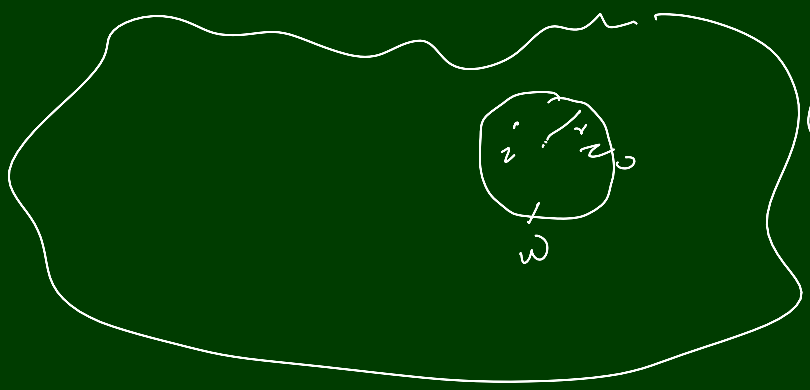


Let $z_0 \in U$

Claim $f \in H(B(z_0, r))$

$\because z_0 \in U$, $\exists\, r > 0$ s.t. $B[z_0, r] \subset U$

Let $z \in B(z_0, r)$.



W.K.T

$$f_n(z) = \frac{1}{2\pi i} \int_\gamma \frac{f_n(\omega)}{\omega - z}\, d\omega$$

$$\epsilon = d(z, [\gamma]) > 0$$

$$f_n \rightrightarrows f \quad \text{on } [\gamma]$$

To S.T $\dfrac{f_n(\omega)}{\omega - z} \rightrightarrows \dfrac{f(\omega)}{\omega - z}$ where $\omega \in [\gamma]$,

$$f_n \rightrightarrows f, \quad g_n \rightrightarrows g$$

$$f_n \cdot g_n \rightrightarrows fg \quad \left[\text{provi } f_n, g_n \text{ are bdd}\right]$$

$\forall\, \omega \in [\gamma]$, $|\omega - z| \geq \epsilon$ $\qquad \epsilon = d(z, [\gamma])$.

$$\frac{1}{|\omega - z|} \leq \frac{1}{\epsilon}$$

$$\therefore \frac{f_n(\omega)}{\omega - z} \rightrightarrows \frac{f(\omega)}{\omega - z} \quad \text{on } [\gamma]$$

$$\therefore \frac{1}{2\pi i}\int_\gamma \frac{f_n(\omega)}{\omega - z}\, d\omega \to \frac{1}{2\pi i}\int_\gamma \frac{f(\omega)}{\omega - z}\, d\omega \quad [\text{By prev}]$$

$$\overset{\shortparallel}{f_n(z)} \longrightarrow f(z)$$

$$\therefore \quad f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}\,dw$$

$$\therefore \quad f \in H(B(z_0, r)) \quad \left[\text{Do as in first thm } f \text{ is hol iff } f \text{ is analytic}\right]$$

$$\left[\frac{1}{w-z} = \frac{1}{w-z_0+z_0-z}\right]$$

Also
$$\frac{f_n(w)}{(w-z)^{k+1}} \rightrightarrows \frac{f(w)}{(w-z)^{k+1}} \quad \text{on } [\gamma]$$

$$\therefore \quad \frac{k!}{2\pi i}\int_\gamma \frac{f_n(w)\,dw}{(w-z)^{k+1}} \to \frac{k!}{2\pi i}\int_\gamma \frac{f(w)}{(w-z)^{k+1}}$$

$$\therefore \quad f_n^{(k)}(z) \to f^{(k)}(z) \quad \forall\, z \in B(z_0, r)$$

To establish uniform convergence.

$$|w-z| \geq \epsilon$$
$$|w-z|^{k+1} \geq \epsilon^{k+1}$$

$$f_n^{(k)}(z) - f^{(k)}(z) = \frac{k!}{2\pi i}\int_\gamma \frac{(f_n(w)-f(w))}{(w-z)^{k+1}}\,dw$$

$$|w-z|^{k+1} \geq (r-\rho)^{k+1} \quad \forall\, z \in B(z_0, r)$$

$$\left|f_n^{(k)}(z) - f^{(k)}(z)\right| \leq \frac{k!}{2\pi i}\,\frac{1}{\epsilon^{k+1}}\,\|f_n - f\|_\infty\, L(\gamma) \longrightarrow 0$$

choose $\rho < r$.

$\forall\, z \in B[z_0, \rho]$ $\quad f_n^{(k)} \rightrightarrows f$

Let $K \subseteq U$

Let $z \in K$

$\exists\, r_z > 0$ $\quad B(z, r_z) \subseteq U$

$\therefore\, B[z, \frac{r_z}{2}] \subseteq U$

$\left\{ B\left(z, \frac{r_z}{2}\right) : z \in K \right\}$ is cover K.

$K \subset \bigcup_{i=1}^{l} B\left[z_i, \frac{r_{z_i}}{2}\right]$

$A_i = B(z_i, r_i)$

$A_i = B\left(z_i, \frac{r_{z_i}}{2}\right)$

$A_1, \ldots A_l$

$B[z_0, \rho] \subseteq B(z_0, r)$

— x —

Complex Analysis; sep 8, 2020

1) Parsevel's identity

Let $f(z) = \sum_n a_n (z-z_0)^n$, $z \in B(z_0, R)$

Let $0 \le r < R$ & $0 \le t \le 2\pi$

Then

(i) $\frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + re^{it})|^2 dt = \sum_{n=0}^{\infty} |a_n|^2 r^{2n}$

(ii) if $\forall\, z$ s.t $|z - z_0| = r$, $|f(z)| \le M(r)$.

Then $\sum_{n=0}^{\infty} |a_n|^2 r^{2n} \le (M(r))^2$

Proof

Observe

(1) $(C[0, 2\pi], \| \|_2)$ is an IPS ✓

(2) $\left\{ \frac{e^{ikt}}{\sqrt{2\pi}} : k \in \mathbb{Z} \right\}$ is an O.N. Set in the I.P.S.

Let $u_k := \frac{e^{ikt}}{\sqrt{2\pi}}$, $k \in \mathbb{Z}$

Then $\langle u_k, u_l \rangle = \frac{1}{2\pi} \int_0^{2\pi} e^{ikt} \cdot e^{-ilt}\, dt$

$= \frac{1}{2\pi} \int_0^{2\pi} e^{i(k-l)} dt$

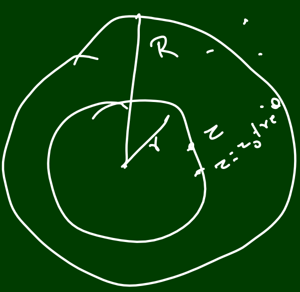
$= \begin{cases} 0 & \text{if } k \ne l \\ 1 & \text{if } k = l. \end{cases}$

$u = \sum_{k=0}^{n} u_k$

$\langle u, u \rangle = \left\langle \sum_{k=0}^{n} u_k, \sum_{j=0}^{n} u_j \right\rangle = \sum_{k=0}^{n} \left\langle u_k, \sum_{j=0}^{n} u_j \right\rangle$

$= \sum_{k=0}^{n} \langle u_k, u_k \rangle$

$= \sum_{k=0}^{n} \| u_k \|^2$ ✓



For $z - z_0 = r e^{it}$

$$f(z) = f(z_0 + re^{it}) = \sum_{n=0}^{\infty} a_n r^n e^{int} \quad \text{———} \ (1)$$

Let $s_n(t) = \sum_{k=0}^{n} a_k r^k e^{ikt}$

$$\therefore \langle s_n(t), s_n(t) \rangle = \left\langle \sum_{k=0}^{n} a_k r^k e^{ikt}, \sum_{j=0}^{n} a_j r^j e^{ijt} \right\rangle$$

$$= \sum_{k=0}^{n} \left\langle a_k r^k e^{ikt}, \sum_{j=0}^{n} a_j r^j e^{ijt} \right\rangle$$

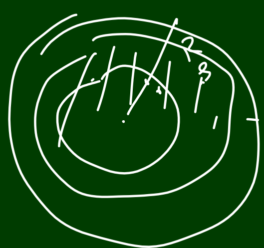
$$= \sum_{k=0}^{n} \langle a_k r^k e^{ikt}, a_k r^k e^{ikt} \rangle$$

$$= \sum_{k=0}^{n} a_k \overline{a_k} r^{2k} \langle e^{ikt}, e^{ikt} \rangle$$

$$= 2\pi \sum_{k=0}^{n} |a_k|^2 r^{2k}$$

$$\therefore \frac{1}{2\pi} \| s_n(t) \|^2 = \sum_{k=0}^{n} |a_k|^2 r^{2k}$$

Also $\| s_n(t) \|^2 = \int_0^{2\pi} |s_n(t)|^2 \, dt$



$$\therefore \frac{1}{2\pi} \int_0^{2\pi} |s_n(t)|^2 \, dt = \sum_{k=0}^{n} |a_k|^2 r^{2k} \quad \text{———} \ (1)$$

W.K.T $s_n(t) \rightrightarrows f(z_0 + re^{it})$ on $[0, 2\pi]$

If $f_n \rightrightarrows f$ and $g_n \rightrightarrows g$ & f & g are bdd.

Then $f_n g_n \rightrightarrows fg$. Try!

$a_n \to a$, $b_n \to b$
$a_n b_n \to ab$

Let $n \geq N = \max\{N_1, N_2\}$

Let $t \in X$.

$$\left| f_n(t) \cdot g_n(t) - f(t) g(t) \right|$$
$$\leq \left| f_n(t)\left(g_n(t) - g(t)\right) \right| + \left| g(t)\left(f_n(t) - f(t)\right) \right|$$
$$\leq |f_n(t)|\, |g_n(t) - g(t)| + |g(t)|\, |f_n(t) - f(t)|$$
$$\leq \|f_n\|\, \|g_n - g\| + \|g\|\, \|f_n - f\|$$
$$\leq M_1 \|g_n - g\| + M_2 \|f_n - f\|, \quad n \geq N_1,\ n \geq N_2$$
$$< M_1 \frac{\epsilon}{2M_1} + M_2 \frac{\epsilon}{2M_2}$$

$\because f_n \rightrightarrows f$ & f is bdd
(f_n) is bdd in $(B(X, \mathbb{R}), \|\ \|_\infty)$

Note that $|f_n(t)| \rightrightarrows |f(z_0 + re^{it})|$

$f_n \to f$

$$\big|\, |f_n(t)| - |f(t)| \,\big| \leq |f_n(t) - f(t)| \leq \|f_n - f\|$$

$$\therefore |f_n(t)|^2 \rightrightarrows |f(z_0 + re^{it})|^2$$

$$\therefore \frac{1}{2\pi}\int_0^{2\pi} |f_n(t)|^2\, dt \to \frac{1}{2\pi}\int_0^{2\pi} |f(z_0 + re^{it})|^2\, dt$$

[By previous thm $f_n \rightrightarrows f$ Then $\int f_n \to \int f$]

i.e
$$\frac{1}{2\pi}\int_0^{2\pi} |f(z_0 + re^{it})|^2\, dt = \lim_{n\to\infty} \frac{1}{2\pi}\int_0^{2\pi} |f_n(t)|^2\, dt$$
$$= \lim_{n\to\infty} \sum_{k=0}^{n} |a_n|^2 r^{2n} \quad (\text{by } (1))$$
$$= \sum_{n=0}^{\infty} |a_n|^2 r^{2n}.$$

(2) is easy follows for monotonicity of the integral.

$$|f(z_0 + re^{it})|^2 \leq M(r)$$

$$\therefore \int_0^{2\pi} |f(z_0+re^{it})|^2 \, dt \le (M(r))^2 \int_0^{2\pi} dt$$

$$\therefore \frac{1}{2\pi}\int_0^{2\pi} |f(z_0+re^{it})|^2 \, dt \le (M(r))^2$$

2) Use notations of (1), $R=\infty$ & $\forall z \in \mathbb{C}$, $|f(z)| \le M$

Then f is constant (This called Liouville's thm)

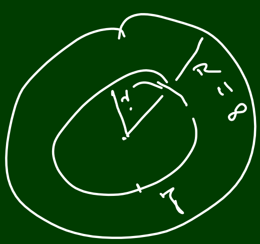
$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n, \quad z \in B(z_0, R)$$

$\forall n \geq 1$, $a_n = 0$

We use (ii) of Parseval's identity

$\forall z$, $|z-z_0| = r$, $|f(z)| \le M(r)$

$$\sum_{n=0}^{\infty} |a_n|^2 r^{2n} \le (M(r))^2 \quad \text{—— } (*)$$



$\forall z \in \mathbb{C}$, $|f(z)| \le M$

$\therefore$ $(*)$ becomes,

$$\sum_{n=0}^{\infty} |a_n|^2 r^{2n} \le M^2$$

$|z-z_0| = r$

$\because R=\infty$, $r>0$ can be be as big as we wish.

Suppose $\exists N \geq 1$ s.t $a_N \neq 0$

We have by $(*)$ $|a_N|^2 r^{2N} \le \sum_{n=0}^{\infty} |a_n|^2 r^{2n} \le M^2$

$M+1 \le M$

Choose $r = \dfrac{(M+1)^{1/N}}{|a_N|^{1/N}}$

Then $\cancel{|a_N|^2} \cdot \dfrac{(M+1)^2}{\cancel{|a_N|^2}} \le M^2$

$\therefore M+1 \le M$

$\Rightarrow \Leftarrow$

(3) Cauchy Inequalities

$$M(r) := \sup\{|f(z)| : |z-z_0| = r\}$$

For $0<r<R$, $\quad |f^{(n)}(z_0)| \le n!\, \frac{M(r)}{r^n}$, $\quad n=0,1,2,\dots$

W.K.T $\quad f^{(n)}(z_0) = n!\, a_n$

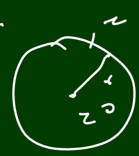
To P.T

$$\frac{|f^{n}(z_0)|}{n!}\, r^n \le M(r)$$

i.e. $\quad |a_n|\, r^n \le M(r)$

i.e. $\quad |a_n|^2 r^{2n} \le (M(r))^2$ ✓

Proof $\quad |a_n|^2 r^{2n} \le \sum_{k=0}^{\infty} |a_k|^2 r^{2k} \le (M(r))^2$



(4) Maximum Modulus thm

Let $0<r<R$

Assume $\quad |f(z_0)| = M(r) := \sup = \{|f(z)| : |z-z_0| = r\}$

Then f is constant.

$\forall n \ge 1$, $\quad a_n = 0$ $\qquad f(z) = \sum_n a_n (z-z_0)^n$

$$|a_0| = |f(z_0)|$$

$$|a_0|^2 = |M(r)|^2$$

$$\sum_{n=0}^{\infty} |a_n|^2 r^{2n} \le M(r)^2$$

$$\underbrace{|a_0|^2}_{= M(r)^2} + \sum_{n=1}^{\infty} |a_n|^2 r^{2n} \le M(r)^2$$

$$\sum_{n=1}^{\infty} |a_n|^2 r^{2n} \le 0$$

$\Rightarrow \forall n \ge 1, \quad |a_n| = 0$

$\Leftrightarrow a_n = 0$ ✓

FTA

Let $p(z)$ be a non-constant polynomial in $\mathbb{C}$

Then $p(z)$ has a root in $\mathbb{C}$

Proof

Suppose $p(z)$ has no roots in $\mathbb{C}$

Then $f(z) = \dfrac{1}{p(z)}$ is entire in $\mathbb{C}$

Suppose we say f is bdd in $\mathbb{C}$

Then f is const $\Rightarrow$ p is const ($\Rightarrow\Leftarrow$)

(i) $\lim\limits_{|z|\to\infty} |f(z)| = 0$

(ii) f is bdd in $\mathbb{C}$ ✓

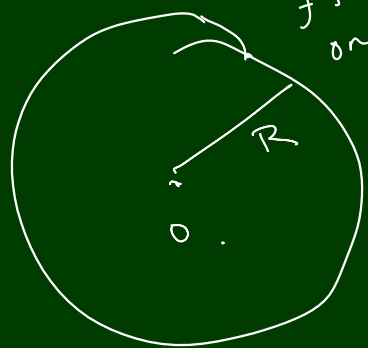
(Cor (i))

$\boxed{\lim\limits_{z\to a} f(z) = l}$

Given $\epsilon > 0$ ($\exists\, R > 0$ ($|z| > R$

$|f(z)| < \epsilon$))

f is bdd on $(B[0,R])^c$



Is f bdd on $B[0,R]$

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