BHARATHIDASAN UNIVERSITY
Tiruchirappalli - 620024
Tamil Nadu, India

Programme : M.Sc. Mathematics
Course Title : Complex Analysis
Course code : 24S3M09CC

UNIT 5

Dr. P. S. Srinivasan
Associate Professor
Department of Mathematics

Complex Analysis, Nov 2, 2020

Isolated Singularities

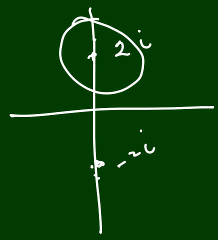
$$f(z) = \frac{z}{z^2+z} \qquad g(z) = \frac{1}{z+1}$$

Let $U \subseteq \mathbb{C}$ open & $a \in U$

A fn $f : U \setminus \{a\} \to \mathbb{C}$ is said to have a isolated singularity at a if $\exists\, r > 0$ s.t. $f \in H(B'(a, r))$

Examples $f : \mathbb{C} \setminus \{\pm 2i\} \to \mathbb{C}$

1) $f(z) = \dfrac{1}{z^2+4}$ ✓



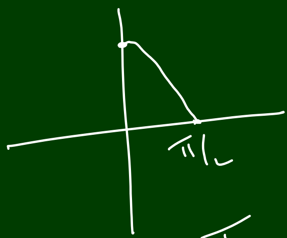
$z^2 + 4 = 0$

$z^2 = -4$

$z = \pm 2i$

$z = \pm 2i$ are isolated sing. of f.

2) $f(z) = \tan\left(\frac{1}{z}\right)$, $z \neq 0$.



$$= \frac{\sin(1/z)}{\cos(1/z)}$$

$\cos 1/z = 0$

$$\frac{\pi}{2} + 2k\pi \qquad \frac{\pi}{2}(4k+1) \qquad \frac{1}{r}$$

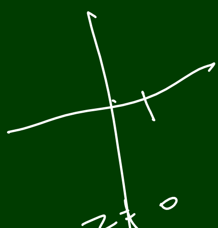
$$\frac{1}{z} = \frac{\pi}{2}(4k+1)$$

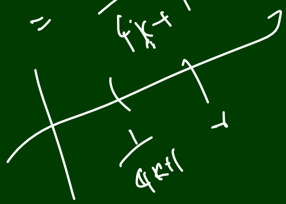
$$z = \frac{2}{\pi(4k+1)} = \frac{c}{4k+1}$$

$$A = \left\{ \frac{2}{\pi(4k+1)} : k \in \mathbb{Z} \right\}$$

0 is dist

$\therefore$ 0 is not an isolated singularity.





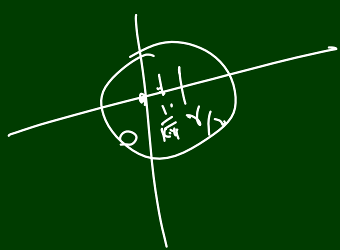
3) $g(z) = \dfrac{1}{\sin(1/z)}$, $z \neq 0$

$\sin\left(\frac{1}{z}\right) = 0$

$$\frac{1}{z} = k\pi$$

$$z = \frac{1}{k\pi}$$

$$A = \left\{ \frac{1}{k\pi} : k \in \mathbb{Z} \right\}$$



$g : \mathbb{C} \setminus A \to \mathbb{C}$

0 is non-isolated singul.

1) $f(z) = \frac{\sin z}{z}, z \neq 0$

$\sin z \in H(\mathbb{C})$

$z \in H(\mathbb{C} \setminus \{0\})$

$f \in H(\mathbb{C} \setminus \{0\})$.

$z = 0$ is isolated sing because there are no other singularities.

$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \dots$

$\frac{\sin z}{z} = \boxed{1 - \frac{z^2}{3!} + \frac{z^4}{5!} + \dots}$

$\cos z = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \dots$

$\frac{\cos z}{z} = \frac{1}{z} + \frac{z}{2!} + \frac{z^3}{4!} + \dots$

$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{1}{z}\right)^n$

2) $g(z) = \frac{\cos z}{z}, z \neq 0$

3) $h(z) = e^{1/z}, z \neq 0$

For the fn f, g, h 0 is the only singular. it is isolated.

Non-isolated are of 3 kinds.

Investig

$f \in H(\mathbb{C} \setminus \{0\})$

$\lim_{z \to 0} \frac{\sin z}{z} = 1$

$$\tilde{f}(z) = \begin{cases} \frac{\sin z}{z}, & z \neq 0 \\ 1, & z = 0 \end{cases}$$

$\tilde{f} \in H(\mathbb{C})$

by Morera's thm (2 Cor)

$\therefore$ We have remove singular of f at 0 by suitably defining f at 0

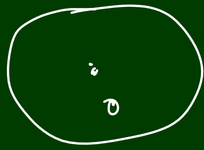
This singul is cal Removable Singularity

2) $g(z) = \frac{\cos z}{z}$

W.P.T $\lim_{z \to 0} \frac{\cos z}{z} = \infty$

given $R > 0$, $\exists\, \delta > 0$ $\forall z$ st $|z| < \delta$, $\left|\frac{\cos z}{z}\right| > R$

Let $R > 0$



$\cos 0 = 1$

Exploit cont of cos at 0

For $\epsilon = 1/2$, $\exists\, \delta > 0$ s.t

$\forall z \in B(0,\delta)$, $\quad |\cos z - 1| < 1/2$

$$\frac{1}{2} > |\cos z - 1| \geq \big||\cos z| - |1|\big|$$

$$\big||\cos z| - 1\big| < 1/2 \qquad \cdots < 1/2$$

$$-1/2 < |\cos z| - 1 < 1/2$$

$$\therefore -1/2 < |\cos z| < 3/2$$

$$1/2 < |\cos z|$$

$\forall z \in B(0,\delta)$, $\quad |\cos z| > 1/2$

$\underline{|\cos z| > 1/2}$

$N = \max\{N_1, N_2\}$

$N_1 > 2R$

$\frac{1}{N_2} < \delta$

Choose $N \geq N_1 > 2R$

$$\leq \frac{1}{N} < \delta$$

Let $|z| < \frac{1}{N_0}$

$$\frac{1}{|z|} > N \quad > R$$

$$\frac{|\cos z|}{|z|} > \frac{1}{2} N$$

$$\therefore \lim_{z \to 0} \frac{\cos z}{z} = \infty$$

(iii) exer

ST $\quad \lim_{z \to 0} e^{1/z}$ does not exist

For $x > 0$, $x \to 0$

$x < 0$, $x \to 0$

$-x > 0$

$e^{1/z} = e^{1/x} \to \infty$

$e^{1/z} = e^{-\frac{1}{x}} \to 0$

$z = iy \quad y \to 0, y \in \mathbb{R}$

$e^{\frac{1}{z}} = e^{-iy}$ dw

Defn $U \overset{\text{open}}{\subseteq} \mathbb{C}$, $a \in U$, $f : U \setminus \{a\} \to \mathbb{C}$ & a be an isolated Sing of f.

Then

a is removable Sing if $\lim_{z \to a} f(z)$ exists a finite complex number

a is pole if $\lim_{z \to a} f(z) = \infty$

a is essential Sing if $\lim_{z \to a} f(z)$ does not exists.

Complex Analysis, Nov 3, 2020.

Riemann's Thm on Removable Singularity

Let $f \in H(B'(a,r))$ and $\lim_{z \to a} (z-a)f(z) = 0$

Then $\lim_{z\to a} f(z)$ exists a complex no & hence 'a' is removable singularity.

Proof

$$\frac{g(z)-g(a)}{z-a} = \frac{\boxed{(z-a)^2 f(z)} - 0}{z-a}$$

$g(z)$

$$= (z-a)f(z) \to 0 \quad \text{as } z \to a$$

$$g'(a) = \lim_{z\to a} \frac{g(z)-g(a)}{z-a}$$

$f(z) = \frac{\sin z}{z}$

$\lim_{z\to 0} z \cdot f(z) = \lim \cancel{z} \cdot \frac{\sin z}{\cancel{z}} = 0$

$g(z) = \frac{\cos z}{z}$

$\lim_{z\to 0} z \cdot g(z) = 1 \neq 0$

Define $g(z) = \begin{cases} (z-a)^2 f(z) & , z \neq a \\ 0 & , z = a \end{cases}$

Then g is diff at a & $g'(a) = 0$

The Taylor's series exp g around a

$$g(z) = \sum_{n=0}^{\infty} \frac{g^{(n)}(a)}{n!}(z-a)^n = g(a) + g'(a)(z-a) + \frac{g''(a)(z-a)^2}{2!} + \cdots$$

$$= 0 + 0 + \frac{g''(a)}{2!}(z-a)^2 + \cdots$$

$$= (z-a)^2 h(z) \quad \text{where } h \text{ is holom at } a$$

$\therefore (z-a)^2 f(z) = (z-a)^2 h(z) \quad \forall z \neq a$

$\therefore f(z) = h(z), \ \forall z, z \neq a$

$\lim_{z\to a} h(z)$ exists

$\therefore \lim_{z\to a} f(z)$ exists

Cor: Let $z=a$ be a iso sing of f. Suppose

(i) f is bdd in a punctured disk around a

or

(ii) f is conts at a

Then a is a rem sing of f.

(i) Suppose f is bdd on $B'(a,r)$ for some $r>0$.

claim: $\lim_{z \to a} (z-a)f(z) = 0$ $\quad \forall z \in B'(a,r) \quad |f(z)| < M$

T.P.T Given $\epsilon > 0$. $\exists \delta > 0$ s.t $\forall z \in B(a,\delta)$

$$|(z-a)f(z)| < \epsilon$$

$$|(z-a)f(z)| = |z-a|\,|f(z)|$$

$$\le M|z-a|$$

$$= M\delta < \epsilon$$

$\boxed{\delta < \frac{\epsilon}{M}}$

$\delta = \min\{\frac{\epsilon}{M}, r\}$

$\lim_{z \to a} (z-a)f(z) = 0$

$(z-a)|f(z)| \le |z-a| M \to 0$

$\to 0$.

Landau Notation (big Oh-notat)

$f,g : B'(a,r) \to \mathbb{C}$

$f = O(g)$ as $z \to a$

means $\exists C > 0$ & $\exists \delta > 0$ $(\forall z \in B'(a,\delta))$

$$|f(z)| \le C\,|g(z)|$$

$$f = O\left(|z-a|^{-1/2}\right)$$

$$|f(z)| \le C\,|z-a|^{-1/2}$$

$$\le \frac{C}{|z-a|^{1/2}}$$

$f(z) = \frac{1}{z} + \frac{1}{z^2} + 1$

$f(z) = O\left(\frac{1}{z^2}\right) = O(z^{-2})$

$f(z) = z^2 + z^3 + z^4$

$g(z) = z^2$

$z \ne 0 \quad \frac{f(z)}{g(z)} = \frac{z^2+z^3+z^4}{z} = z + z^2 + z^3$

$1 + z + z^2$

$f(z) = \frac{1}{(z-a)^m} + \cdots + \frac{1}{(z-a)}$

$f = O\left((z-a)^{-m}\right) \le C$

Complex Analysis Nov 10, 2020

Lemma Assume $f \in H(B'(a,R))$, f has a pole of order m at a, $g(z) = (z-a)^m f(z)$ in $B'(a,r)$

Then $\forall\, \eta \in B'(a,R)$, $f(z) = \dfrac{a_{-m}}{(z-a)^m} + \dfrac{a_{-m+1}}{(z-a)^{m-1}} + \cdots + \dfrac{a_{-1}}{(z-a)} + h(z)$

where $h \in H(B(a,R))$, $a_k = \dfrac{g^{(m+k)}(a)}{(m+k)!} = \displaystyle\int_\gamma \frac{f(w)}{(w-a)^{k+1}}\,dw$

where $\gamma = S(a,r)$, $0 < r < R$.

Proof By P.T $\exists\, \phi \in H(B(a,R))$ s.t. $\forall\, \eta \in B'(a,R)$, $f(z) = \dfrac{\phi(z)}{(z-a)^m}$

$\forall\, z \in B'(a,R)$, $\phi(z) = (z-a)^m f(z)$

$\therefore$ on $B'(a,R)$, $\phi = g$.

$\therefore$ g has a rem sing at a

By redef g at a

we get g is holomorphic around a

$$g(z) = \sum_{k=0}^{m-1} \frac{g^{(k)}(a)(z-a)^k}{k!} + (z-a)^m h(z)$$

$$\therefore\quad f(z) = \frac{g(z)}{(z-a)^m} = \frac{g(a)}{(z-a)^m} + \frac{g'(a)}{2!\,(z-a)^{m-1}} + \quad + \frac{g^{(m-1)}(a)}{(m-1)!\,(z-a)} + h(z)$$

$$g(z) = \sum a_k (z-a)^k$$

$$\frac{g^{(m+k)}(a)}{(m+k)!} = \frac{1}{2\pi i}\int_\gamma \frac{g(w)}{(w-a)^{m+k+1}}\,dw$$

$$= \frac{1}{2\pi i}\int_\gamma \frac{(w-a)^m f(w)}{(w-a)^{m+k+1}}\,dw \quad \checkmark$$

$$= \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{(w-a)^{k+1}}\,dw$$

Thm Let $z=a$ be a isol sing of f.

Then f has pole of order m at $z=a$ iff $\exists\, C_1, C_2 > 0$ $\exists\, r>0$ s.t

$\forall z \in B'(a,r)$,

$$\frac{C_1}{|z-a|^{m_1}} \le |f(z)| \le \frac{C_2}{|z-a|^{m_2}}$$

Proof

f has a pole of order $m \Rightarrow \exists\, R>0\ \exists\, g \in H(B(a,R))$ st $\forall z \in B'(a,R)$, $f(z) = \dfrac{g(z)}{(z-a)^m}$ & $g(a) \neq 0$

Find $r \in (0,R)$ s.t $g \in H(B[a,r])$ & $\forall z \in B[a,r]$, $g(z) \neq 0$

$$z \in B'(a,r) \begin{cases} (z-a)^m f(z) = g(z) \\ |z-a|^m |f(z)| = |g(z)| \end{cases}$$

$C_2 = \text{lub}\{|g(z)| : z \in B[a,r]\}$

$C_1 = \text{glb}\{|g(z)| : z \in B[a,r]\}$

$r \neq g(a)$ $\quad 0 < |g(a)| \leq C_2$

$\therefore C_2 > 0$

$\forall z \in B[a,r]$ $\quad |z-a|^m |f(z)| \leq C_2$

$$|f(z)| \leq \frac{C_2}{|z-a|^m}$$

$\exists z_0 \in B[a,r]$

$C_1 = |g(z_0)| > 0$

$\forall z \in B'[a,r]$ $\quad \dfrac{C_1}{|z-a|^m} \leq |f(z)|$

$\Leftarrow$ $\forall z \in B'(a,r)$ $\quad |f(z)||z-a|^m \leq C_2$

$\therefore g(z) = (z-a)^m f(z)$ has removable singu at $z=a$

$$|(z-a)^m f(z)| \geq C_1$$

$$|(z-a)^{m-1} f(z)| \geq \frac{C_1}{|z-a|}$$

$$\parallel$$

$$|h(z)| \geq \frac{C_1}{|z-a|}$$

$$\lim_{z\to a} |h(z)| \geq \lim_{z\to a} \frac{C_1}{|z-a|} = \infty.$$

$$\therefore \lim_{z\to a} h(z) = \infty$$

$h(z)$ has pole at $z=a$

Claim $g(a) \neq 0$

$$(z-a)h(z) = (z-a)(z-a)^{m-1} f(z)$$
$$= (z-a)^m f(z)$$
$$= g(z)$$

$$\lim_{z \to a} (z-a)h(z) = \lim_{z\to a} g(z) = g(a).$$

If $g(a) = 0$, h has rem sing at a
$\Rightarrow \Leftarrow$ h has a pole at a

$\therefore$ $g(a) \neq 0$. $g \in H(B(a,r))$

$\therefore$ $z \in B'(a,r)$, $f(z) = \dfrac{g(z)}{(z-a)^m}$, $g(a) \neq 0$

1. Prove the f has a pole of order m to $z=a$ iff $\frac{1}{f}$ has zero of order m to $z=a$

2. P.T f has a pole of order m at a iff $\lim_{z\to a} (z-a)^{m+1} f(z) = 0$
$\lim_{z\to a} (z-a)^m f(z) \neq 0$

$a_0 = \frac{1}{g(a)} \neq 0$

$h = \frac{1}{g}$

$h(z) = \sum_k a_k (z-a)^k$

$\frac{1}{f}(a) = (z-a)^m \left[a_0 + a_1(z-a) + \cdots \right]$
$= (z-a)^m \phi(z)$, $\phi(a) = a_0 \neq 0$

Complex Analysis, Nov 11, 2020.

Defn (Meromorphic fn)

A fn which is holomorphic in a region except for poles is said to be meromorphic.

Every ration fn is meromorphic

A fn $f(z) = \frac{P(z)}{Q(z)}$ P & Q are polynomials

$f(z) = \frac{z}{z^2+2}$ $\quad z = \pm\sqrt{2}i$

$= \frac{z}{(z+\sqrt{2}i)(z-\sqrt{2}i)} = \frac{\phi(z)}{z-\sqrt{2}i}$

$\lim_{z\to\sqrt{2}i} f(z) = \phi(\sqrt{2}i)\cdot\infty = \infty$

$f(z) = \frac{\sin z}{\cos z}$

$\cos z = 0$

$z = \pm n\pi$

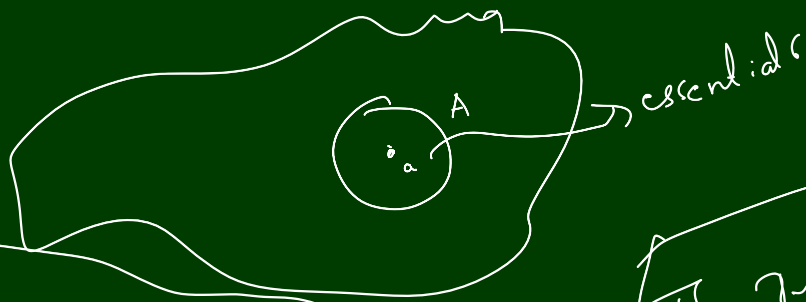
$f(z) = \exp(1/z)$ $\quad z=0$ is an essential singl.

Thm (Casorati – Wiestress Thm) Let U be a region $f: U \to \mathbb{C}$

Let a be essential Sing of f in U

Then $\forall\, r>0$ s.t $B(a,r) \subseteq U$, $f(B'(a,r))$ is dense in $\mathbb{C}$

$f(A)$ is dense in $\mathbb{C}$



[Thm (Big Picard's Thm)

$\forall\, r>0$, s.t $B(a,r) \subseteq U$ $\quad f(B'(a,r))$ misses at most one point of $\mathbb{C}$]

The proof requires addit. results done in 2nd ...

Proof: Suppose not.

$\exists r>0$, with $B(a,r)\subset U$, $\exists\, \alpha\in\mathbb{C}$ & $\epsilon>0$

s.t $B(\alpha,\epsilon)\cap f(B'(a,r))=\phi$.

i.e $\forall\, z\in B'(a,r)$, $|f(z)-\alpha|\geq\epsilon$

$$g: B'(a,r)\to\mathbb{C}$$

$$g(z)=\frac{1}{f(z)-\alpha}$$

$\forall z\in B', z\neq a$ $\quad |g(z)|=\dfrac{1}{|f(z)-\alpha|}<\dfrac{1}{\epsilon}$

$\therefore$ g is bdd in $B'(a,r)$

$\therefore$ g has a removable sing at $z=a$

$\lim\limits_{z\to a} g(z)$ exists say λ

$\therefore \lim\limits_{z\to a} g(z)=\lambda$

$\lim\limits_{z\to a}\dfrac{1}{f(z)-\alpha}=\lambda$

Case(i) $\lambda\neq 0$

$\therefore \lim\limits_{z\to a} f(z)-\alpha=\dfrac{1}{\lambda}$

$\therefore \lim\limits_{z\to a} f(z)=\alpha+\dfrac{1}{\lambda}$

$\therefore \lim\limits_{z\to a} f(z)$ exists $\Rightarrow\Leftarrow$ a is essential

Case (ii) $\lambda=0$

i.e $\lim\limits_{z\to a}\dfrac{1}{f(z)-\alpha}=0$ Then $\lim\limits_{z\to a}|f(z)-\alpha|=\infty$

$\therefore \lim\limits_{z\to a}|f(z)|=\infty$

$\therefore z=a$ is pole $\Rightarrow\Leftarrow$

Cor: f is non const entire fn.

f is not a poly iff $g(z) := f(1/z)$, $z \in \mathbb{C}^*$ has a essential sing at $z = 0$

Proof

Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$

Suppose 0 is not an essential singularity for g

$\exists N \in \mathbb{N}$ s.t $\forall n \geq N$, $z^n g(z)$ has removable singularity

$z^n f(1/z)$ has " " "

$$0 = \int_{S(0,1)} w^n f(1/w)\, dw = \int_{S(0,1)} w^n \cdot \sum_k a_k \frac{1}{w^k}\, dw$$

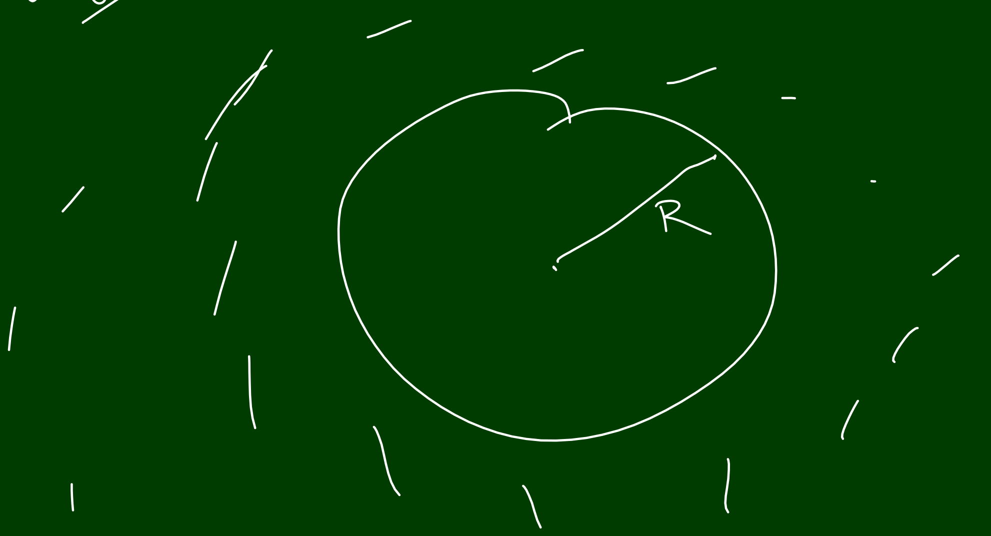
$$= \sum_k a_k \int_{S(0,1)} \frac{1}{w^{k-n}}\, dw \qquad k = n+1$$

$$= 2\pi i\, a_{n+1}$$

$\therefore$ f is poly of deg. at most n

Ex! 1. $f : \mathbb{C} \to \mathbb{C}$ non const entire s.t. $f(\mathbb{C})$ is dense in $\mathbb{C}$

2. ~~Let f be~~ entire & not a poly. Then $\forall R > 0$, $f(\mathbb{C} \setminus B(0,R))$ is dense



Complex Analysis, Nov 12, 2020

$\sum_{n=-\infty}^{\infty} a_n$ is convergent if

$S_0 = a_0$

$S_1 = a_{-1} + a_0 + a_1$

Wrong under.

~~$S_n = \sum_{k=-n}^{n} a_k$~~

$S_{m,n} = \sum_{k=-m}^{n} a_k$

$\exists s \in \mathbb{C}$ $\forall \epsilon > 0$ $\exists N \in \mathbb{N}$ $\forall m, n \geq N$ $|S_{mn} - s| < \epsilon$

Def $\sum_{n=-\infty}^{\infty} a_n$ is said to be convergent if $\exists s \in \mathbb{C}$

$\forall \epsilon > 0 \, (\exists N \in \mathbb{N} \, (\forall m, n \geq N$

$|(a_{-m} + a_{-m+1} + \cdots + a_{-1} + a_0 + \cdots + a_n) - s| < \epsilon))$

Let $b_n = a_{-n}$

$\sum_{n=-\infty}^{\infty} a_n$ is convergent to s iff $\sum_{n=1}^{\infty} b_n$ & $\sum_{n=0}^{\infty} a_n$ are convergent to the same s_1 & s_2 & $s = s_1 + s_2$.

$\Rightarrow$ Let $t_m = \sum_{n=1}^{m} b_n$ $\quad s_n = \sum_{k=0}^{n} a_n$

Given $\epsilon > 0$, $\exists N \in \mathbb{N}$ $\forall n, m \geq N$

$|t_m + s_n - s| < \frac{\epsilon}{2}$ ——— (1)

$|t_p - t_q|$

$= |t_p + s_n - s_n + s - s - t_q|$

$= |(t_p + s_n - s) - (t_q + s_n - s)|$

$\leq |t_p + s_n - s| + |t_q + s_n - s|$

choose $p, q \geq N$: $n \geq N$

$< \quad \epsilon/_2 + \epsilon/_2 = \epsilon$

but

$\therefore \quad t_n \to s_1 \quad \& \quad s_n \to s_2$

$t_n + s_n \to s_1 + s_2$

$\Downarrow \qquad ''$

$s \qquad \checkmark$

$n \geq N_1$

$n \geq N_2$

$N = \max\{N_1, N_2\}$.

$$|t_n + s_n - s| = |t_n - s_1 + s_n - s_2|$$
$$\leq |t_n - s_1| + |s_n - s_2|$$
$$< \epsilon/_2 + \epsilon/_2$$

Give an example to say that $\sum_{n=-\infty}^{\infty} a_n$ is convergent

is not same as $\underline{a_{-n} + \cdots + a_0 + \cdots + a_n} \to s$.

$-1 + 1 + 1 + 1 + 1$

$$a_n = \begin{cases} 1 & n \geq 0 \\ -1 & n < 0 \end{cases}$$

$a_{-n} + \cdots + a_n = 1$

Define

$\sum_{n=-\infty}^{\infty} f_n(z)$ is said converge on U

Complex Analysis, Nov 16, 2020

$\sum_{n=-\infty}^{\infty} a_n \to s$ iff $\sum_{n=1}^{\infty} b_n \to s_1$ & $\sum_{n=0}^{\infty} a_n \to s_2$, $b_n = a_{-n}$ & $s = s_1 + s_2$

Lemma

Let $R > 0$ be radius of conv of $f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$.

Then $\left(g(z) =\right) \sum_{n=0}^{\infty} c_n (z-a)^{-n}$ is convergent for $|z-a| > 1/R$

In fact in the set $\{ z : |z-a| \geq r \}$ where $r > 1/R$

the convergence is uniform

Also $g'(z) = -\sum_{n=1}^{\infty} n c_n (z-a)^{-n-1}$ $\forall z$ s.t. $|z-a| > 1/R$

Proof:

$g(z) := f\left(\frac{1}{z-a}\right)$ for $|z-a| > 1/R$.

Given $\epsilon > 0$, & for $0 < \rho < R$

By the unif conv of $f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$, $\exists N \in \mathbb{N}$

s.t. $\forall n \geq N$, $\forall z \in B(a, \rho)$, $\left| f(z) - \sum_{k=0}^{n} c_n (z-a)^k \right| < \epsilon$

$\forall n \geq N$, $|z-a| < \rho$, $\frac{1}{|z-a|} > \frac{1}{\rho}$, $\left| g(z) - \sum_{k=0}^{n} c_n (z-a)^{-k} \right| < \epsilon$ (!)

$f(z) = c_0 + c_1 (z-a) + c_2 (z-a)^2 + \cdots$

$g(z) = c_0 + \frac{c_1}{z-a} + \frac{c_2}{(z-a)^2} + \cdots$

$\left| f(z) - \sum_{k=0}^{n} c_k (z-a)^k \right| < \epsilon$ for $|z-a| < \rho$

$\frac{1}{|z-a|} > \frac{1}{\rho}$

$$\left|\sum_{k=n+1}^{\infty} c_k (z-a)^k\right| < \epsilon \quad \text{for} \quad |z-a| < \rho$$

$$\left|\sum_{k=n+1}^{\infty} c_k w^{-k}\right| < \epsilon \qquad |z-a| > \frac{1}{\rho}$$

$$\text{i.e.} \quad \left|\sum_{k=n+1}^{\infty} c_k (z-a)^{-k}\right| < \epsilon \qquad |w| > \frac{1}{\rho}, \quad \frac{1}{|w|} < \rho$$

$$\text{i.e.} \quad \left|g(z) - \sum_{k=0}^{n} c_k (z-a)^{-k}\right| < \epsilon \quad \forall\, z \text{ s.t. } |z-a| \geq 1/\rho$$

$$f'(z) = \sum_{n=1}^{\infty} n c_n (z-a)^{n-1} \qquad g(z) = f\left(\frac{1}{z-a}\right)$$

By chain rule

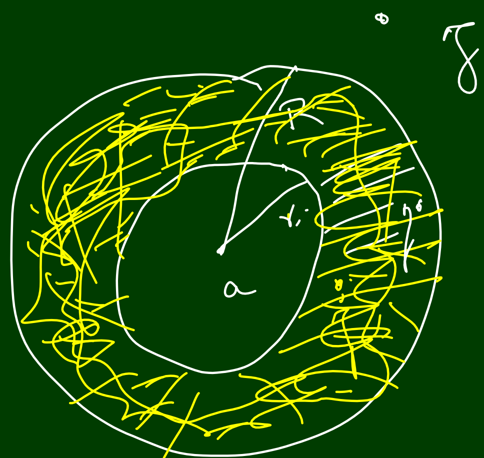
$$\therefore g'(z) = f'\left(\frac{1}{z-a}\right)\frac{(-1)}{(z-a)^2}$$

$$= -\sum_{n=1}^{\infty} n c_n \frac{1}{(z-a)^{n+1}}$$

Defn: Let $0 < r < R \leq \infty$

$$A(a; r, R) := \{ z \in \mathbb{C} : r < |z-a| < R \}$$

is called annulus



$|z-a| >$

If $R = \infty$

~~then~~ $A(a; r, R) = \mathbb{C} \setminus B[a, r]$

Lemma: Let $F : B'(a, R) \to \mathbb{C}$ be holomorphic. Let $0 < r < R$

Let $\gamma_r = a + re^{it}$, $0 \leq t \leq 2\pi$.

Then the fn $r \xrightarrow{\phi} \int_{\gamma_r} F$ is constant.

In particular $0 < r, s < R$

$\phi : (0, R)$

$\phi(r) = \phi(s)$

$$\int_{\gamma_r} F = \int_{\gamma_s} F \quad \checkmark$$

Proof: $\phi : (0, R) \longrightarrow \mathbb{C}$ by $\gamma_r = a + re^{it}, \quad 0 \le t \le 2\pi$

$$\phi(r) = \int_{\gamma_r} F(w)\, dw$$

EST $\phi' = 0$. (Hint Use Leibnitz rule of diff)

$$\phi(r) = \int_0^{2\pi} F(\gamma_r)\, \gamma_r'\, dt$$

$$= \int_0^{2\pi} F(\gamma_r)\, r \cdot i e^{it}\, dt$$

$\phi'(r) =$ ~~~ . Review by Tom completed it.

Complex Analysis, Nov 17, 2020.

$F : B'(a,R) \to \mathbb{C}$ is holomorphic.

Let $0 < r < R$, $\gamma_r = a + re^{it}$, $0 \le t \le 2\pi$

$\phi : (0, R) \to \mathbb{C}$

$$\phi(r) = \int_{\gamma_r} F(w)\,dw \text{ is constant.}$$

Claim $\phi' = 0$

$$\gamma_r'(t) = rie^{it}$$

$$\phi(r) = \int_0^{2\pi} F(\gamma_r)\cdot \gamma_r' \, dt$$

$$\phi(r) = \int_0^{2\pi} \left[F(a+re^{it})\, rie^{it}\right] dt$$

$$\phi'(r) = \int_0^{2\pi} \frac{\partial\left[F(a+re^{it})\, rie^{it}\right]}{\partial r}\, dt$$

$$= \int_0^{2\pi} \left[F'(a+re^{it})\, e^{it}\, ire^{it} + F(a+re^{it})\, ie^{it}\right] dt$$

$$= \int_0^{2\pi} \left[F'(a+re^{it})\, ire^{2it} + F(a+re^{it})\, ie^{it}\right] dt \quad \text{--- (1)}$$

$$\frac{\partial}{\partial t}\left[F(a+re^{it})\cdot e^{it}\right] = \underline{F'}(a+re^{it})\, rie^{2it} + F(a+re^{it})\, ie^{it}$$

$$= \int_0^{2\pi} \frac{\partial}{\partial t}\left[F(a+re^{it})\cdot e^{it}\right] dt$$

$$= F(a+re^{i2\pi})\cdot e^{i2\pi} - F(a+r)$$

$$= F(a+r) - F(a+r) = 0$$

Lemma Cauchy Integral formula for annulus.

Let $f \in H(B'(a,R))$.

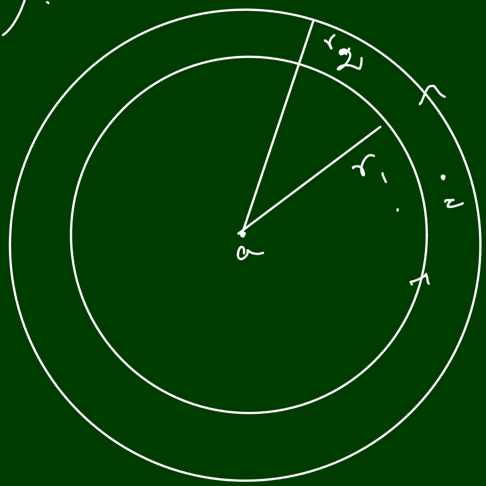
& $z \in B'(a,R)$.

Choose r_1 & r_2 s.t

$0 < r_1 < |z-a| < r_2 < R$

For $j = 1,2$

$\gamma_j = a + r_j e^{it} \quad 0 \le t \le 2\pi$



Then for $r_1 < |z-a| < r_2$

$$f(z) = \frac{1}{2\pi i}\int_{\gamma_2} \frac{f(\omega)}{\omega - z}\,d\omega - \frac{1}{2\pi i}\int_{\gamma_1} \frac{f(\omega)}{\omega - z}\,d\omega$$

// Define

$$F(\omega) = \begin{cases} \dfrac{f(\omega)-f(z)}{\omega - z} & \omega \neq z,\ \omega \in B'(a,R) \\ f'(z), & \omega = z \end{cases}$$

Note that $F : B'(a,R) \to \mathbb{C}$

F has rem sing at z by Riemann's thm. (or Morera's thm).

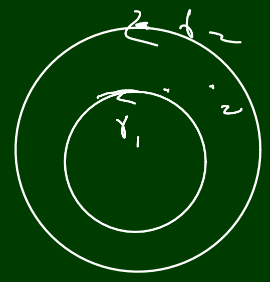
$\therefore F \in H(B'(a,R))$

Apply previous lemma to $\frac{1}{2\pi i} F$

$$\int_{\gamma_2} \frac{1}{2\pi i} F = \int_{\gamma_1} \frac{1}{2\pi i} F$$

$$\frac{1}{2\pi i}\int_{\gamma_2} F = \frac{1}{2\pi i}\int_{\gamma_1} F \qquad \text{———} (*)$$

Since $z \notin \{\gamma_j\}$



$$\frac{1}{2\pi i}\int_{\gamma_j} F = \frac{1}{2\pi i}\int_{\gamma_j}\frac{f(\omega)-f(z)}{\omega - z}\,d\omega$$

$$= \frac{1}{2\pi i}\int_{\gamma_j}\frac{f(\omega)}{\omega-z}\,d\omega - \frac{f(z)}{2\pi i}\int_{\gamma_j}\frac{d\omega}{\omega-z}$$

$\frac{1}{2\pi i}\int_{\gamma_1}\frac{d\omega}{\omega-z}=0$, $\frac{1}{2\pi i}\int_{\gamma_2}\frac{d\omega}{\omega-z}=1$ (using this in $(*)$)

$$\frac{1}{2\pi i}\int_{\gamma_2}\frac{f(\omega)}{\omega-z}\,d\omega - f(z) = \frac{1}{2\pi i}\int_{\gamma_1}\frac{f(\omega)}{\omega-z}\,d\omega - 0$$

$$f(z) = \frac{1}{2\pi i}\int_{\gamma_2}\frac{f(\omega)}{\omega-z}\,d\omega - \frac{1}{2\pi i}\int_{\gamma_1}\frac{f(\omega)}{\omega-z}\,d\omega$$

Laurent's Series

Thm Let f be Holomorphic in the annulus $A(a;r,R)$.

Then f is given by

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n, \quad z \in A(a;r,R).$$

& $$a_n = \int_{\gamma}\frac{f(\omega)}{(\omega-a)^{n+1}}\,d\omega$$ where $\gamma = \gamma_\rho$ For $r<\rho<R$.

$\gamma_\rho = a + \rho e^{it}$, $0 \le t \le 2\pi$

// Further the Series conv unif & absolutely on the Compact Subsets of $A(a;r,R)$.

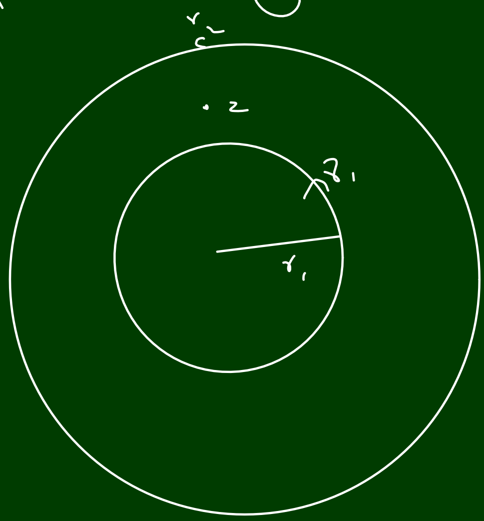
Proof Choose r_1 & r_2 s.t $r < r_1 < |z-a| < r_2 < R$.

By previous Lemma,

$$f(z) = \frac{1}{2\pi i}\int_{\gamma_2} \frac{f(w)}{w-z}\,dw - \frac{1}{2\pi i}\int_{\gamma_1} \frac{f(w)}{w-z}\,dw$$

$$= f_2 + f_1 \quad \text{—— (1)}$$

For $w \in \{\gamma_2\}$



$$\frac{1}{w-z} = \frac{1}{w-a+a-z}$$

$$= \frac{1}{(w-a)\left[1 - \frac{(z-a)}{(w-a)}\right]}$$

$$= \frac{1}{w-a}\left[1 - \frac{z-a}{w-a}\right]^{-1}$$

$$= \frac{1}{w-a}\left[\sum_{n=0}^{\infty}\left(\frac{z-a}{w-a}\right)^n\right]$$

$$= \sum_{n=0}^{\infty} \frac{(z-a)^n}{(w-a)^{n+1}}$$

$$\frac{1}{w-z} = \frac{1}{w-a+a-z}$$

$r_1 < |z-a| < r_2 = |w-a|$

$$= \frac{-1}{(z-a)\left[1 - \frac{w-a}{z-a}\right]}$$

$|w-a| = r_1 < |z-a|$

$$= \frac{-1}{z-a} \sum_{n=0}^{\infty} \frac{(w-a)^n}{(z-a)^n}$$

$$= -\sum_{n=0}^{\infty} \frac{(w-a)^n}{(z-a)^{n+1}}$$

$$f(z) = \frac{1}{2\pi i}\int_{\gamma_2} \frac{f(\omega)}{\omega - z}\, d\omega \quad - \frac{1}{2\pi i}\int_{\gamma_1} \frac{f(\omega)}{\omega - z}\, d\omega$$

$$= \frac{1}{2\pi i}\int_{\gamma_2} f(\omega) \sum_{n=0}^{\infty} \frac{(z-a)^n}{(\omega-a)^{n+1}}\, d\omega + \frac{1}{2\pi i}\int_{\gamma_1} f(\omega) \cdot \sum_{n=0}^{\infty} \frac{(\omega-a)^n}{(z-a)^{n+1}}\, d\omega$$

$$= \sum_{n=0}^{\infty} \frac{1}{2\pi i}\int_{\gamma_2} \frac{f(\omega)\, d\omega}{(\omega-a)^{n+1}} (z-a)^n + \sum_{K=0}^{\infty} \left[\frac{1}{2\pi i}\int_{\gamma_1} f(\omega) \cdot (\omega-a)^K d\omega\right] \cdot \frac{1}{(z-a)^{K+1}}$$

$n = -K - 1$

$$= \quad " \quad + \sum_{-1}^{-\infty} \boxed{\frac{1}{2\pi i}\int_{\gamma_1} \frac{f(\omega)\, d\omega}{(\omega-a)^{n+1}}} \cdot (z-a)^n$$

$$f(z) = \sum_n c_n (z-a)^n$$ by Lemma 1 where $r < \rho < R$.

where $c_n = \boxed{\frac{1}{2\pi i}\int_{\gamma_\rho} \frac{f(\omega)\, d\omega}{(\omega-a)^{n+1}}}$ (*)

Uniqueness Let f be holo in $A(a; r, R)$.

Let $\sum_{n=-\infty}^{\infty} a_n (z-a)^n \rightrightarrows f$ on compact subsets $A(a; r, R)$

Then $a_n = \boxed{}$ (*)

Obtain Laurent series exp f at 0

$$f(z) = \frac{e^z}{z}$$

$$e^z = 1 + z + \frac{z^2}{2!} \cdots \quad \sim \mathring{B}(0,1)$$

$$\boxed{f(z) = \frac{e^z}{-} = \frac{1}{z} + 1 + \frac{z}{2!} + \cdots}$$

$$c_n = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w^{n+1}}\, dw$$

$$= \frac{1}{2\pi i} \int_{\gamma} \frac{e^{w}}{w^{n+2}}\, dw$$

$n \geq 0$ $\gamma(t) = e^{it}$

$$\frac{(n+1)!}{2\pi i} \int_{\gamma} \frac{g(w)}{w^{n+2}}\, dw = g^{(n+1)}(0)$$

$$c_n = \frac{1}{(n+1)!}$$

$n = -1$

$$c_{-1} = 1$$

$$c_{-2} = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w^{-2+1}}\, dw$$

$$= \frac{1}{2\pi i} \int_{\gamma} w f(w)\, dw$$

$n \leq -2$

$c_n = 0$

Complex Analysis, 18 Nov 2020

Prob Laurent's Series

1) $f(z) = \dfrac{1}{z(1-z)}$, $z \in \mathbb{C} \setminus \{0, 1\}$

Find L.S ex for f at $z = 0$.

$$-A + B = 0$$
$$A = 1$$

$$\frac{1}{z(1-z)} = \frac{A}{z} + \frac{B}{1-z}$$
$$= \frac{A(1-z) + Bz}{z(1-z)}$$

$$\frac{1}{z(1-z)} = \frac{1}{z} + \frac{1}{1-z}$$
$$= \frac{1}{z} + (1-z)^{-1}$$
$$= \frac{1}{z} + \sum_{n=0}^{\infty} z^n \qquad 0 < |z| < 1$$

Alternatively

$$\gamma(t) = \frac{e^{it}}{2} \qquad 0 \le t \le 2\pi$$

$C_n, n \geq -1$

$$c_n = \frac{1}{2\pi i} \int_\gamma \frac{dw}{w^{n+2}(1-w)}$$
$$= \frac{1}{2\pi i} \int_\gamma \frac{g(w)}{w^{n+2}} \, dw \quad , \; g(w) = (1-w)^{-1}$$
$$g^{n+1}(0) = (n+1)! \cdot 1$$
$$= \frac{g^{(n+1)}(0)}{(n+1)!}$$
$$= 1$$

1) $f(z+1) = \dfrac{(z+1)^2}{z}$, $|z| > 0$, at $z = 0$

2) $f(z) = \dfrac{1}{z^2(z-1)}$, at $z = 1$

3. $f(z) = \dfrac{1}{z(z-1)(z-2)}$

a) $0 < |z| < 1$

b) $1 < |z| < 2$

c) $|z| > 2$

1) $f(z) = \frac{1}{z} + z + z \quad , \quad |z| > 0$

2) $f(z) = \dfrac{-1}{z^2(1-z)}$

$= \dfrac{-1}{z^2}(1-z)^{-1}$

$= \dfrac{-1}{z^2}\left[1 + z + z^2 + \dots\right]$

$= -\left\{\dfrac{1}{z^2} + \dfrac{1}{z} + 1 + z + \dots\right.$

b) $f(z) = \dfrac{1}{z^2(z-1)}$

$= \dfrac{1}{(z-1)\left[(z-1)+1\right]^2}$

$= \dfrac{-1}{1-z}\left[1-(1-z)\right]^{-2}$

$= \dfrac{-1}{1-z}\left[1 + 2(1-z) + 3(1-z)^2 + \dots\right]$

$= -\left\{\dfrac{1}{1-z} + 2 + 3(1-z) + \dots\right]$

$|1-z| < 1$

3) $f(z) = \dfrac{1}{2z} - \dfrac{1}{z-1} + \dfrac{1}{2(z-2)}$

$|z| < 1 < 2$

$0 < |z| < 1$

$f(z) = \dfrac{1}{2z} + \dfrac{1}{1-z} - \dfrac{1}{2(2-z)}$

$= \dfrac{1}{2z} + \sum_{n=0}^{\infty} z^n$

$$\frac{1}{2-z} = \frac{1}{2\left(1-\frac{z}{2}\right)}$$

$$= \frac{1}{2}\left(\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n\right.$$

$$\frac{1}{2}\left(\frac{1}{2-z}\right) = \frac{1}{2 \cdot 2}\left(\sum_{n=0}^{\infty}\frac{z^n}{2^n}\right)$$

$$= \sum_{n=0}^{\infty}\frac{z^n}{2^{n+2}}$$

$|z| < 1 < 2$

$|z| < 2$

$\frac{|z|}{2} < 1$

$$\therefore f(z) = \frac{1}{2z} + \sum_{n=0}^{\infty}\left(1-2^{-n-2}\right)z^n$$

$$f(z) = \frac{1}{2z} - \frac{1}{z-1} + \frac{1}{2(z-2)}$$

$$= \frac{-1}{z-1} + \frac{1}{2z} + \frac{1}{2(z-2)}$$

$$\frac{1}{2z} = \frac{1}{2}\cdot\frac{1}{z-1+1}$$

$$= \frac{1}{2}\cdot\frac{1}{1-(1-z)}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}(1-z)^n \quad = \frac{1}{2}\left(1+(1-z)+(1-z)^2+\cdots\right)$$

$$\frac{1}{2(z-2)} = \frac{1}{2}\cdot\frac{1}{(z-1-1)}$$

$0 < |z-1| < 1$

$$= \frac{-1}{2}\cdot\frac{1}{(1-(z-1))}$$

$$= \frac{-1}{2}\left(\frac{1}{1+(1-z)}\right)$$

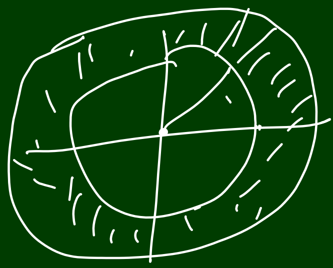
$$= \frac{-1}{2}\left(1-(1-z)+(1-z)^2-(1-z)^3+\cdots\right]$$

$$f(z) = \frac{1}{1-z} + (1-z) + (1-z)^3 + (1-z)^5.$$

Ans $f(z) = \frac{-1}{2z} - \sum_{n=2}^{\infty}\frac{1}{z^n} - \frac{1}{4}\sum_{n=0}^{\infty}2^{-n}z^n$

$1 < |z| < 2$

$$f(z) = \frac{1}{z^2} - \frac{1}{z-1} + \frac{1}{z(z-2)}$$



$1 < |z| < 2$

$|z| > 1$

$\frac{1}{|z|} < 1$

$$\frac{1}{z-1} = \frac{1}{z\left(1-\frac{1}{z}\right)}$$

$$= \frac{1}{z}\sum_{n=0}^{\infty}\frac{1}{z^n} = -\left(\frac{1}{z} + \frac{1}{z^2} + \dots\right)$$

$$= -\sum_{n=0}^{\infty}\frac{1}{z^{n+1}}$$

$$-\frac{1}{z-1}$$

$|z| < 2$

$\frac{|z|}{2} < 1$

$$\frac{1}{z(z-2)} = \frac{1}{z^2\left(\frac{z}{2}-1\right)}$$

$$= \frac{-1}{z^2\left(1-\frac{z}{2}\right)}$$

$$= \frac{-1}{z^2}\left(1-\frac{z}{2}\right)^{-1}$$

$$= \frac{-1}{z^2}\sum_{n=0}^{\infty}\frac{z^n}{2^n}$$

$$= -\sum_{n=0}^{\infty}\frac{z^n}{2^{n+2}}$$

$$= -\left(\frac{1}{2^2} + \frac{z}{2^3} + \frac{z^2}{2^4}\right)$$

$$f(z) = \frac{1}{z^2} = \left(\frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \dots\right) - \left(\frac{1}{2^2} + \frac{z}{2^3} + \frac{z^2}{2^4} + \dots\right)$$

$$= -\sum_{n=2}^{\infty}\frac{1}{z^n} \qquad -\frac{1}{2^2} - \frac{1}{2^2}\sum_{n=0}^{\infty}\frac{z^n}{2^n}$$

$$f(z) = \frac{1}{2z} - \frac{1}{z-1} + \frac{1}{2(z-2)}$$

$$2 < |z|$$

$$\frac{2}{|z|} < 1$$

$$-\frac{1}{z-1} = \frac{-1}{z\left(1-\frac{1}{z}\right)}$$

$$= -\sum_{n=0}^{\infty} \frac{1}{z^{n+1}}$$

$$\frac{1}{2(z-2)} = \frac{1}{2z\left(1-\frac{2}{z}\right)}$$

$$= \frac{1}{2z} \sum_{n=0}^{\infty} \left(\frac{2}{z}\right)^n$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} \frac{2^n}{z^{n+1}}$$

$$f(z) = \frac{1}{2z} - \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} + \frac{1}{2} \sum_{n=0}^{\infty} \frac{2^n}{z^{n+1}}$$

$$= \frac{1}{2z} - \left(\frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4}\right) + \frac{1}{2}\left(\frac{1}{z} + \frac{2}{z^2} + \frac{2^2}{z^3} + \cdots\right)$$

$$= \frac{(2-1)}{z^3} + \frac{(2^2-1)}{z^4} + \frac{(2^3-1)}{z^5}$$

$$= \sum_{n=1}^{\infty} \frac{(2^n-1)}{z^{n+2}}$$

Complex Analysis, Nov 20, 2020.

Classification of Singularities using Laurent's Series

Thm $f \in H(B'(a,R))$. Let $c_j, j \in \mathbb{Z}$ be the coeff of Laurent's Series of f around a

Then

(i) f has a removable singular at a iff $\forall j < 0$, $c_j = 0$

i.e. f has no principal part

(ii) f has a pole at a iff $\exists N \in \mathbb{N}$ s.t. $\forall j > N$, $c_{-j} = 0$ & $c_{-N} \neq 0$.

(iii) f has a essential sing at a iff $\{j \in \mathbb{N} : c_{-j} \neq 0\}$ is infinite

i.e. $c_{-j} \neq 0$ for infinitely many $j \in \mathbb{N}$.

Proof

(i) $\Rightarrow$

f has a removable sing at a $\Rightarrow$ $\exists\, (a_n)$ in $\mathbb{C}$ & $\exists r \in (0,R)$, $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$, $\forall z \in B'(a,r)$

Then $f \in H(B(a,R))$, $\Rightarrow$ $\forall$, $B'(a,r)$

Then Laurent Series exp of f at a in $B'(a,R)$ is taylor series of f at a (why?) $\boxed{a_n = c_n}$

$\therefore c_{-j} = 0$ $\forall j \in \mathbb{N}$

$\Leftarrow$

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n, \quad z \in B'(a,R)$$

Define $f(a) := c_0$ $\quad f \in H(B(a,R))$

Then $\lim_{z \to a} f(z) = f(a) = c_0$

$\therefore$ f has a removable sing at a.

(ii) $\Rightarrow$ f has a pole at a say with order m

Then $g(z) = (z-a)^m f(z)$

Then g has a rem sing at a

Give the proof of the same lemma done during this singularity

$$f(z) = \frac{a_{-m}}{(z-a)^m} + \cdots + \frac{a_{-1}}{z-a} + h(z).$$

$\Leftarrow$ $g(z) = (z-a)^N f(z)$. $g(z)$ is bdd in disk $B(a,r)$

Then $\lim_{z\to a} g(z)$ is ...

& $\lim_{z\to a} (z-a)^{N-1} f(z)$ is not ...

$\therefore$ f has pole of order N.

Meromorphic fn:

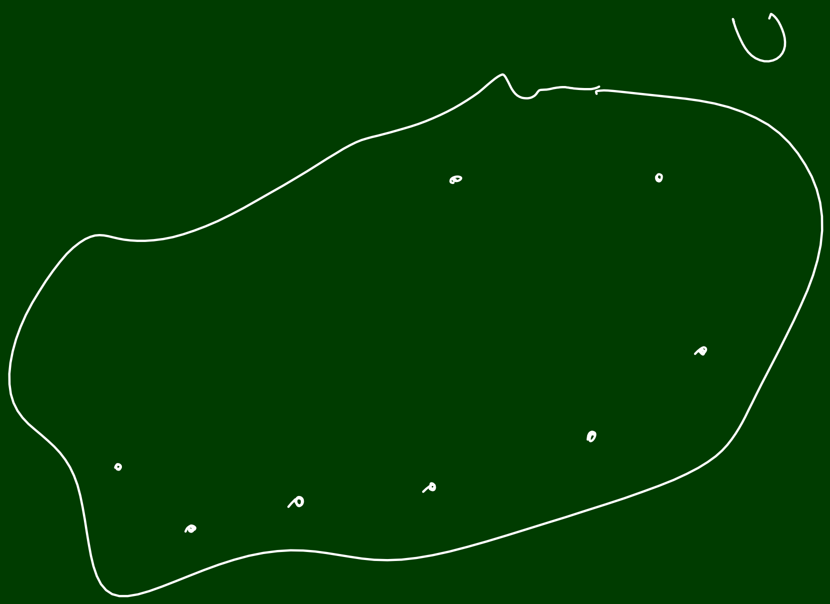
Recall defn.

Let $U \subseteq \mathbb{C}$ be open

~~Def 1 $f: U \to \mathbb{C}$ is mero if f is holo except for isolated sing which are poles~~

Def 2 $U \subseteq \mathbb{C}$ open

$f: U \to \mathbb{C}$ is said to be meromorphic if $\exists$ open set $V \subset U$ s.t. f is holomor on V $\forall z \in U \setminus V$, z is a pole of f



Def 2 $\Rightarrow$ Def 1

Def 1 $\Rightarrow$ Def 2?

We take Def 2 to be the defn for me.

$M(U) = \{ f : U \to \mathbb{C} \; : \; f \text{ is meromorphic on } U \}$

<u>Lemma</u> $f \in M(U)$ & $K \subset U$ be compact, then set of poles of f in K is finite

i.e. $P_f \cap K$ is finite

Proof: Suppose $P_f \cap K$ is infinite

Produce a distinct sequence say (α_n) in $P_f \cap K$

$$\alpha_n \in P_f \cap K \subseteq K$$

$\therefore \exists \alpha \in \mathbb{R}$ & (α_{n_k}) of (α_n) s.t. $\alpha_{n_k} \to \alpha$.

Can $\alpha \in V$? suppose $\alpha \in V$

$\exists r > 0$ s.t. $B(\alpha, r) \subset V$

$\exists k_0 \in \mathbb{N}$ s.t. $\alpha_{n_{k_0}} \in B(\alpha, r) \subset V$

$\Rightarrow\Leftarrow$ $\alpha_{n_{k_0}}$ is a pole for f

Can $\alpha \in P_f$ $\quad$ $\alpha \in P_f$

α is isolated $\exists r > 0$, on $B'(\alpha, r)$ f is holomorphic

$\alpha_{n_k} \to \alpha$.

$\alpha_{n_k} \in B(\alpha, r)$.

$\exists N \quad \forall k \geq N, \quad \alpha_{n_k} = \alpha$ //

$P_f := \{ z \in U : z \text{ is a pole of } f \}$

P_f for f on U is countable $\quad$ P_f is countable.

$$P_f = \bigcup_n P_f^n \leadsto \text{finite}$$

countable

$$U_n = B(0,n) \cap U \qquad \subset U$$

$n \Rightarrow$ open.

$$K_n = B\left[0, \frac{n}{2}\right] \cap$$

compact

$$P_f \cap K_n \text{ is finite}$$

$\forall n \in \mathbb{N}$

$$P_f = \bigcup_{n \in \mathbb{N}} (P_f \cap K_n)$$

Complex Analysis, Nov 23, 2020

Ex: The set of poles P_f of a meromorphic fn on an open set U is countable.

$$P_f = \bigcup_n P_f \cap K_n$$

[$K \overset{cpt}{\subseteq} U$

$P_f \cap K$ is finite]

$K_n := \{ z \in U : |z| \le n \ \& \ d(z, \mathbb{C} \setminus U) \ge 1/n \}$

K_n cpt sets

$K_n \subset B[0,n]$ $K_n \subset K_{n+1}^{\circ}$ open set

$\boxed{\bigcup_n K_n = U}$

$P_f = \bigcup_n (P_f \cap K_n)$

Another proof:

Suppose P_f is uncountable subset of U

$\therefore$ P_f has a cluster point in U (?)

$\exists \alpha \in U$ s.t α is cluster point of P_f

$\forall r > 0$, $B(\alpha, r) \cap P_f$ is infinite

Let $r > 0$

$B[\alpha, r/2] \cap P_f$

$U \subseteq \mathbb{C}$

$B(\alpha, \frac{r}{4}) \subset B[\alpha, \frac{r}{2}]$

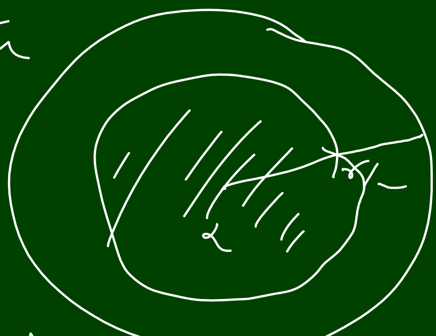
$K \subseteq U$ is cpt iff $K \subseteq \mathbb{C}$ is cpt

$$\underbrace{P_f \cap B(\alpha, r/4)}_{\text{infinite}} \subset P_f \cap \underbrace{B[\alpha, \tfrac{r}{2}]}_{K}$$

infinite

$A \subseteq Y \subseteq X$ then A is closed in X then A is closed Y

$B[\alpha, r/2] \cap U = B_U[\alpha, r/2]$



$\exists (z_n)$ distinct seq in P_f s.t

$z_n \to \alpha$ $\qquad\qquad P_f = U \setminus V$

Case(i)

$\alpha \in V$ $\quad \exists\, r > 0$ s.t $B(\alpha, r) \subset V$ & f is holo on $B(\alpha, r)$ —— ①

$\exists\, N \in \mathbb{N}$ s.t $\forall n \geq N$ $z_n \in B(\alpha, r)$

In fact $z_N \in B(\alpha, r)$

$\Rightarrow \Leftarrow$ to ① $\because z_N$ is a pole.

Case(ii)

$\alpha \in P_f$

$\alpha \in P_f$ then $\exists$ dist seq (z_n) in P_f

s.t $z_n \to \alpha$.

$\because \alpha \in P_f$ $\exists\, r > 0$ s.t on $B'(\alpha, r)$ f is holomorphic.

$\Rightarrow (z_n)$ is eventually constant

$\forall n \geq N$, $z_n \in B'(\alpha, r)$.

In fact $z_N \in B'(\alpha, r)$.

$\therefore z_N = \alpha$.

$\forall n \geq N$, $z_n = \alpha$

Any uncountable subset of $\mathbb{R}$ or $\mathbb{C}$ has a cluster point.

$$\mathbb{C} = \bigcup_{n\in\mathbb{N}} B[0,n]$$

$$A = \bigcup_{n\in\mathbb{N}} \left[B[0,n] \cap A\right] = \bigcup_{n\in\mathbb{N}} B[0,n] \cap A$$

$$= \bigcup_{n\in\mathbb{N}} A_n$$

$\exists\, N \in \mathbb{N}$ s.t. A_N is uncountable

$A_N \subset B[0,N] \rightsquigarrow$ cpt

$\rightsquigarrow$ infinite

$\therefore \exists\, \alpha \in \mathbb{C}$ s.t. α is cluster point A_N

$\Rightarrow \alpha$ is cluster point of A

$$A = \bigcup_{n\in\mathbb{N}} A_n.$$

$A_n = B[0,n] \cap A \subseteq A$

Let $z \in A$

T.P.: $\exists\, n \in \mathbb{N}$ s.t. $z \in B[0,n]$

By A.P. $\exists\, n \in \mathbb{N}$, $n \geq |z|$

$\therefore z \in B[0,n] \cap A = A_n$

$f \in M(U)$ Then P_f is countable $P_f = U \setminus V$

$B_U[0,n] = B[0,n] \cap U$

$A_n = P_f \cap B_U[0,n]$

$$\boxed{P_f = \bigcup_{n \in \mathbb{N}} A_n}$$

$z \in P_f$

$\exists n \in \mathbb{N}$ s.t $n > |z|$

$n \geq |z|$

$z \in B[0,n]$

Also $z \in U$

$z \in B[0,n] \cap U = B_U[0,n]$

A. Tem 23/11/2020

Recall

$$K(x,t) = \begin{cases} (x+1)t, & 0 \le x \le t \\ (t+1)x, & t \le x \le 1 \end{cases}$$

FHIE $\quad g(x) = \lambda \int_0^1 K(x,t)\, g(t)\, dt$

$$K(t,x) = \begin{cases} (t+1)x, & 0 \le t \le x \\ (x+1)t, & x \le t \le 1 \end{cases} = K(x,t)$$

$$g(x) = \lambda \int_0^1 K(t,x)\, g(t)\, dt$$

$$g(x) = \lambda \int_0^x (t+1)x\, g(t)\, dt + \lambda \int_x^1 (x+1)t\, g(t)\, dt$$

L.F Diff w.r.t x

$$g'(x) = \lambda \left\{ \int_0^x (t+1) g(t)\, dt + (x+1)x g(x) \frac{d}{dx}(x) - 0 \right\}$$

$$+ \lambda \left\{ \int_x^1 t g(t)\, dt + 0 - (x+1)x\, g(x) \frac{d}{dx}(x) \right\}$$

$$g'(x) = \lambda \int_0^x (t+1) g(t)\, dt + \lambda \int_x^1 t g(t)\, dt$$

Diff again w.r.t x

$$g''(x) = \lambda \left\{ 0 + (x+1) g(x) \frac{d}{dx}(x) - 0 \right\}$$

$$+ \lambda \left\{ 0 + 0 - x g(x) \frac{d}{dx}(x) \right\}$$

$$= \lambda \left\{ (x+1)g(x) - x g(x) \right\}$$

$$g''(x) = \lambda g(x) \Rightarrow \boxed{g''(x) - \lambda g(x) = 0}$$

$$\left.\begin{array}{l} g(0) = \lambda \int_0^1 t g(t)\,dt \\ g'(0) = \lambda \int_0^1 t g(t)\,dt \end{array}\right\} \Rightarrow g(0) = g'(0)$$

i.e, $\boxed{g(0) - g'(0) = 0}$

$$\left.\begin{array}{l} g(1) = \lambda \int_0^1 (t+1) g(t)\,dt \\ g'(1) = \lambda \int_0^1 (t+1) g(t)\,dt \end{array}\right\} \Rightarrow g(1) = g'(1)$$

i.e, $\boxed{g(1) - g'(1) = 0}$

The eigenvalue problem is :

$$\left\{\begin{array}{l} g''(x) - \lambda g(x) = 0 \\ g(0) - g'(0) = 0 \\ g(1) - g'(1) = 0 \end{array}\right.$$

W.K.T

× Case(i) $\lambda = 0 \Rightarrow$

✓ Case(ii) $\lambda > 0 : \lambda = \alpha^2$

✓ Case(iii) $\lambda < 0 : \lambda = -\alpha^2$

× Case(i) st. line $g = Ax + B$

Case(ii) exponents $g = Ae^{\lambda x} + Be^{-\lambda x}$

Case(iii) $\sin x, \cos x$

$\lambda = 1$ $(\lambda > 0)$

$$g'' - g = 0$$
$$g(x) = c_1 e^{x} + c_2 e^{-x}$$
$$g'(x) = c_1 e^{x} - c_2 e^{-x}$$

$g(0) = g'(0)$

$\Rightarrow c_1 + c_2 = c_1 - c_2$

$\Rightarrow 2c_2 = 0 \Rightarrow c_2 = 0$

$g(1) = g'(1)$

$\Rightarrow c_1 e = c_1 e \Rightarrow c_1$ is arbitrary (free)

$\therefore g(x) = c_1 e^{x}$ $\therefore$ eigenfunction corresponding to the characteristic number $\lambda = 1$ is e^{x}.

Option 1 is correct

$\lambda = -\pi^2$ $(\lambda < 0)$

$g'' + \pi^2 g = 0$ $\Rightarrow g(x) = c_1 \cos\pi x + c_2 \sin\pi x$

$g'(x) = -\pi c_1 \sin\pi x + \pi c_2 \cos\pi x$

$g(0) = g'(0) \Rightarrow c_1 = \pi c_2$

$g(1) = g'(1) \Rightarrow -c_1 = -\pi c_2$

$\Rightarrow c_1 = \pi c_2$

$\therefore g(x) = c_2 \pi \cos\pi x + c_2 \sin\pi x$

Option 2 is wrong : Option 4 is correct

eigen fn :

$\pi \cos\pi x + \sin\pi x$

$(\lambda = -\pi^2)$

Check $\lambda = -4\pi^2$

$g'' + 4\pi^2 g = 0$

$\Rightarrow g(x) = c_1 \cos 2\pi x + c_2 \sin 2\pi x$

$g'(x) = -2\pi c_1 \sin 2\pi x + 2\pi c_2 \cos 2\pi x$

$g(0) = g'(0) \Rightarrow c_1 = 2\pi c_2$

$g(1) = g'(1) \Rightarrow c_1 = 2\pi c_2$

$c_1 = 2\pi c_2$

$g(x) = c_2 2\pi \cos 2\pi x + c_2 \sin 2\pi x$

eigen function: $\pi \cos 2\pi x + \sin 2\pi x$

Option 3 is wrong

Ans options 1 and 4

Recall

$$p(x) = a_0 + a_1 x + a_2 x^2$$

$$y(x) = p(x) + \int_0^x y(t)\sin(x-t)\,dt$$

$y(0) = p(0)$

$y(0) = a_0$

Diff w.r.t x

$$y'(x) = p'(x) + \int_0^x \cos(x-t)\,y(t)\,dt + y(x)\sin 0\,\frac{d}{dx}(x) - 0$$

$$y'(x) = p'(x) + \int_0^x \cos(x-t)\,y(t)\,dt$$

Again diff^ing w.r.t x

$$y''(x) = p''(x) + \int_0^x -\sin(x-t)\,y(t)\,dt + \cos 0\cdot y(x)\frac{d}{dx}(x) - 0$$

$$y''(x) = p''(x) - \int_0^x \sin(x-t)\,y(t)\,dt + y(x)$$

$$y''(x) = p''(x) - (y(x) - p(x)) + y(x)$$

$$\boxed{y''(x) = p''(x) + p(x)}$$

$p(x) = a_0 + a_1 x + a_2 x^2$

$p'(x) = a_1 + 2a_2 x$

$p''(x) = 2a_2$

$(*) \leftarrow \quad y''(x) = 2a_2 + a_0 + a_1 x + a_2 x^2$

$$y'(0) = p'(0) \Rightarrow \boxed{y'(0) = a_1}$$

$$y''(0) = p''(0) + p(0) \Rightarrow y''(0) = 2a_2 + a_0$$

From(*) $\quad y'(x) = 2a_2 x + a_0 x + \frac{a_1}{2}x^2 + \frac{a_2}{3}x^3 + c_1$ $\quad a_1$

also, $\quad y(x) = a_2 x^2 + \frac{a_0}{2}x^2 + \frac{a_1}{6}x^3 + \frac{a_2}{12}x^4 + \cancel{c_1} x + c_2$ $\quad a_1$

$y'(0) = a_1 \Rightarrow \boxed{c_1 = a_1}$ $\qquad y(0) = a_0 \Rightarrow a_1 + c_2 = a_0$

$$c_2 = a_0 - a_1$$

Option 1 – wrong / Option 2 – correct

Option 3 – wrong

$y'(0) = a_1$ if $a_1 \neq 0$, $y'(0) \neq 0$

✓ Option 4 → correct

$y''(0) = 2a_2 + a_0$ if $2a_2 + a_0 = 0$

$y''(0) = 0$

—— o ——

CSIR

FHIE with kernel

$$K(x,t) = \begin{cases} x(t-1), & 0 \le x \le t \\ t(x-1), & t \le x \le 1 \end{cases}$$

$$\phi(x) = \lambda \int_0^1 K(x,t)\, \phi(t)\, dt$$

$y'' + \lambda y = 0$

$\lambda = n^2 : n = 1, 2, 3$

$\lambda = 1$ $\sin x, \cos x$

$\lambda = 4$ $\sin 2x, \cos 2x$

$\lambda = 9$ $\sin 3x, \cos 3x$

✓ (1) $\lambda = -\pi^2$, $\phi(x) = \sin \pi x$

(2) $\lambda = -2\pi^2$, $\phi(x) = \sin 2\pi x$

(3) $\lambda = -3\pi^2$, $\phi(x) = \sin 3\pi x$

✓ (4) $\lambda = -4\pi^2$, $\phi(x) = \sin 2\pi x$

—— o ——

CSIR: Assume that h_1, h_2, g_1 and $g_2 \in C[a,b]$.

Let $\phi(x) = f(x) + \int_a^b \left[h_1(t) g_1(x) + h_2(t) g_2(x) \right] \phi(t)\, dt$

be an integral equation.

Consider the following statements:

S_1 : If the given integral equation has a solution for some $f \in C[a,b]$ then

$$\int_a^b f(t)\underbrace{g_1(t)}_{a_1(t)}\,dt = 0 = \int_a^b f(t)\underbrace{g_2(t)}_{a_2(t)}\,dt$$

$a_2(t) \leftrightarrow a_2(s)$

S_2 : The given integral equation has a unique solution for every $f \in C[a,b]$ if λ is not a characteristic number of the corresponding homogeneous equation.

Then,

(1) both S_1 and S_2 are true

(2) S_1 is true but S_2 is false

✓ (3) S_1 is false but S_2 is true

(4) both S_1 and S_2 are false

— o —

HINT $\quad K(s,t) = \sum_{i=1}^{n} a_i(s) b_i(t)$ (separable kernel)

$$K(x,t) = h_1(t)g_1(x) + h_2(t)g_2(x)$$

— o —

Remark

FNHIE $\quad \phi(x) = f(x) + \lambda^* \int_a^b K(x,t)\,\phi(t)\,dt \quad — ①$

FHIE $\quad \phi(x) = \lambda \int_a^b K(x,t)\,\phi(t)\,dt$

eigenvalue $\rightarrow \lambda_k$ $\qquad D(\lambda) = 0$

eigenfunction $\rightarrow \phi_k(x)$

Case (i)

$\lambda^* \neq \lambda_k$ (Note)

(1) has a unique solution.

Case (ii)

$\lambda^* = \lambda_k$ (Note)

Subcase (i)

$\int_a^b f(x)\phi_k(x)\,dx = 0$

infinitely many solutions.

Subcase (ii)

$\int_a^b f(x)\phi_k(x)\,dx \neq 0$

no solution.

Check S_1 with a solved problem

HW (separable kernel)

—o—

$\boxed{\text{'c'}}$

HW CSIR Consider the integral equation

$$\phi(x) - \frac{e}{2}\int_{-1}^{1} x e^t \phi(t)\,dt = f(x).$$ Then

(1) $\exists$ a continuous function $f : [-1,1] \to (0,\infty)$ for which solution exists

(2) $\exists$ a continuous function $f : [-1,1] \to (-\infty,0)$ for which solution exists

(3) for $f(x) = e^{-x}(1-3x^2)$, a solution exists

(4) for $f(x) = e^{x}(x+x^3+x^5)$, a solution exists.

—o—

HINT $\phi(x) = f(x) + \frac{e}{2}\int_{-1}^{1} x e^t \phi(t)\,dt$ | $k(s,t) = a_1(s)b_1(t)$

$$\frac{e^{x}}{2} \underbrace{\int_{-1}^{1} e^{t}\varphi(t)\,dt}_{`c'}$$

$\phi(x)$

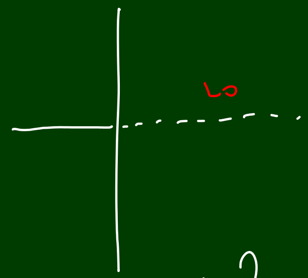
Complex Analysis, Aug 16, 2021.

$L_\alpha = \{te^{i\alpha} : t \geq 0\}$

Recall

$\arg_0 : \mathbb{C} \setminus L_0 \to (0, 2\pi)$

$\arg_0$ is a conts fn.



$A : \mathbb{C}^* \to \mathbb{R}$

$A(z) = \{t \in \mathbb{R} : z = |z| e^{it}\}$

$t \in A(z)$, t is called argument of z.

$\forall z \in \mathbb{C}^*$, $\exists!\, \theta \in [0, 2\pi)$, $z = |z| e^{i\theta}$

$A(z) = \{\arg_0(z) + 2k\pi : k \in \mathbb{Z}\}$

$\arg_0 : \mathbb{C}^* \to [0, 2\pi)$

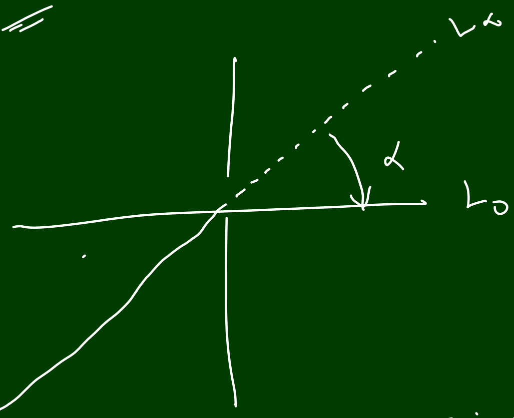
This a fn,

$\arg_\alpha : \mathbb{C} \setminus L_\alpha \to (\alpha, \alpha + 2\pi)$ is a conts fn?

Let $\alpha \in \mathbb{R}$

$f : \mathbb{C} \setminus L_\alpha \to \mathbb{C} \setminus L_0$

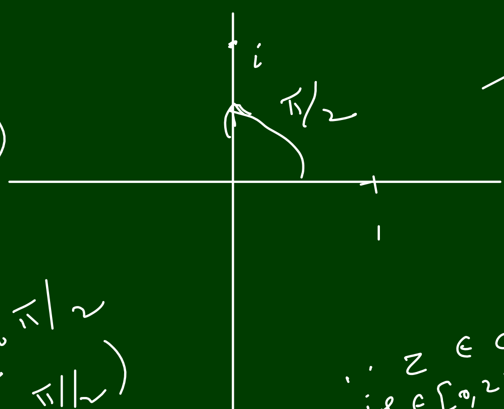
$f(z) = z e^{-i\alpha}.$



$z_1 = r_1 e^{i\theta_1}$

$z_2 = r_2 e^{i\theta_2}$

$z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$



$T : \mathbb{C} \to \mathbb{C}$

$T(1) = i$

$T(z) = T(z \cdot 1)$

$= z\, T(1)$

$= z \cdot i$

$z = e^{i\theta}$

$w = z \cdot e^{i\pi/2}$

$= e^{i(\theta + \pi/2)}$

$e^{i\pi/2} = i$

$\therefore z \in \mathbb{C}^*$

$\exists!\, s \in [0, 2\pi)$

$z = |z| e^{is}$

$z e^{-i\alpha} = |z| e^{i(s-\alpha)}$

Then $t - \alpha = 0$

$t = \alpha$

$f : \mathbb{C} \setminus L_\alpha \to \mathbb{C} \setminus L_0$

$f(z) = z e^{-i\alpha}$

Let $z_0 \in \mathbb{C} \setminus L_\alpha$

f is cont at z_0

Let $z \in \mathbb{C} \setminus L_\alpha$

$\therefore z \notin L_\alpha$.

$f(z) = z e^{-i\alpha} \notin L_0$

Suppose $z e^{-i\alpha} \in L_0$

$$|f(z)-f(z_0)| = |ze^{i\alpha} - z_0 e^{i\alpha}|$$
$$= |e^{i\alpha}(z-z_0)|$$
$$= |z-z_0|\,|e^{i\alpha}|$$
$$= |z-z_0|$$

$\therefore |f(z)-f(z_0)| \le |z-z_0| < \delta$

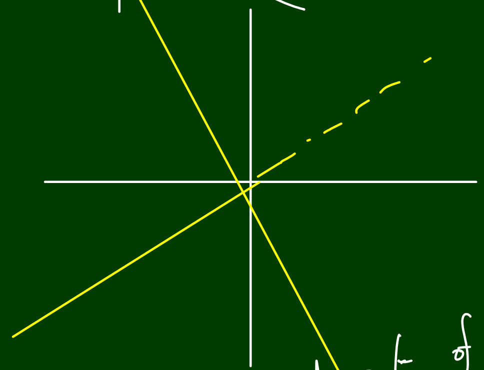
$= \epsilon$

$e^{i\alpha} = \cos\alpha + i\sin\alpha$.

βe^{α} $\quad z \mapsto \beta z$

$\phi : \mathbb{R}^2 \to \mathbb{R}$

$\phi(x,y) = x+y$



✓ $\arg_\alpha(z) = \alpha + \arg_0(f(z))$

$= \alpha + (\arg_0 \circ f)(z)$

$= \alpha + g(z)$

As $\arg_\alpha$ is composition of conts

[To do alg of cont fns on $C(X;\mathbb{R})$]

Let $x \in \mathbb{R}$

$$e^x := \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$\exp : \mathbb{R} \to \mathbb{R}$

Not $\boxed{\exp(x) = e^x}$

$\boxed{0! := 1}$ $\quad x^0 = 1$

$\forall x \in \mathbb{R}$

The series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges (will give a proof later)

If $x \ge 0$ then $e^x = \text{lub}\{s_n : n \in \mathbb{N}\}$

$$s_n = \sum_{k=0}^{n} \frac{x^k}{k!}$$

Property

(i) If $x > 0$, $e^x > 1 + x > x$

(ii) $\forall x \in \mathbb{R}$, $e^x > 0$

References

1. S.Kumaresan, A Pathway to Complex Analysis, Techno world Publications, 2021.

2. Bak, J., Newman and D.J, Complex Analysis, 3rd edition, Springer Nature, New York, 2015.

3. R. Priestely, Introduction to Complex Analysis, Oxford India, 2008.

4. Theodore W. Gamelin, Complex Analysis, Springer Verlag, 2003.

5. Lars V. Ahlfors, Complex Analysis, Third Ed. McGraw-Hill Book Company, Tokyo, 2017.

6. R.V. Churchill & J.W. Brown, Complex Variables and applications, 8th edition, McGraw-Hill, 2017.

7. L.S. Hahn and B. Epstein, Classical Complex analysis, Jones and Barlett Student Edition, 2011.

8. J.B. Conway, Functions of One Complex Variable, Narosa, 2 edn., 2000.