BHARATHIDASAN UNIVERSITY
Tiruchirappalli - 620024
Tamil Nadu, India

Programme : M.Sc. Mathematics
Course Title : Complex Analysis
Course code : 24S3M09CC

UNIT 4

Dr. P. S. Srinivasan
Associate Professor
Department of Mathematics

Complex Analysis, Oct 8 2020

Zeros of $f \in H(U)$. Let $z_0 \in U$.

$m(f : z_0)$ is the order of zeros of f at z_0

✓ $\begin{cases} f(z) = (z-z_0)^m g(z) \\ \text{where } g(z_0) \neq 0 \text{ \& } g \text{ is holom at } z_0. \\ \end{cases}$

$Z(f) = \{z \in \mathbb{C} : f(z) = 0\}$

Thm Let U be a region. & if $Z(f)$ has a cluster point

Then $f = 0$ on U

✓ Observation

if z_0 is cluster point of $Z(f)$ i.e. $E \subset Z(f)$

Then, $f(z_0) = 0$

$E = \{z \in U : z \text{ is a cluster point of } Z(f)\}$

E ST $E = U$. (Then $U \subset E \subset Z(f)$ $\therefore$ $z \in U$ Then $z \in Z(f)$ $U \subset Z(f)$ $\therefore Z(f) = U$)

By hypo $E \neq \phi$.

Claim E is clopen

Let, $z_0 \in E$

$\therefore z_0$ is clustpt of $Z(f)$

f is holo at z_0 $\therefore$ f is analytic at z_0

$\therefore \exists\, r > 0$ s.t $B(z_0, r) \subset U$ & $\exists\, (a_n)_{n \in \mathbb{Z}_+}$ s.t

$$f(z) = \sum_n a_n (z-z_0)^n, \quad z \in B(z_0, r)$$

Claim $\forall n \in \mathbb{Z}_+$, $a_n = 0$

Proof

Sup $\exists\, m \in \mathbb{Z}_+$ s.t $a_m \neq 0$

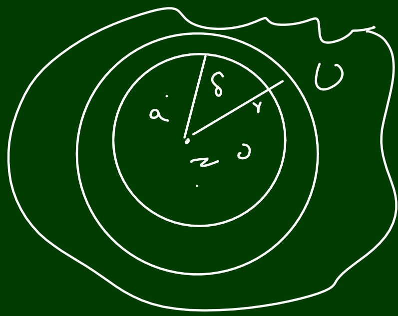
choose m be smallest such integer [by WOP]

$a_m \neq 0$ & $\forall k < m$, $a_k = 0$

$\therefore z_0$ is zero of order m

$f(z) = (z-z_0)^m g(z)$ s.t $g(z_0) \neq 0$
& g is holo at z_0

$z_0 \in (Z(f))'$
$\therefore \forall s>0 \quad B'(z_0, s) \cap Z(f) \neq \phi$ —— $(*)$



g is conts at z_0
$\therefore \exists \delta > 0$ s.t on $B(z_0, \delta)$
$g \neq 0$ —— $(**)$

$\therefore$ Apply $(*)$ for $s = \delta$; $\therefore$ choose $a \in B'(z_0, \delta) \cap Z(f)$
$\therefore f(a) = 0$

On the other hand
$g(a) \neq 0$
$a \neq z_0$
$\therefore (a - z_0)^m \neq 0$
$\therefore f(a) = (a-z_0)^m g(a) \neq 0$

— x —

$\therefore f = 0$ on $B(z_0, r)$

$\therefore B(z_0, r) \subset E$ (?) $\therefore B(z_0, r) \subseteq Z(f)$

Reason
$w \in B(z_0, r) \Rightarrow f(w) = 0$, $w \in Z(f)$
~~Claim~~ ~~$f(w) = \sum_n a_n (w - z_0)^n$~~ $B(z_0, r) \subseteq Z(f)$
$f(w) = 0$

w is cluster point of $B(z_0, r)$.
$\therefore w$ is cluster point of $Z(f)$
$\therefore w \in E$.

— x —

$\therefore E$ is open

G
$\|$
w is cluster poi. $B(z_0, r)$
$\forall s > 0$, $B(w, s) \cap G \neq \phi$
$B(w, s) \cap Z(f) \neq 0$.

$X = \mathbb{R}^n$
$r < s$
$B(a, r) \subsetneq B(a, s)$

Complex Analysis, Oct 9, 2020.

Thm Let U be open & connected sub of $\mathbb{C}$ & $f \in H(U)$.
Assume $Z(f)$ has a cluster point in U.
Then $f = 0$ on U.

Proof:
$E = \{ z \in U : z \text{ is cluster point of } Z(f) \}$

Claim $E = U$

For this EST E is clopen & $E \neq \phi$.

E is open done.

Recall: Let $z_0 \in E$

TPT $\exists r > 0$ s.t $B(z_0, r) \subset E$

$f \in H(U)$. $\therefore f$ is analytic at z_0

$\therefore \exists r > 0$ s.t $B(z_0, r) \subset U$ & $\exists (a_n)_{n \in \mathbb{Z}^+}$

s.t $f(z) = \sum_n a_n (z - z_0)^n$, $z \in B(z_0, r)$

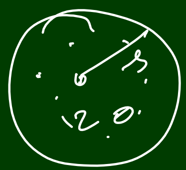
Claim $\forall n \in \mathbb{Z}^+$, $a_n = 0$

$\exists m \in \mathbb{Z}_+$ s.t $a_m \neq 0$
m be least such.

Then z_0 is a zero of f with order m

$f(z) = (z - z_0)^m g(z)$ where $g(z_0) \neq 0$
g is holo at z_0
$g(z_0) \neq 0$

z_0 is cluster point of $Z(f)$



$f = 0$ on $B(z_0, r)$

Let $z \in B(z_0, r)$ Then z is cluster pt of $B(z_0, r) \subset Z(f)$

$\therefore z$ is cluster point of $Z(f)$
$z \in E$

E is closed

Let (z_n) in E & $z_n \to z$ where $z \in U$

<u>Claim</u> $z \in E$. Case(i) $\forall n \in \mathbb{N}, z_n \neq z$

Let $r > 0$ <u>Claim</u> $B'(z, r) \cap Z(f) \neq \emptyset$

<u>Proof</u>

$\because z_n \to z$, $\exists\, n_0 \in \mathbb{N}$ s.t $z_{n_0} \in B'(z, r)$

$z_{n_0} \in E \Rightarrow f(z_{n_0}) = 0$

$z_{n_0} \in B'(z, r) \cap Z(f)$

$\therefore z$ is cluster point of $Z(f)$. $\therefore z \in E$

<u>Case(ii)</u> $\exists n \in \mathbb{N}$, $z_n = z$

then $z \in E$

Cor: (Identity thm)

Let U be open, connected subset of $\mathbb{C}$

Let $f, g \in H(U)$ & let $A \subset U$ s.t $\forall z \in A$, $f(z) = g(z)$

Assume A has a cluster point U

Then <u>on U $f = g$</u>

ie $\forall z \in U$, $f(z) = g(z)$

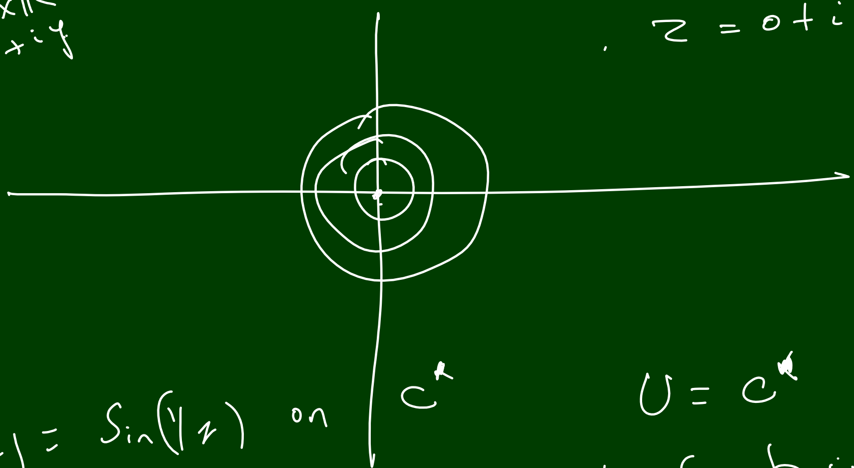
(Principle of Analytic Cont).

1) $f \in H(\mathbb{C})$, $\forall x \in \mathbb{R}$, $f(x) = e^x$ Then <u>$\forall z \in \mathbb{C}, f(z) = e^z$</u>

$\mathbb{R} \times \{0\}$, $\mathbb{R} \times \mathbb{R}$

$z \in \mathbb{C}$; $z = x + iy$

$z = 0 + i0$



2) $f(z) = \sin(1/z)$ on $\mathbb{C}^*$ $U = \mathbb{C}^*$

Find $Z(f)$ Does it have cluster point

$Z(f) = \left\{ \frac{1}{n\pi} : n \in \mathbb{Z} \setminus \{0\} \right\}$

$= \left\{ \frac{c}{n} : n \in \mathbb{Z} \setminus \{0\} \right\}$ $\qquad 0 \notin U$

0 is a cluster point

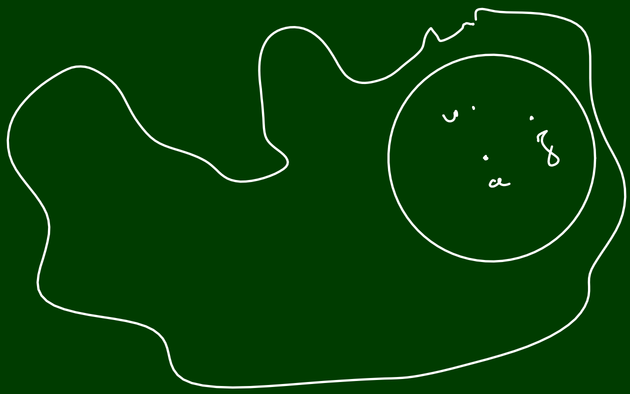
— x —

Complex Analysis, OCT 15, 2020

Maximum Modulus Thm

U - open connected, $f \in H(U)$ & $B(a,r) \subset U$

Assume $\forall z \in B(a,r)$, $|f(a)| \geq |f(z)|$. Then f is Constant



Case(i) $f(a) = 0$

$\forall z \in B(a,r)$ $|f(z)| \leq 0$

$\therefore$ $z \in B(a,r)$, $f(z) = 0$

$f = 0$ on $B(a,r)$

$f: U \to \mathbb{C}$ $\quad f|_{B(a,r)} = 0$

$g: U \to \mathbb{C}$

$\forall z \in U$, $g(z) = 0$

choose $A = B(a,r)$, $\quad g = 0$

$\therefore$ By identity thm $f = 0$ on U

Recall:-

$H(U) := \{ f: U \to \mathbb{C} : f$ is holomorphic on $U \}$

open connected

$H(U)$ is an Integral domain or not?

Ans: Yes.

$H(U) \subsetneq C(U)$

$f: \mathbb{C} \to \mathbb{C}$ by $f(z) := |z| = \sqrt{x^2+y^2}$

where $z = x+iy \in \mathbb{C}$.

Find the region of diff. of f ?

Ans:-

$z_0 \neq 0$

$$u_x = \frac{1}{2}\frac{1}{\sqrt{x^2+y^2}} \neq v_y = 0$$

$$\frac{f(z)-f(0)}{z-0} = \frac{|z|}{z} \to \begin{cases} 1 & \text{if } z \to 0 \text{ on +ve real} \\ -1 & \text{if } z \to 0 \text{ -ve real} \end{cases}$$

is nowhere diff. fn. on $\mathbb{C}$.

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nowhere diff. fn's $\left\{ f(z) := \operatorname{Re} z \text{ or } \operatorname{Im} z \quad |z| , \ \bar{z} \right.$

$H(U)$ is an integral domain.

(i.e) it has no zero divisors!

by Identity thm.

$fg = 0 \Rightarrow$ either $f = 0$ or $g = 0$!

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pbm: 1

$$Z(f) := \{ z \in U : f(z) = 0 \}$$

if $f \equiv 0$, $Z(f) = U$

if $f \equiv z$, $Z(f) = \{0\}$

if $f(z) = e^{z}$, $Z(f) = \varphi$.

if f is any poly. of degree 'n'

$|Z(f)| \leq n$. $\hookrightarrow$ finite

If $f : \underset{\text{open connected}}{\overset{\circ}{U}} \to \mathbb{C}$ holomorphic $\underline{fn}$.

then $Z(f)$ is atmost Countable

(ie either finite or countable) infinite set.

Suppose $Z(f)$ is uncountable.

$\rightarrow$ Any uncountable subset of $\mathbb{C}$

has a cluster point (??).

$\rightarrow$ Bolzano-weierstrass thm.

(Any infinite and bdd subset of $\mathbb{R}^n$

has a cluster point.

Subclaim: if $A \subseteq \mathbb{C}$ uncountable.

then A has a cluster point.

$A = \bigcup_{n=1}^{\infty} A_n$ where $A_n = A \cap \overline{B}[0, n]$ for $n \in \mathbb{N}$.

$\bigcup_n A_n \subseteq A$ ✓

$A \subseteq \bigcup_n A_n$??

(By A.P.)

$\exists N \in \mathbb{N}$ s.t. A_N is uncountable

$\Rightarrow A_N$ is infinite & bdd set.

Conclusion: Any nonzero entire fn. has atmost countable number of zeros.

$f(z) = \sin z \rightarrow Z(f) = \{ n\pi : n \in \mathbb{Z} \}$

$f(z) = e^z \rightarrow Z(f) = \varphi$

Suppose f is an poly. entire fn.

$\rightarrow Z(f)$ is nonempty finite set existence of an analytic (or holom.) fn. guaranteed by identity thm.

Eg: 5 (1) Does there exists an holomorphic fn. f on $\mathbb{D}$-unit disc s.t. $f\left(\frac{1}{n}\right) = \frac{(-1)^n}{n}$ for $n \in \mathbb{N}$.

Ans: Suppose $\exists$ exists an holom. fn. s.t. $f\left(\frac{1}{n}\right) = \frac{(-1)^n}{n}$

$$f\left(\frac{1}{2n}\right) = \frac{1}{2n} \longrightarrow 1$$

Define $g(z) = z$,

$$f\left(\frac{1}{2n}\right) = \frac{1}{2n} = g\left(\frac{1}{2n}\right)$$

Since f, g are both holom. and define $A = \left\{\frac{1}{2n} : n \in \mathbb{N}\right\}$

By identity thm. $f = g$ on $\mathbb{D}$

$f(z) = z$ on $\mathbb{D}$

By hypo. $f(1) = -1$

$\Rightarrow \Leftarrow$.

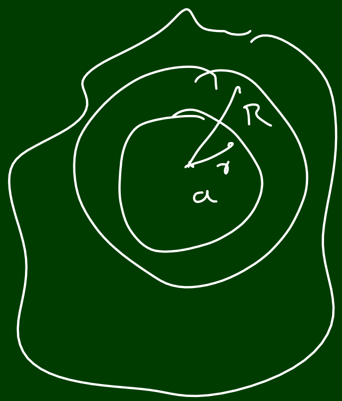
Complex Analysis, OCT 19, 2020

Maximum modulus Principle

Let $f \in H(U)$, U open connected, $B(a,R) \subset U$

Assume $\forall \, z \in B(a,R)$, $a \in U$, $|f(a)| \geq |f(z)|$. Then f is Constant on U

Proof



Case (i) $f(a) = 0$ Then done

Case (ii) $f(a) \neq 0$

Let $0 < r < R$

By MVT, $f(a) = \frac{1}{2\pi} \int_0^{2\pi} f(a + re^{it})\, dt$

$$|f(a)| = \frac{1}{2\pi} \left| \int_0^{2\pi} f(a + re^{it})\, dt \right|$$

$$\leq \boxed{\frac{1}{2\pi} \int_0^{2\pi} |f(a+re^{it})|\, dt}$$

$$\leq |f(a)| \cdot \frac{1}{2\pi} \int_0^{2\pi} dt$$

$$\leq |f(a)|$$

$f : [a,b] \to \mathbb{C}$

$\left| \int f \right| \leq \int |f|$

$|f(a+re^{it})| \leq |f(a)|$

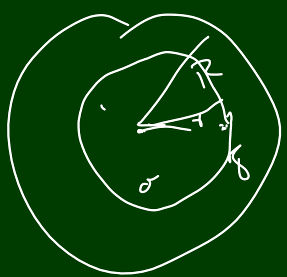
$$\therefore \frac{1}{2\pi} \int_0^{2\pi} |f(a+re^{it})|\, dt = |f(a)| = \frac{1}{2\pi} \int_0^{2\pi} |f(a)|\, dt$$

$$\int_0^{2\pi} \left[|f(a+re^{it})| - |f(a)| \right] dt = 0$$

$g \geq 0$, $g \in C(U)$

$\int_0^{2\pi} g = 0$

$g = 0$ on U



$\therefore \forall t \in [0, 2\pi]$, $|f(a+re^{it})| = |f(a)|$

$\because 0 < r < R$, is arbitrary.

$\forall z \in B(a,R)$, $|f(z)| = |f(a)|$

$\Rightarrow |f|$ is constant on $B(a,R)$

$\because f \in H(B(a,R))$, $|f|$ is constant By C-R equations

f is constant on $B(a,R)$

By Id. $\therefore f$ is constant on U.

$\therefore$ Proof uses

(1) MVT ✓

(2) C-R equations Ex. ✓

(3) Identity thm (2 twice)

$f(z) = u + iv$

$|f| = u^2 + v^2$

$u^2 + v^2 = c$

$2u u_x + 2v v_x = 0$

$2u u_y + 2v v_y = 0$

$2u^2 \cdot u_x - 2uv u_y = 0$

$2uv u_y + 2v^2 u_x = 0$

$4(u^2+v^2) u_x = 0$

$4c\, u_x = 0$

$u_x = 0$

$v_x = 0$

$f'(z) = u_x + i v_x = 0$

Exercise

Parseval's identity

Cor: Let U be a bounded and open connected in $\mathbb{C}$

Let $f \in H(U)$ and $f \in C(\overline{U})$

Then either f is constant or $|f|$ attains maximum on the boundary of U

Proof

Assume f is non-constant $\because \overline{U} \subset \overline{B[\alpha, \ell]} = B[\alpha, \ell]$

$\therefore f(\overline{U})$ is bdd

Then $M = \text{lub}\{|f(z)| : z \in \overline{U}\}$

$\because f \in C(\overline{U})$, f attains its bounds namely M

$\therefore \exists\, a \in \overline{U}$ s.t. $|f(a)| = M$

If $a \in U$

By Maximum modulus principle

f is constant on U $\Rightarrow \Leftarrow$

$\therefore a \in \overline{U} \setminus U \therefore a \in \partial U$

Ex 1

$U = \{ z \in \mathbb{C} : \operatorname{Re} z > 0 \}$

$f : U \to \mathbb{C}$

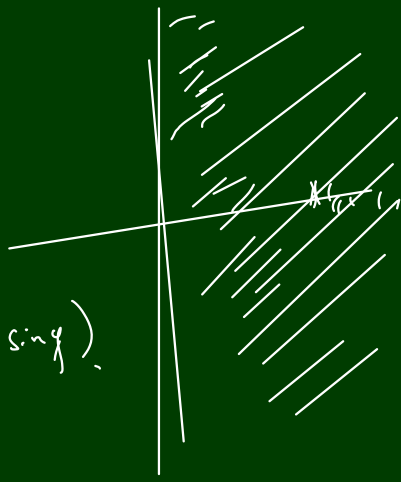
$f(z) = e^z$

$= e^{x+iy}$

$= e^x (\cos y + i \sin y)$.

$|f(z)| = |e^x|$

$= e^x$



Minimum Modulus principle?

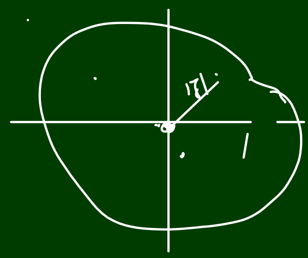
$U = B(0,1)$.

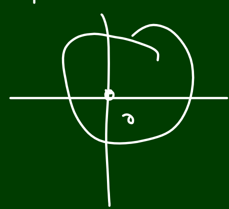
$f(z) = z$

$|f(z)| = |z|$.

max $|f|$ attained in bdry

$f(0) = 0$





$f(z) = z+1$

$B(0,1)$.

$|f(z)|^2 = |z+1|$

$= |x+iy+1|$

$f(x,y) = (1+x)^2 + y^2$

$r^2 t - s^2$

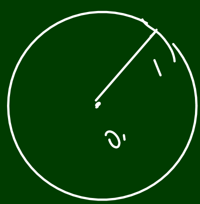
$\frac{\partial f}{\partial x} = 2(1+x)$.

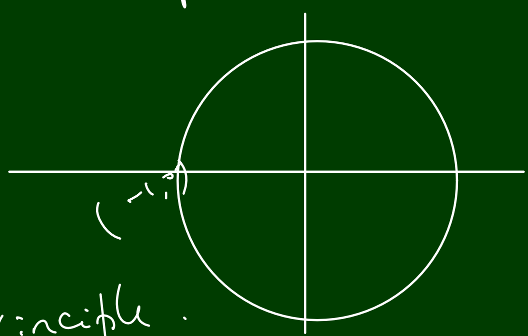
$\frac{\partial f}{\partial y} = 2y$

$4x = 0$

$x = -1$

$(-1, 0)$





Cor: Minimum Modulus Principle.

Let $g \in H(U)$, U open connected

& $\exists\, a \in U$ s.t. $\forall z \in B(a,R) \subset U$, $|g(a)| \le |g(z)|$.

Then g is constant.

This fails for $g(z) = z$.

Let $f = \frac{1}{g}$ $\qquad 0 < |g(a)| \le |g(z)|$

$$\frac{1}{|g(a)|} \ge \frac{1}{|g(z)|}$$

$\therefore \forall z \in B(a,R)$, $|f(a)| \ge |f(z)|$

$\Rightarrow f$ is constant
By MMP.

$\therefore g$ is constant

State hypo carefully.

Cor: Let $f \in H(U)$, f is non-constant.
Then $z \in U$ cannot be a relative local minimum of $|f|$
unless $f(z) = 0$.

Proof: $\exists z \in U$ s.t.
$f(z) = 0$

Then $|f|(z) = 0$

$\therefore$ That is the local minim for $|f|$

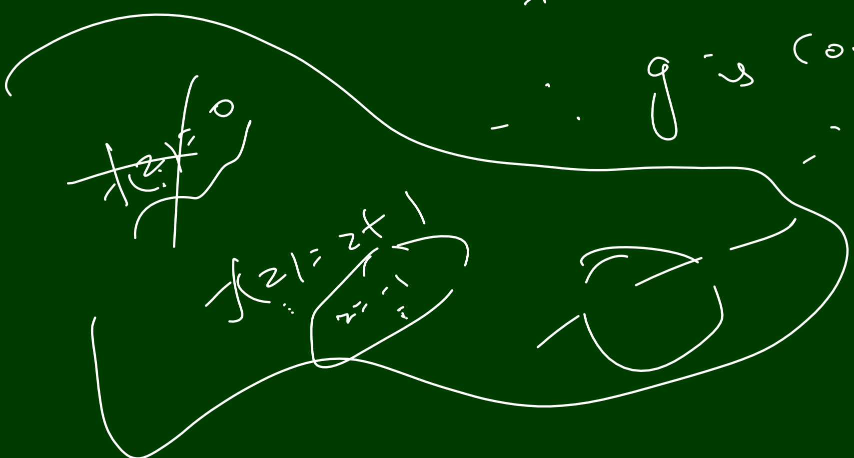
$\therefore$ Assume $\forall z \in U$, $f(z) \neq 0$. Suppose $\exists a \in U$ s.t. $|f(a)| \le |f(z)|$ $\forall z \in U$

Take $g(z) = \frac{1}{f(z)}$

Then $|g(a)| \ge |g(z)|$
$\forall z \in U$

$\therefore g$ is constant by m.m.p

$\therefore f$ is const $\Rightarrow\Leftarrow$



Ex

$f(z) = \frac{z^2}{z+2}$ Find the max of $|f|$ on $B[0,1]$

Complex Analysis, OCT 23, 2020

1) Let $f(z) = \frac{z}{z^2+2}$. Find the max of $|f|$ on $B[0,1]$

First
1
2
4
5

3 & 6 Ex

2) Find the max & min moduli of $z^2 - z$ on $B[0,1]$

3) Find the max mod $f(z) = e^{z^2}$ on $B(0,1)$

4) For $|\alpha| < 1$, $\phi_\alpha(z) := \frac{z-\alpha}{1-\bar{\alpha} z}$

s.t $\phi_\alpha(B(0,1)) \subset B(0,1)$ open

5) Let f be entire s.t $|f(z)| \to \infty$ as $|z| \to \infty$
S.T $f(\mathbb{C})$ is closed & hence conclude that f is onto.

6) Prove FTA using OMT Hint Ex 5.

Let me try (2)

$$f(z) = z^2 - z \quad \text{on } B[0,1]$$

$|z| = 1$

(x, y)

$$|f(z)| = |z^2 - z| = |z(z-1)| = |z||z-1|$$

$|f(z)| \le |z| + 1 = 2$

on $|z|=1$ $|f(z)| = |z-1|$

on $|z|=1$ $g(z) = |f(z)|$

$g: B[0,1] \to \mathbb{R}$

Lagrange multiplier $\Rightarrow \not\Leftarrow$

$f(z) = z(z-1)$

$z = 0$

$z = 1$

Complex Analysis, 28 Sep 2020 - Class 2

Mean Value Property:

Let $f \in H(U)$, $B(a,R) \subset U$. Let $0 < r < R$

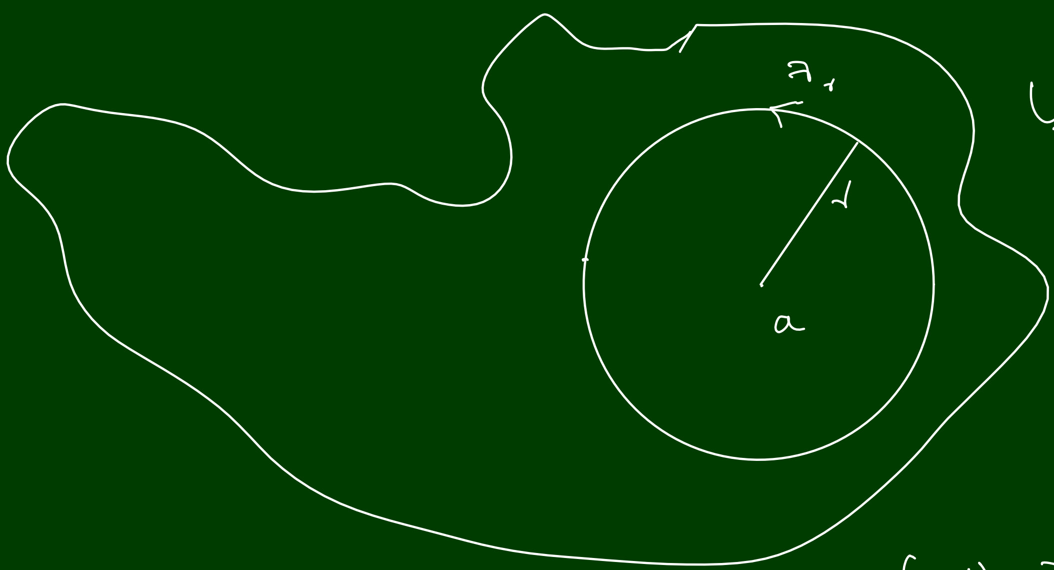
& $\gamma_r(t) = a + re^{it}$

Then $f(a) = \frac{1}{2\pi}\int_0^{2\pi} f(a+re^{it})\,dt$

Understand & give me Real Analogue.

$f \in H(U)$
Mean value

$$f(a) = \frac{1}{2\pi}\int_0^{2\pi} f(a+re^{it})\,dt$$



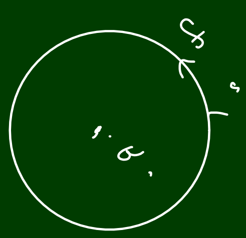
i.e.
The average of f on the circle γ_r centred at a is $f(a)$

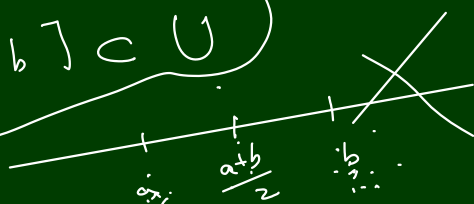
$f : U \to \mathbb{R}$

f is diff on U

& $[a, b] \subset U$

$f(a) = \frac{1}{2\pi}\int_0^{2\pi} \cancel{f(s)\,ds}$





$$f\left(\frac{a+b}{2}\right) = \frac{f(a)+f(b)}{2}$$

Proof: By CIF

$$f(a) = \frac{1}{2\pi i} \int_{\gamma_r} \frac{f(\omega)}{\omega - a}\, d\omega$$

$$g(\omega) = \frac{f(\omega)}{\omega - a}$$

$$= \frac{1}{2\pi i} \int_0^{2\pi} g(\gamma(t)) \cdot \gamma'(t)\, dt$$

$$= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(\gamma(t))}{\gamma(t) - a} \cdot \gamma'(t)\, dt$$

$$= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(a + re^{it})}{\cancel{re^{it}}}\, i\, \cancel{r}e^{it}\, dt$$

$$= \frac{1}{2\pi} \int_0^{2\pi} f(a + re^{it})\, dt$$

Ex:

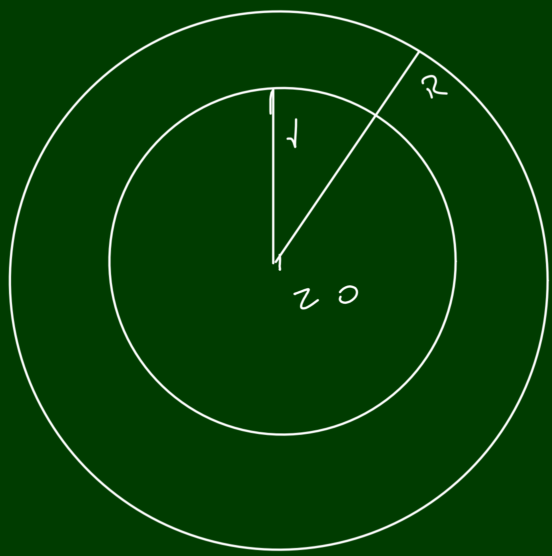
f is holo & nonconst on $B(z_0, R)$

Assume $f(z_0) = 0$

S.T $\exists\, r \in (0, R)$ s.t on the circle $S(z_0, r)$, Re f assumes both +ve & −ve values.

Proof: Suppose not

$\forall\, r \in (0, R)$, on the circle $S(z_0, r)$, Re f assumes only +ve



$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it})\, dt$$

$$= \frac{1}{2\pi}\left[\int_0^{2\pi} \mathrm{Re} f(z_0 + re^{it}) + i \int_0^{2\pi} \mathrm{Im} f(z_0 + re^{it}) dt \right]$$

$$\therefore \left| \int_0^{2\pi} \mathrm{Re} f(z_0 + re^{it}) \right| \leq 0$$

$$\mathrm{Re} f \geq 0$$

$$\int_0^{2\pi} \mathrm{Re} f(z_0 + re^{it}) dt \geq 0$$

$$\int_0^{2\pi} \mathrm{Re} f(z_0 + re^{it}) dt = 0$$

$$\Rightarrow \mathrm{Re} f(z_0 + re^{it}) = 0, \quad \forall t \in [0, 2\pi]$$

$$\mathrm{Re} f = 0$$

$\therefore$ $\forall r \in (0, R)$, on $S(z_0, r)$, $\mathrm{Re} f = 0$

Let $z \in B(z_0, R)$

Let $r = |z - z_0|$

on γ_r, $\mathrm{Re} f = 0$

$$\Rightarrow \mathrm{Re} f(z) = 0$$

$\therefore \forall z \in B(z_0, R)$, $\operatorname{Re} f = 0$

By C-R eq. $\Rightarrow$ $f = 0$

Parseval's identity

Let $f \in H(U)$, $B(z_0, R) \subset U$

Let $r \in (0, R)$, $\gamma_r(t) = z_0 + re^{it}$

$$\frac{1}{2\pi}\int_0^{2\pi} |f(z_0+re^{it})|^2 = \sum_{n=0}^{\infty} |a_n|^2 r^{2n}$$

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

Already done

Refer previous notes

Observe, $\|s_n\|^2 = \frac{1}{2\pi}\int_0^{2\pi} |s_n|^2 = \sum_{k=0}^{n} |a_k|^2 r^{2k}$ ✓

✓ Liouville's Thm is next class

C.A, OCT 21, 2020

OMT $f \in H(U)$, U open connected then f is a open map

i.e $V \overset{\text{open}}{\subseteq} U$ then $f(V)$ is open.

Proof Uses - { Minimum modulus principle
Identity thm Exercise.

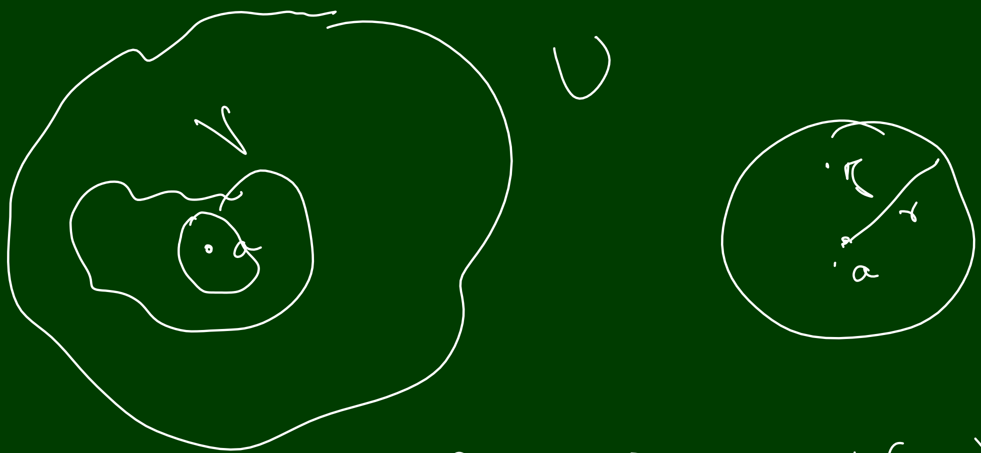
Find int. $B(a,\epsilon) \subset V$

$B(a,r) \subset V$

$f(B(a,\epsilon)) \subset f(V)$

Let $f(a) \in f(V)$ where $a \in V$

$\exists \epsilon > 0$ s.t $B(f(a), \epsilon) \subset f(B(a,r))$



$f(z) \neq f(a)$

$w \in S(a,r)$

$|f(w) - f(a)| = \delta$

$\delta \neq 0$

$\delta > 0$

$\delta > 0$

$\exists r > 0$ s.t on $B'[a,r]$, $f(z) \neq f(a)$

$f(z) - f(a) \neq 0$

$\delta = \min\limits_{z \in S(a,r)} |f(z) - f(a)|$ $\boxed{\delta > 0}$

Then $\epsilon = \frac{\delta}{2}$

$B(f(a), \epsilon) \subseteq f(B(a,r))$

Let $w \in B(f(a), \epsilon)$

Use map Construct $g: B(a,r) \to \mathbb{C}$

$g(z) = f(z) - w$

$\forall z \in S(a,r)$ $\quad |g(z)| = |f(z) - f(a) + f(a) - w|$

$$\geq |f(z) - f(a)| - |f(a) - w|$$

$$\geq 2\epsilon - \epsilon = \epsilon \qquad |f(a) - w| < \epsilon$$

$z = a$

$$|g(a)| = |f(a) - w| < \epsilon$$

$\therefore |g|$ attains minimum somewhere in $B(a,r)$

$(\exists z_0 \in B(a,r)$ s.t $|g(z_0)|$ is minimum for $|g|$ on $\overline{B}(a,r))$

By MMP $\exists z \in B(a,r)$, $g(z) = 0$ $\quad [\forall z \in B(a,r),\ g(z) \neq 0$ In fact $g(z_0) \neq 0]$ $\Rightarrow \Leftarrow$ M.M.P

$$f(z) - w = 0$$

$$\therefore\ w = f(z).$$

$$w \in B(f(a), \epsilon)$$

$$w \in f(B(a,r))$$

$$B(f(a), \epsilon) \subseteq f(B(a,r)) \subseteq f(V)$$

Thm (A) Schwarz lemma

Let $f : B(0,1) \to B(0,1)$ be holomorphic

Assume $f(0) = 0$. Then

(i) $\forall z \in B(0,1)$, $|f(z)| \leq |z|$

(ii) $|f'(0)| \leq 1$

Further In (i) the equality occurs iff $f(z) = cz$ for c s.t $|c| = 1$

$g : B(0,1) \to \mathbb{C}$

$\lim_{z\to 0} \frac{f(z)-f(0)}{z-0} = \lim_{z\to 0} \frac{f(z)}{z} = g$

$$g(z) = \begin{cases} \frac{f(z)}{z} & z \neq 0 \\ f'(0) & z = 0 \end{cases}$$

$g \in C(B(0,1))$

$g \in H(B'(0,1))$

g is holomorphic in $B(0,1)$

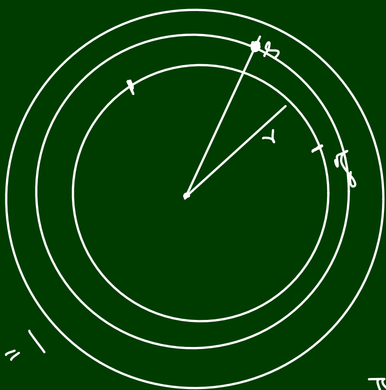
By morera's thm

$g \in H(B(0,1))$?

$r < 1$

$B[0,r]$

On $S[0,r]$ $|g(z)| = \frac{|f(z)|}{|z|} \leq \frac{1}{|z|} \leq \frac{1}{r}$



By M.M.T.

$\forall z \in B[0,r]$, $|g(z)| \leq \frac{1}{r}$

Let $0 < r < s$, $B[0,r] \subseteq B[0,s]$

Then

On $B[0,s]$, $|g(z)| \leq \frac{1}{s}$, $\forall z \in B[0,s]$, $|g(z)| \leq \frac{1}{s}$

$\lim_{s\to 1} \frac{1}{s} = 1$

$\therefore \forall z \in B[0,r]$, $|g(z)| \leq 1$

$1-\delta$ $\quad 1$

For any r $\therefore \forall z \in B(0,1)$, $|g(z)| \leq 1$

Publish on $\forall \epsilon > 0 \left(\exists \delta > 0 \left(\forall s \in (1-\delta, 1), \right.\right.$ $\left|\frac{1}{s} - 1\right| < \epsilon$

Let $\epsilon > 0$

$\left|\frac{1}{s} - 1\right| = \frac{1}{s} - 1$

$s < 1$

$\frac{1}{s} > 1$

$s > 1-\delta$

$\frac{1}{s} < \frac{1}{1-\delta}$

$\frac{1}{s} - 1 < \frac{1}{1-\delta} - 1 = \frac{\delta}{1-\delta} < \epsilon$

< 1

$1-\delta < s$

$1-\delta \quad s \quad 1 \quad 1/s$

?

$\delta < \frac{\epsilon}{1+\epsilon}$

$\delta < \epsilon(1-\delta)$

$\delta + \epsilon\delta < \epsilon$

$\delta(1+\epsilon) < \epsilon$

$$\lim_{s \to 1} \frac{1}{s} = 1$$

Suppose $\exists z_0 \in B(0,1)$ $|f(z_0)| = |z_0|$

$\underbrace{\exists z_0 \in B(0,1)}$, $\underbrace{|g(z_0)| = 1}$

$\boxed{g = c}$

$g(z_0) = c$

$|g(z_0)| = |c|$

$1 = |g(z_0)|$

By maxim

$$g(z) = \begin{cases} \frac{f(z)}{z} & z \neq 0 \\ f'(0), & z = 0 \end{cases}$$

$\underbrace{f(z) = gz}$ when $z \neq 0$

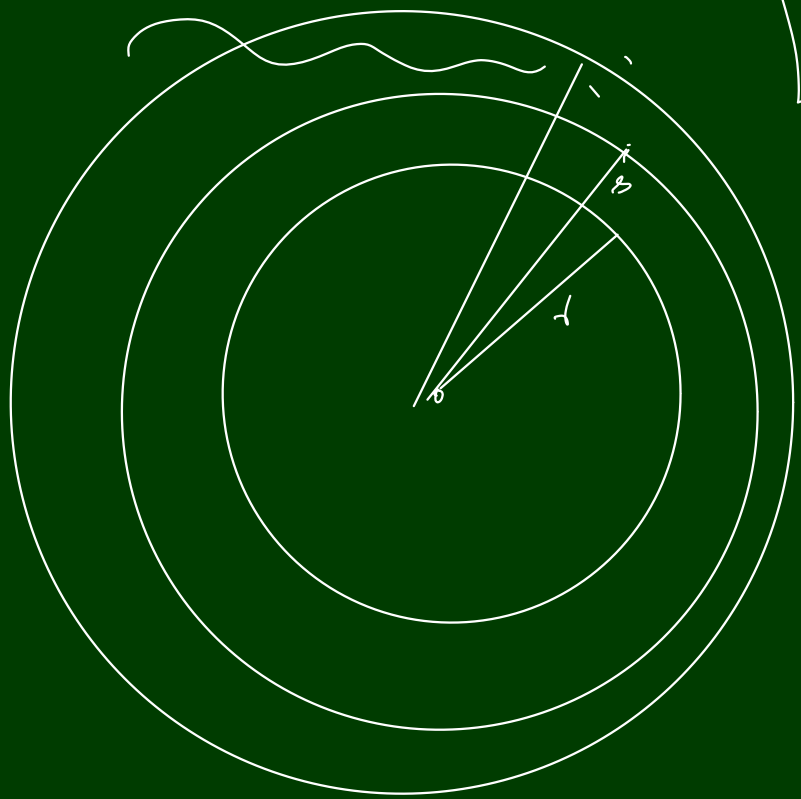
Also when $z = 0$

$f(0) = 0$

$\forall z \in B(0,1)$

$\therefore f(z) = cz$ where $|c| = 1$

$$\boxed{\begin{gathered} f(t) \leq g(t) \\ \lim_{t \to a} f(t) \leq \lim_{t \to a} g(t) \end{gathered}}$$



$\forall s > r$

$\forall |z| < r$, $|g(z)| \leq \frac{1}{s}$

$\lim_{s \to 1^-} |g(z)| \leq \lim_{s \to 1^-} \frac{1}{s}$

$0 < r < 1$

$\forall |z| < r$ $|g(z)| \leq 1$

$z \in B(0,1)$

$|z| = r < 1$

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