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Unit I

Lebesgue Outermeasure of Intervals

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Example

Show that for any set E, $\quad m^*(E) = m^*(E + x)$.

Proof.

$\Rightarrow$ Let $\epsilon > 0$ be any number. Then there $\exists \{I_k\}_{k=1}^{\infty}$ such that $E \subseteq \bigcup_{k=1}^{\infty} I_k$ and $m^*(E) + \epsilon \geq \sum_{k=1}]^{\infty} l(I_k)$. Since, $E + x \subseteq \bigcup_{k=1}^{\infty}(I_k + x)$.

$$
\begin{aligned}
m^*(E + x) &\leq \sum_{k=1}^{\infty} l(I_k + x). \\
&= \sum_{k=1}^{\infty} l(I_k) \\
m^*(E + x) &\leq m^*(E) + \epsilon \\
m^*(E + x) &\leq m^*(E).
\end{aligned}
$$

Theorem

If I is an interval then $m^*(I) = l(I)$.

Proof.

case(i) $I = [a, b]$. Let $\epsilon > 0$ be any number then
$I_1 = [a, b + \epsilon), \quad I_2 = I_3 = \cdots = \emptyset$. Then

$$m^*(I) \leq \sum_{k=1}^{\infty} l(I_k) = (b - a) + \epsilon.$$
$$= l(I) + \epsilon.$$
$$\textit{Therefore} \quad m^*(I) \leq l(I) + \epsilon.$$

□

Proof.

$$m^*(I) \leq l(I). \tag{1}$$

$$m^*(I) = inf\{\sum l(I_k) : \quad E \subset \bigcup_{k=1}^{\infty} I_k, \quad I_k \subseteq [a,b)\}$$

$\Leftarrow$ For any $\epsilon > 0$, then there $\exists$ $I_k = [a_k, b_k)$ such that $I \subseteq \bigcup_{k=1}^{\infty} I_k$. and

$$m^*(I) + \epsilon \geq \sum_{k=1}^{\infty} l(I_k).$$

$\sum l(I_k) = \sum l(I_k) + \epsilon.$

Proof.

Set ${I_k}' = (a_k + \frac{\epsilon}{2^k}, b_k)$, for all k, I_k is open

$$\sum_{k=1}^{\infty} l({I_k}') + \epsilon = \sum_{k=1}^{\infty} l(I_k).$$

$I \subseteq \bigcup_{k=1}^{\infty} {I_k}'$.
Since I is compact, we choose a finite subcover $\{J_1, J_2, \cdots, J_N\}$ of I in ${I_k}_{k=1}^{\infty}$ in such a way that no $J's$ is contained in the other.
Let $J_k = (c_k, d_k), k = 1, 2, \cdots, N$.
Now we rearrange $c_1 \leq c_2 \leq \cdots \leq c_N$.

Proof.

$$
\begin{aligned}
d_N - c_1 &= d_N - c_N + c_N - d_{N-1} + d_{N-1} - c_{N-1} + c_{N-1} - \cdots + \\
&\qquad c_2 - d_1 + d_1 - c_1 \\
&= (d_N - c_N) + (c_N - d_{N-1}) + (d_{N-1} - c_{N-1}) + \cdots \\
&\qquad + (c_2 - d_1) + (d_1 - c_1). \\
&= \sum_{k=1}^{N} (d_k - c_k) + \sum_{k=1}^{N-1} (c_{k=1} - d_k) \\
&= \sum_{k=1}^{N} (d_k - c_k) - \sum_{k=1}^{N-1} (d_k - c_{k+1}), \\
&< \sum_{k=1}^{N} (d_k - c_k) = \sum_{k=1}^{N} l(J_k) \leq \sum_{k=1}^{N} l({I_k}').
\end{aligned}
$$

$b - a < d_N - c_1$

Proof.

$$\leq \sum_{k=1}^{\infty} l(I_k{}')$$

$$= \sum_{k=1}^{\infty} (b_k - a_k + \epsilon | 2^k)$$

$$= \sum_{k=1}^{\infty} (b_k - a_k) + \sum_{k=1}^{\infty} \epsilon | 2^k.$$

$$= \sum_{k=1}^{\infty} l(I_k) + \epsilon.$$

$$b - a \leq m^*(I) + 2\epsilon.$$

Proof.

Since $\epsilon > 0$ is arbitrary,

$$b - a \leq m^*(I) \tag{2}$$

from (6) and (7),

$$m^*(I) = l(I).$$

case(ii) $I = [a, b)$.
Let $\epsilon > 0$, be given , set $I' = [a, b + \epsilon]$. Then

$$m^*(I) \leq m^*(I') = (b - a) + \epsilon. \tag{3}$$

Proof.

set $I^{''} = [a, b - \epsilon]$

$$m^*(I) \geq m^*(I^{''}) = (b - a) - \epsilon. \tag{4}$$

from (1.3) and (1.4)

$$-\epsilon \leq m^*(I) - (b - a) \leq \epsilon$$

$$\Rightarrow |m^*(I) - (b - a)| \leq \epsilon$$

Since ϵ is arbitrary,

$$m^*(I) = b - a = l(I).$$

For other finite interval, the proof are simalar to case (ii)

Proof.

case(iii) $I = (-\infty, a)$.

$$m^*(I) = +\infty.$$

Let us take a number $M > 0$. Set $J = [a - M, a)$

$$J \subseteq I \Rightarrow m^*(J) \leq m^*(I),$$

$$\Rightarrow M \leq m^*(I).$$

$$m^*(I) = +\infty = l(I).$$

Similarly we prove the result for other infinite intervals.