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Properties of Length function

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Definition

Let I be an interval with end points a and b with $a \leq b$. Then the length of the interval I is

$$l(I) = \begin{cases} b - a & \text{if } -\infty < b \text{ and } a < \infty \\ \infty & \text{otherwise} \end{cases}$$

Properties of Length Function

(1) $l(\emptyset) = 0.$

(2) $l(\{x\}) = 0.$

(3) For any $x \in \mathbb{R}, \quad l(I + x) = l(x).$

(4) If $I = \bigcup_{k=1}^{n} I_k, \quad I_k \in \mathcal{I} \quad \text{and} \quad I \in \mathcal{I}, \quad l(I) \leq \sum_{k=1}^{n} l(I_k).$

(5) If $I = \bigcup_{k=1}^{n} I_k, \quad I_k \in \mathcal{I}, \quad I_k \cap I_l = p, \quad k \neq l \quad \text{and}$
$I \in \mathcal{I}, \quad l(I) = \sum_{k=1}^{n} l(I_k).$

(6) If $I, J \in \mathcal{I}$ such that $I \subseteq J, \quad l(I) \leq l(J).$

Definition

IF $E = I_1 \cup I_2 \cup \cdots \cup I_n, \quad I_k \cap I_l = \emptyset$ with $k \neq l$. Then the measure of E is defined by

$$m(E) = l(I_1) + l(I_2) + \cdots + l(I_n).$$

$\mathcal{A}$= is the collection of subsets of E of $\mathbb{R}$ such that E is a union of finitely many disjoint intervals.

$$\mathcal{A} = \{E \subseteq \mathbb{R} / E = \bigcup_{k=1}^{n}, \quad I_k \in \mathcal{I}, \quad I_k \cap I_l = \emptyset, \quad k \neq l\}.$$

It is clear that $I \subseteq \mathcal{A}$. and

$$m : \mathcal{A} \to \mathbb{R}^*$$

$$m(E) = \sum_{k=1}^{n} l(I_k).$$

Definition

A collection $\mathcal{A}$ of subsets of $\mathbb{R}$ is called an algebra of sets in $\mathbb{R}$.

(1) $\phi, \mathbb{R} \in \mathcal{A}$.

(2) If $E, F \in \mathcal{A}, \quad E|F \in \mathcal{A}$

(3) If $E_1, E_2, \cdots, E_n \in \mathcal{A}$, then $\bigcup_{k=1}^{n} E_k \in \mathcal{A}$.

Definition

A measure is a set function $m : \mathcal{A} \to \mathbb{R}$ such that $m(\emptyset) = 0$; $m(E) \geq 0$; and whenever $E_1, E_2, \cdots, E_n$ are disjoint collection of set in $\mathcal{A}$ then

$$m\left(\bigcup_{k=1}^{n} E_k\right) = \sum_{k=1}^{n} m(E_k). \tag{1}$$

This property (1) is called subadditive.

For any E_k, E_l such that $E_k \cap E_l = \emptyset, \quad k \neq l$.

$$m(\bigcup_{k=1}^{n} E_k) = \sum_{k=1}^{n} m(E_k). \tag{2}$$

This property (2) is called additive.
Define the family

$$\mathcal{A}^* = \{E \subseteq \mathbb{R} | E = \bigcup_{k=1}^{n} I_k, I_k \in \mathcal{I}. \tag{3}$$

For any st $E \subseteq \mathbb{R}$, define the inner and outer measures

$$m^*(E) = inf\{\sum_{k=1}^{n} l(I_k) \quad |E \subseteq \bigcup_{k=1}^{n} I_k\}, \tag{4}$$

$$m_*(E) = sup\{\sum_{k=1}^{n} l(I_k) \quad |\bigcup_{k=1}^{n} I_k \subseteq E\}. \tag{5}$$

Definition

If the inner measure $m_*(E) = m^*(E)$ then E is called Jordan measurable set in $\mathbb{R}$.

$$\mathcal{M} = \{E \subseteq \mathbb{R} : E \quad is \quad Jordan \quad measurable\}.$$

$$m_*(E) = m(E) = m^*(E).$$

Definition

A set E is Jordan measurable iff for any $A \subseteq \mathbb{R}$

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c).$$

Definition

countable sub- additive: If $I = \bigcup k = 1^{\infty} I_k, \quad I_k \in \mathcal{I}, \quad I \in \mathcal{I}$, then $l(I) \leq \sum_{n=1}^{\infty} l(I_k)$,
If $I = \bigcup_{n=1}^{\infty}, \quad I_n \in \mathcal{I}, \quad I_n \cap I_m = \emptyset$ with $n \neq m$ and $I \in \mathcal{I}$, then $l(I) = \sum_{n=1}^{\infty} l(I_n)$.

Definition

A set E is Jordan measurable iff for any $A \subseteq \mathbb{R}$

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c).$$

Definition

Let $E \subseteq \mathbb{R}$. Define $m^*(E) = inf\{\sum_{k=1}^{\infty} l(I_k) | E \subseteq \bigcup_{k=1}^{\infty} I_K\}$, then prove that m^* is called the lebesgue outer measure.

Theorem

(i) $m^*(\emptyset) = 0$.

(ii) $m^*(E) \geq 0$, *for all* $E \subseteq \mathbb{R}$.

(iii) $m^*(\{x\}) = 0$.

Proof

(i)Given $\epsilon > 0$, $I_1 = [0, \epsilon]$, $I_2 = I_3 = \cdots = \emptyset$.

$$m^*(\emptyset) \leq \sum_{n=1}^{\infty} l(I_n) \leq \epsilon.$$

$\Rightarrow m^*(\emptyset) \leq \epsilon$, for all $\epsilon > 0 \Rightarrow m^*(\emptyset) = 0$.

Proof

(ii) is trivial.
(iii) For any $\epsilon > 0$, $I_1 = [x, x+\epsilon), \quad I_2 = I_3 = \cdots = \emptyset.$

$$\{x\} \subseteq \bigcup_{k=1}^{\infty} I_k.$$

$$m^*(\{x\}) = \sum_{k=1}^{\infty} l(I_k) = \epsilon.$$

$$m^*(\{x\}) \leq \epsilon \Rightarrow m^*(\{x\}) = 0.$$