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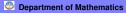
Programme: M.Sc., Mathematics

Course Title :Measure Theory and Integration COurse Code : 21M11CC

Properties of Length function

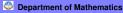
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Let *I* be an interval with end points *a* and *b* with $a \le b$. Then the length of the interval *I* is

$$I(I) = \begin{cases} b - a & \text{if } -\infty < b \text{ and } a < \infty \\ \infty & \text{otherwise} \end{cases}$$



(a)

(1) $l(\emptyset) = 0$. (2) $l(\{x\}) = 0$. (3) For any $x \in \mathbb{R}$, l(l+x) = l(x). (4) If $l = \bigcup_{k=1}^{n} l_k$, $l_k \in \mathcal{I}$ and $l \in \mathcal{I}$, $l(l) \le \sum_{k=1}^{n} l(l_k)$. (5) If $l = \bigcup_{k=1}^{n} l_k$, $l_k \in \mathcal{I}$, $l_k \cap l_l = p$, $k \ne l$ and $l \in \mathcal{I}$, $l(l) = \sum_{k=1}^{n} l(l_k)$. (6) If $l, J \in \mathcal{I}$ such that $l \subseteq J$, l(l) < l(J).

IF $E = I_1 \cup I_2 \cup \cdots \cup I_n$, $I_k \cap I_l = \emptyset$ with $k \neq I$. Then the measure of E is defined by

$$m(E) = I(I_1) + I(I_2) + \cdots + I(I_n).$$

A= is the collection of subsets of E of \mathbb{R} such that E is a union of finitely many disjoint intervals.

$$\mathcal{A} = \{ E \subseteq \mathbb{R}/E = \bigcup_{k=1}^{n}, \quad I_k \in \mathcal{I}, \quad I_k \cap I_l = \emptyset, \quad k \neq l \}.$$

It is clear that $I \subseteq A$. and

$$m: \mathcal{A} \to \mathbb{R}^*$$

 $m(E) = \sum_{k=1}^n l(I_k).$

Definition of Algebra and Measure

Definition

A collection \mathcal{A} of subsets of \mathbb{R} is called an algebra of sets in \mathbb{R} .

(1) $\phi, \mathbb{R} \in \mathcal{A}$. (2) If $E, F \in \mathcal{A}$, $E|F \in \mathcal{A}$ (3) If $E_1, E_2, \cdots, E_n \in \mathcal{A}$, then $\bigcup_{k=1}^n E_k \in \mathcal{A}$.

Definition

A measure is a set function $m : A \to \mathbb{R}$ such that $m(\emptyset) = 0$; $m(E) \ge 0$; and whenever E_1, E_2, \dots, E_n are disjoint collection of set in A then

$$m\left(\bigcup_{k=1}^{n} E_{k}\right) = \sum_{k=1}^{n} m(E_{k}).$$
(1)

This property (1) is called subadditive.

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For any E_k , E_l such that $E_k \cap E_l = \emptyset$, $k \neq l$.

$$m(\bigcup_{k=1}^{n} E_k) = \sum_{k=1}^{n} m(E_k).$$
 (2)

This property (2) is called additive. Define the family

$$\mathcal{A}^* = \{ E \subseteq \mathbb{R} | E = \bigcup_{k=1}^n I_k, I_k \in \mathcal{I}.$$
(3)

For any st $E \subseteq \mathbb{R}$, define the inner and outer measures

$$m^{*}(E) = \inf\{\sum_{k=1}^{n} I(I_{k}) | E \subseteq \bigcup_{k=1}^{n} I_{k}\},$$

$$m_{*}(E) = \sup\{\sum_{k=1}^{n} I(I_{k}) | \bigcup_{k=1}^{n} I_{k} \subseteq E\}.$$
(5)

Department of Mathematics

If the inner measure $m_*(E) = m^*(E)$ then *E* is called Jordan measurable set in \mathbb{R} .

 $\mathcal{M} = \{ E \subseteq \mathbb{R} : E \text{ is Jordan measurable} \}.$

 $m_*(E) = m(E) = m^*(E).$

Definition

A set *E* is Jordan measurable iff for any $A \subseteq \mathbb{R}$

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c).$$

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countable sub-additive: If $I = \bigcup k = 1^{\infty} I_k$, $I_k \in \mathcal{I}$, $I \in \mathcal{I}$, then $I(I) \leq \sum_{p=1}^{\infty} I(I_k),$ If $I = \bigcup_{n=1}^{\infty}$, $I_n \in \mathcal{I}$, $I_n \cap I_m = \emptyset$ with $n \neq m$ and $I \in \mathcal{I}$, then $I(I) = \sum_{n=1}^{\infty} I(I_n).$

Definition

A set *E* is Jordan measurable iff for any $A \subseteq \mathbb{R}$

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c).$$



Let $E \subseteq \mathbb{R}$. Define $m^*(E) = inf\{\sum_{k=1}^{\infty} I(I_k) | E \subseteq \bigcup_{k=1}^{\infty} I_K\}$, then prove that m^* is called the lebesgue outer measure.

Theorem

(i) $m^*(\emptyset) = 0.$ (ii) $m^*(E) \ge 0$, for all $E \subseteq \mathbb{R}$. (iii) $m^*(\{x\}) = 0.$

(日)

(i) Given
$$\epsilon > 0$$
, $l_1 = [0, \epsilon]$, $l_2 = l_3 = \cdots = \emptyset$.
 $m^*(\emptyset) \le \sum_{n=1}^{\infty} l(l_n) \le \epsilon$.

 $\Rightarrow m^*(\emptyset) \leq \epsilon$, for all $\epsilon > 0 \Rightarrow m^*(\emptyset) = 0$.



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Proof

(ii) is trivial. (iii) For any $\epsilon > 0$, $l_1 = [x, x + \epsilon)$, $l_2 = l_3 = \cdots = \emptyset$.

$$\{x\}\subseteq \bigcup_{k=1}^{\infty}I_k.$$

$$m^*(\lbrace x\rbrace) = \sum_{k=1}^{\infty} l(l_k) = \epsilon.$$
$$m^*(\lbrace x\rbrace) \le \epsilon \Rightarrow m^*(\lbrace x\rbrace) = 0.$$

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