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Unit I
Vector fields of Surfaces and Orientation

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Vector Fields on surfaces; orientation

Definition

A vector field $\vec{X}$ on an surface $S \subset \mathbb{R}^{n+1}$ is a function which aassigns to each point p in S a vector $\vec{X}(p) \in \mathbb{R}^{n+1}_p$ at p.

Definition

If $\vec{X}(p)$ is tangent to S (ie, $\vec{X}(p) \in S_p$) for each $p \in S$, $\vec{X}$ is said to be a tangent vector feilds on S.

Definition

If $\vec{X}(p)$ is orthogonal to S (ie $\vec{X}(p) \in S_p^{\perp}$ for each $p \in S$, $\vec{X}$ is said to be a normal vector feild on S.

Definition

A function $g : S \to \mathbb{R}^k$, where S is an n- surface in $\mathbb{R}^{n+1}$, is smooth if it is the restriction to S of a smooth function $\tilde{g} : V \to \mathbb{R}^k$ defined on some open set V in $\mathbb{R}^{n+1}$ containing S.
Similarly, a vector field $\vec{X}$ on S is smooth, if it is the restriction to S of a smooth vector field defined on some open set containing S.

Remark

Thus $\vec{X}$ is smooth if and only if $X : S \to \mathbb{R}^{n+1}$ is smooth, where $\vec{X}(p) = (p, X(p))$ for all $p \in S$.

Theorem

Let S be an n- surface in $\mathbb{R}^{n+1}$, *let* $\vec{X}$ *be a smooth tangent vector field on S, and let* $p \in S$. *Then there exists an open interval I containing O and a parametrized curve* $\alpha : I \to S$ *such that*

(i) $\alpha(0) = p$,

(ii) $\alpha(t) = \vec{X}(\alpha(t))$ *for all* $t \in S$

(iii) *If* $\beta : \tilde{I} \to S$ *is any other parametrized curve in S satisfying (i) and (ii), then* $\tilde{I} \subset I$ *and* $\beta(t) = \alpha(t)$ *for all* $t \in \tilde{I}$.

Proof.

Since $\vec{X}$ is smooth,there exists an open set V containing S and a smooth vector field $\vec{X}$ on V such that $\tilde{X}(q) = \vec{X}(q)$ for all $q \in S$. Let $f : U \to \mathbb{R}$ and $c \in \mathbb{R}$ be such that $S = f^{-1}(c)$ and $f(q) \neq 0$ for all $q \in S$. □

Proof.

Let $W = \{q \in U \cap V | f(q) \neq 0\}$. Then W is a open set containing S, and both $\tilde{X}$ and f are defined on W, everywhere tangent to the level sets of f, defined by

$$\vec{Y}(q) = \tilde{X}(q) - \frac{(\tilde{X}(q).\nabla f(q))}{\|\nabla f(q)\|^2} \nabla f(q).$$

Note that $Y(q) = X(q)$ for all $q \in S$. Let $\alpha : I \to W$ be the maximal integral curve of Y through p. Then α actually maps I into S because

$$\begin{aligned}(f \circ \alpha)'(t) &= \nabla f(\alpha(t)).\alpha(t) \\ &= \nabla f(\alpha(t)).Y(\alpha(t)) = 0.\end{aligned}$$

Proof.

and $f \circ \alpha(0) = f(p) = c$, so $f \circ \alpha(t) = c$, for all $t \in I$ conditions (i) and (ii) are clearly satisfied and condition(iii) is satisfied because any $\beta : \tilde{I} \to S$ satisfying (i) and (ii) is also an integred curve of the vector field $\vec{Y}$ on W so the theorem

Corollary

Let $S = f^{-1}(c)$ be an n - surface in $\mathbb{R}^{n+1}$, where $f : U \to \mathbb{R}$ is such that $\nabla f(q) \neq 0$ for all $q \in S$, and let X be a smooth vector fields on U whose restriction to S isa tangent vector field on S. If $\alpha : I \to U$ is an integral curve of X such that $\alpha(t_0) \in S$ for some U, then $\alpha(t) \in S$ for all $t \in I$.

Proof.

Suppose $\alpha(t) \in S$ for some $t \in I, t > t_0$. Let t_1 denote the greatest lower bound of the set $\{t \in I | t > t_0 \quad and \quad \alpha(t) \in S\}$. Then $f(\alpha(t)) = c$ for all $t_0 \leq t \leq t_1$ so , by continuity $f(\alpha(t_1)) = c$; (ie) $\alpha(t_1) \in S$.
Let $\beta : \tilde{I} \to S$ be an integral curve through $\alpha(t_1)$ of the restriction of X to S. Then β is also an integral curve of X, sending 0 to $\alpha(t_1)$, as is the curve $\tilde{\alpha}$ defined by , $\tilde{\alpha}(t) = \alpha(t + t_1)$.
By uniqueness of integral curves, $\tilde{\alpha}(t) = \alpha(t - t_1) = \beta(t - t_1) \in S$ for all t such that $t - t_1$ is in the common domain of $\tilde{\alpha}$ and β.
But this contradicts the fact that $\alpha(t) \in S$ for values of t arbitrary close to t_1.
Hence $\alpha(t) \in S$ for all $t \in I$ with $t > t_0$ the proof for $t < t_0$ is similar. $\square$

Definition

A subset S of $\mathbb{R}^{n+1}$ is said to be connected for each pair p, q of points in S there is continuous map $\alpha : [p, q] \to S$, from some closed interval $[a, b]$ into S, such that $\alpha(a) = p$ and $\alpha(b) = q$.

Remark

Thus S is connected if each pair of points in S can be joined by a continuous , but not necessarily smooth, curve which lies completely in S. Note, for example, that the n- sphere is connected if and only if $n \geq 1$.

Theorem

Let $S \subset \mathbb{R}^{n+1}$ be a connected n- surface in $\mathbb{R}^{n+1}$. Then there exists on S exactly two smooth unit normal vector feilds N_1 and N_2 and $N_2(p) = -N_1(p)$ for all $p \in S$

Proof.

Let $f : U \to \mathbb{R}$ and $c \in \mathbb{R}$ be such that $S = f^{-1}(c)$ and $\nabla f(p) \neq 0$ for all $p \in S$. Then the vector field N_1 on S defined by

$$N_1(p) = \frac{\nabla f(p)}{\|\nabla f(p)\|} \qquad p \in S$$

clearly has the required properties as does the vector field N_2 defined by $N_2(p) = -N_1(p)$ for all $p \in S$.
To show that there are the only two such vector feilds ,suppose N_3 were another then for each $p \in S$, $N_3(p)$ must be a multiple of $N_1(p)$ since both lie in the 1- dimension subspace $S_p^{\perp} \subset \mathbb{R}_p^{n+1}$. Thus

$$N_3(p) = g(p)N_1(p)$$

□

Proof.

where $g : S \to \mathbb{R}$ is smooth function on S $(g(p) = N_3(p).N_1(p) \quad for \quad p \in S)$. Since $N_1(p)$ and $N_3(p)$ are both unit vectors, $g(p) = \pm 1$ for each $p \in S$.
Finally, since g is smooth and S is connected, g must be constant on C. Thus either $N_3 = N_1$ or $N_3 = N_2$. Note:

(i) A smooth unit normal vector field on a n - surface S in $\mathbb{R}^{n+1}$ is called an orientation on S.

(ii) According to the theorem just proved, each connected n - surface in $\mathbb{R}^{n+1}$ has exactly two orientations.

(iii) An n- surface together with a choice of orientation is called an orientation n -surface.

Remark

There are subsets of $\mathbb{R}^{n+1}$ which must people would agree should be called n -surfaces but on which there exists no orientations.

Example

An example is the mobius band B, the surface in $\mathbb{R}^{n+1}$ obtained by taking a rectangular strip of paper, twisting one end through 180°, and taping the ends together that there is no smooth unit normal vector field on B can be seen by picking a unit normal vector at some point on the centeral circle and trying to extend it continuously to a unit normal vector field along this circle. After going around the circle once the normal vector is pointing in the opposite direction.
Since there is a smooth unit normal vector fields on B, B cannot be expressed as a level set $f^{-1}(c)$ of some smooth function $f : U \to \mathbb{R}$ with $\nabla f(p) \neq 0$ for all $p \in S$, and hence B is not a 2 - surface according to our definition. B is an example of an unorintable 2- surface.

Definition

A unit vector in $\mathbb{R}^{n+1}_p$ ($p \in \mathbb{R}^{n+1}$) is called a direction at p. Thus an orientation on an n- surface S in $\mathbb{R}^{n+1}$ is by definition, a smooth choice of normal direction at each point of S.

On a plane curve, an orientation can be used to define a tangent direction at each point of the curve .

Remark

The positive tangent direction at the point p of the oriented plane curve c is the direction obtained by rotating the orientation normal direction at p through an angle of $-\pi|2$ where the direction of positive rotation is counterclockwise.

Example

On a 2- surface in $\mathbb{R}^3$, an orientation can be used to define a direction of rotation in the tangent space at each point of the surface. Given $\theta \in \mathbb{R}$, the positive θ- orientation at the point p of the oriented 2 - surface S is the linear transformation $R_\theta : S_p \to S_p$ defined by $R_\theta(v) = (\cos\theta)v + (\sin\theta)N(p) \times v$ where $N(p)$ is the orientation normal direction at p. R_θ is usually described as the right handed rotation about $N(p)$ through the angle θ.
On a 3 - surface in $\mathbb{R}^4$, an orientation can be used to define a sense of handedness in the tangent space at each point of the surface.

Example

Given an oriented 3- surface S and a point $p \in S$, an ordered orthonormal basis $\{e_1, e_2, e_3\}$ for the tangent space S_p to S at p is said to be right-handed if the determinant $det \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ N(p) \end{pmatrix}$ is positive, where $\vec{N}(p) = (p, N(p))$ is the orientation normal direction at p and $\vec{e}_i = (p, e_i)$ for $i = \{1, 2, 3\}$; the basis is lefthanded if the determinant is negative

Example

On a n- surface in $\mathbb{R}^{n+1}$ (n arbitrary), an orientation can be used to partition the collection of all ordered bases for each tangent space into two subsets, those consistent with the orientation and those inconsistent with the orientation.

An ordered basis $\{v_1, v_2 \cdots, v_n\}$ (not necessarily orthonormal) for the tangent space S_p at the point p of the oriented n- surface S is said to be consistent with the orientation N on S if the determinant

$$det \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \\ N(p) \end{pmatrix}$$

is positive; the basis is inconsistent with N if the determinant is negative. Here, as usual $\vec{v_i} = (p, v_i)$ and $\vec{N(p)} = (p, N(p))$.