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Programme: M.Sc., Mathematics

Course Title : Differential Geometry
COurse Code : 21M13CC

Unit I
Level Sets and Vector fields

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Level Sets

Definition

Given a funcion $f : U \to \mathbb{R}$, where $U \subset \mathbb{R}^{n+1}$,, its **level sets** are the sets $f^{-1}(c)$ defined, for each real number c, by

$$f^{-1}(c) = \{(x_1, x_2, \cdots, x_{n+1}) \in U : f(x_1, x_2, \cdots, x_{n+1}) = c\}$$

Remark

Then the numebr c is called the height of the level set, and $f^{-1}(c)$ is called the level set at theight c.

Remark

Since $f^{-1}(c)$ is the solution set of the equation $f(x_1, x-2, \cdots, x_{n+1}) = c$, the level set $f^{-1}(c)$ is often described as " the set $f(x_1, x-2, \cdots, x_{n+1}) = c$."

Definition

The **graph** of a function $f : U \to \mathbb{R}$ is the subset of $\mathbb{R}^{n+2}$ defined by

$$\begin{aligned} \text{graph}(f) &= \{(x_1, x_2, \cdots, x_{n+2}) \in \mathbb{R}^{n+2} : (x_1, x_2, \cdots, x_{n+1}) \in U \\ \text{and } x_{n+2} &= f(x_1, x_2, \cdots, x_{n+1})\} \end{aligned}$$

Remark

1. For $c \geq 0$, the level set of f at height c is just the set of all points inthe domanin of f over which the graph is at distance c.
2. For $c < 0$, the level set of f at height c is just the set of all points inthe domanin of f under which the graph is at distance $-c$.

Example

consider the function $f; \mathbb{R}^{n+1} \to \mathbb{R}$ defined by $f(x_1, x_2, \cdots, x_{n+1}) = x_1^2 + \cdots + x_{n+1}^2$. The level sets $f^{-1}(c)$ are empty for $c < 0$, consist of a single point (the origin) if $c = 0$, and for $c > 0$ consist of two points if $n = 0$, circle centered at the origin with radius $\sqrt{c}$ if $n = 1$, spheres centered at the origin with radius $\sqrt{c}$ if $n = 2$, etc.

Example

Consider the function $f : \mathbb{R}^3 \to \mathbb{R}$ defined by $f(x, y, z) = x^2 + y^2 + z^2$. Then the level sets for the values of c are as follows:

Value of c	Level set
$c < 0$	empty set
$c = 0$	$\{(0,0)\}$
$c > 0$	sphere of radius $\sqrt{c}$ with center at origin

Sphere $\mathbb{S}^2$

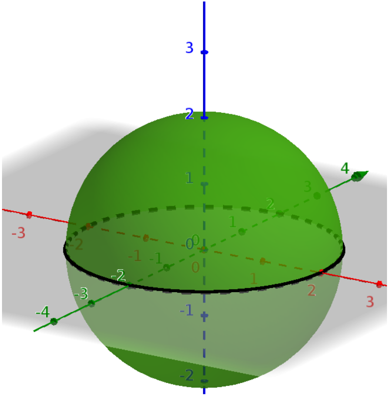


Figure: Level set of $(x, y, z) = x^2 + y^2 + z^2$ at $c = 4$

Example

Consider the function $f : \mathbb{R}^3 \to \mathbb{R}$ defined by $f(x, y, z) = x^2 + y^2 - z^2$. Then the level sets for the values of c are as follows:

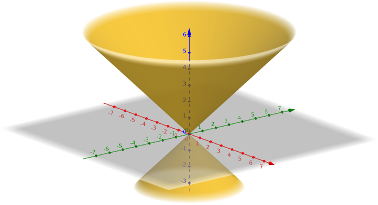


Figure: Level set of $f(x, y, z) = x^2 + y^2 - z^2$ at $c = 0$

Hyperpoloid of one sheet in $\mathbb{R}^3$

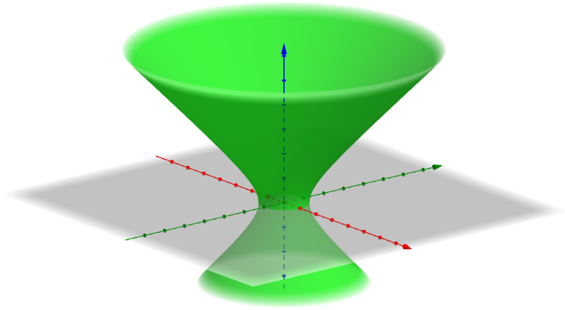


Figure: Level set of $f(x, y, z) = x^2 + y^2 - z^2$ at $c = 1$

Hyperpoloid of two sheet in $\mathbb{R}^3$

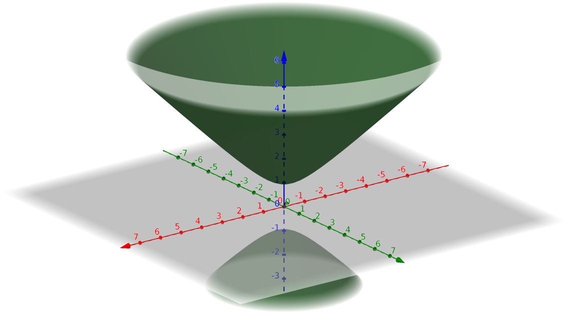


Figure: Level set of $f(x, y, z) = x^2 + y^2 + z^2$ at $c = -1$

Vector Fields

Definition (Vector)

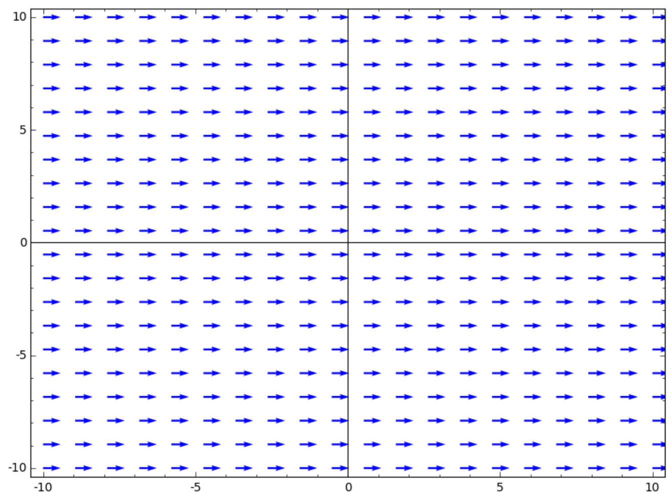
A vector at a point $p \in \mathbb{R}^{n+1}$ is a piar $\overrightarrow{v} = (p, v)$ where $v \in \mathbb{R}^{n+1}$.

Definition

A vector field $\overrightarrow{X}$ on an open set $U \subset \mathbb{R}^{n+1}$ is a function assignes to each point of U a vector at that point. A vector filed is represented by $\overrightarrow{X}(p) = (p, X(p))$ for some function $X : U \to \mathbb{R}^{n+1}$.

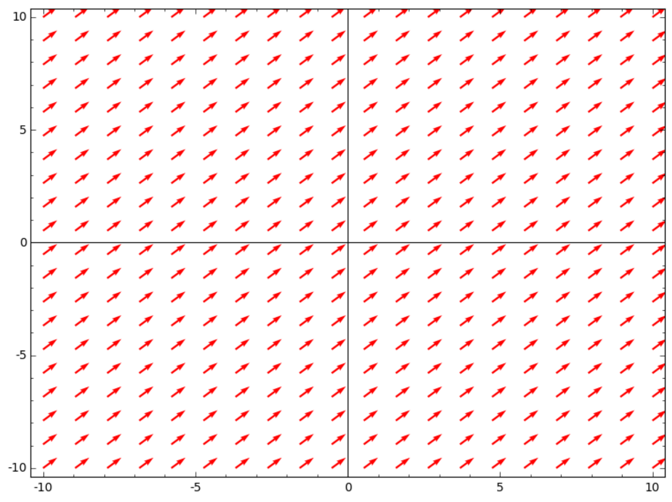
Example $X : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $X(p) = (1, 0)$

Example

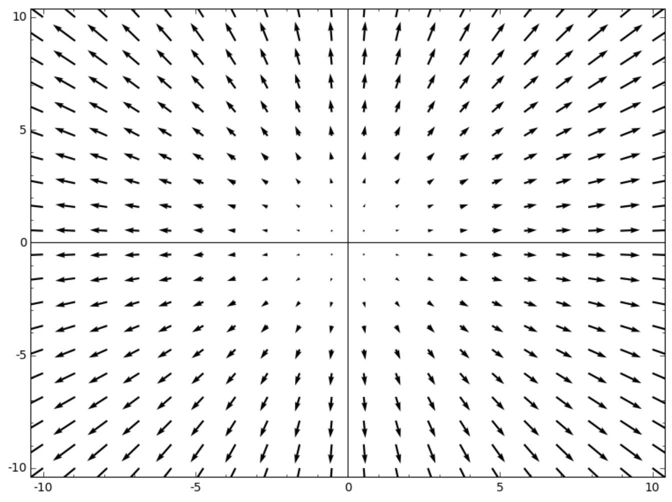


Example $X : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $X(p) = (1, 1)$

Example

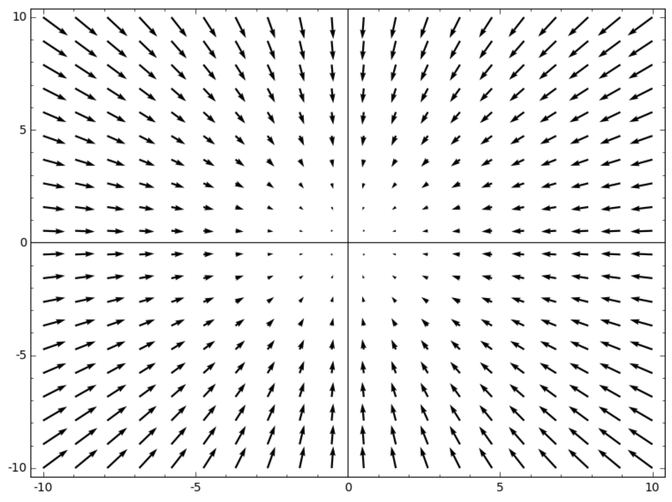


Example

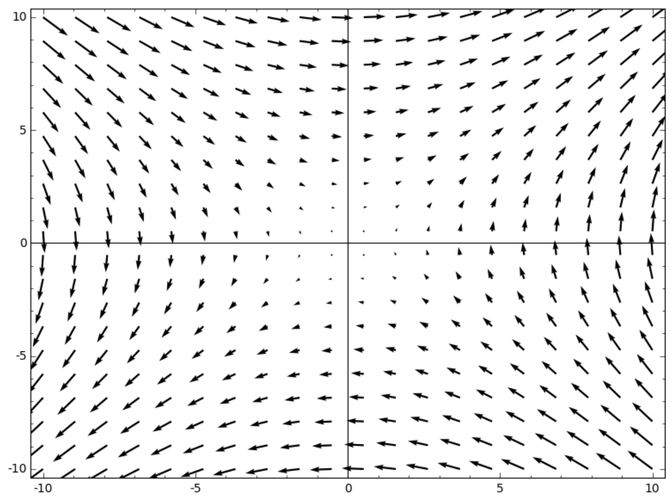


Example $X : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $X(p) = (p, -p)$

Example



Example



Example $X : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $X((x, y)) = ((x, y), (-y, -x))$

Example