UNIT-III

Compactness

Compact Metric Spaces

A metric space (M, d) is said to be **compact** if it is both complete and totally bounded. As you might imagine, a compact space is the best of all possible worlds.

Examples 8.1

1. (a) A subset K of $\mathbb{R}$ is compact if and only if K is closed and bounded. This fact is usually referred to as the Heine–Borel theorem. Hence, a closed bounded interval $[a, b]$ is compact. Also, the Cantor set Δ is compact. The interval $(0, 1)$, on the other hand, is *not* compact.
2. (b) A subset K of $\mathbb{R}^n$ is compact if and only if K is closed and bounded. (Why?)
3. (c) It is important that we not confuse the first two examples with the general case. Recall that the set $\{e_n : n \geq 1\}$ is closed and bounded in ℓ_∞ but not totally bounded – hence not compact. Taking this a step further, notice that the closed ball $\{x : \|x\|_\infty \leq 1\}$ in ℓ_∞ is not compact, whereas any closed ball in $\mathbb{R}^n$ is compact.
4. (d) A subset of a discrete space is compact if and only if it is *finite*. (Why?)

Just as with completeness and total boundedness, we will want to give several equivalent characterizations of compactness. In particular, since neither completeness nor total boundedness is preserved by homeomorphisms, our newest definition does not appear to be describing a topological property.

Compactness is defined in terms of open covers, which we have talked about before in the context of bases but which we formally define here.

Definition *Let $(X, \mathcal{T})$ be a topological space, and let $\mathcal{U} \subseteq \mathcal{T}$ be a collection of open subsets of X. We say $\mathcal{U}$ is an open cover of X if $X = \bigcup \mathcal{U}$.*

If $\mathcal{U}$ is an open cover of X and $\mathcal{V} \subseteq \mathcal{U}$ is a subcollection of $\mathcal{U}$ that is also an open cover of X, we say $\mathcal{V}$ is a subcover of $\mathcal{U}$.

Though the technical term is *open cover*, we will often refer to "covers" since open covers are the only sorts of covers we will discuss.

Compactness is a property in metric spaces. Before discussing the compactness of metric spaces, we must know what a cover, sub cover, and finite is. The definition of compactness is based on these concepts.

Cover of a metric space (X, d) means, for collection **C** = $\{G_\alpha \mid \alpha \in I;\ I \text{ is an index set}\}$ of subsets of X such that $\bigcup_{\alpha \in I} G_\alpha = X$, the **C** is called the cover of X.

If **C’** is a sub-collection of **C**, such that C’ itself covers X then **C’** is said to be a subcover of **C**. If **C’** is a collection of finite elements then it is called finite subcover.

The collection **C** = $\{G_\alpha \mid \alpha \in I$; I is an index set$\}$ is such that every G_α is an open set, then **C** is called open cover of X.

Definition of Compactness

The compactness of a metric space is defined as, let (X, d) be a metric space such that every open cover of X has a finite subcover.

A non-empty set Y of X is said to be compact if it is compact as a metric space.

For example, a finite set in any metric space (X, d) is compact. In particular, a finite subset of a discrete metric (X,d) is compact.

Sequentially Compact: A metric space (X, d) is said to be sequentially compact if every sequence in X has a subsequence that converges in X.

With the concept of compactness and sequential compactness, there is a significant result.

Properties of Compactness

Let (X, d) be a metric space with metric d such that (X, d) is a compact metric space, then

- If Y is a closed subset of X, Y is also compact.
- The union of two compact subsets of a metric space is compact.
- Every compact metric space has the Bolzano-Weierstrass Property (BWP). A metric space is said to have Bolzano-Weierstrass Property if every infinite subset of X has a limit point. That is, every sequence within that infinite subsets converges to a point in it.
- A compact subset of a metric space is closed and bounded.

Example 1:

Prove that the usual metric space (R, d) is not compact.

Solution:

We have to prove that the usual metric space (R, d) is not compact, where R is the set of real numbers.

Consider C = {(-n, n) | n ∈ N}. Then C is an open cover of R as

(i) each element in C is an open set.

(ii) if x ∈ R, then there exist n such that x ∈ (-n, n), so

{x} ⊆ (-n, n), n ∈ N

⇒ ∪ {x} ⊆ ∪ (-n, n)

⇒ R ⊆ ∪ (-n, n)

Again, (-n, n) ⊆ R, ∀ n ∈ N so ∪ (-n, n) ⊆ R

Hence R = ∪ (-n, n)

Now to prove (R, d) is not we shall show that there does not exist any finite sub cover of R.

Let {$(-n_i, n_i)$ | 1 ≤ i ≤ p} be a finite sub-collection of C and let n' = max {n_1, n_2, …, n_p}

Then n' ∈ N ⊆ R be such that

n' ∉ ⋃ $(-n_i, n_i)$

Thus, there is no finite subcollection of C which covers R.

Hence, R is not compact.

Open cover

In topology, an **open cover** of a set S refers to a collection of open sets whose union contains S. More formally, if $\{U_i\}_{i \in I}$ is a collection of open sets in a topological space X, then this collection is called an open cover of $S \subseteq X$ if:

$$S \subseteq \bigcup_{i \in I} U_i$$

Here are some key points regarding open covers:

1. **Open Sets**: The sets U_i in the cover must be open sets in the topology of the space X.
2. **Union**: The union of all the sets in the collection must include every point in the set S.
3. **Finite Subcover**: A related concept is that of a **finite subcover**. If an open cover contains a finite subcollection of open sets that still covers S, then we say that S has a finite subcover. This is an important concept in various topological properties, such as compactness. A set S is called compact if every open cover of S has a finite subcover.

Example

Consider the set $S = [0, 1]$ in the standard topology on the real numbers $\mathbb{R}$. An open cover of S could be the collection of open intervals:

$$\{(-0.1, 0.5), (0.4, 1.1)\}$$

The union of these two sets is $(-0.1, 1.1)$, which contains $[0, 1]$, thus forming an open cover. However, it does not have a finite subcover that is also open and still covers S without exceeding the bounds of S.

Open covers play a vital role in various areas of topology and analysis, particularly in discussing convergence, continuity, and compactness.

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2. A subset K of $\mathbb{R}n$ is compact if and only if K is closed and bounded. (Why?)
3. It is important that we not confuse the first two examples with the general case. Recall that the set $\{e_n : n \geq 1\}$ is closed and bounded in ℓ_∞ but not totally bounded – hence not compact. Taking this a step further, notice that the closed ball $\{x : \|x\|_\infty \leq 1\}$ in ℓ_∞ is not compact, whereas any closed ball in $\mathbb{R}n$ is compact.
4. A subset of a discrete space is compact if and only if it is *finite*.

Definition of the Riemann integral

We say that two intervals are almost disjoint if they are disjoint or intersect only at a common endpoint. For example, the intervals $[0, 1]$ and $[1, 3]$ are almost disjoint, whereas the intervals $[0, 2]$ and $[1, 3]$ are not.

Definition . Let I be a nonempty, compact interval. A partition of I is a finite collection $\{I_1, I_2, \ldots, I_n\}$ of almost disjoint, nonempty, compact subintervals whose union is I.

A partition of $[a, b]$ with subintervals $I_k = [x_{k-1}, x_k]$ is determined by the set of endpoints of the intervals

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$

Abusing notation, we will denote a partition P either by its intervals

$$P = \{I_1, I_2, \ldots, I_n\}$$

or by the set of endpoints of the intervals

$$P = \{x_0, x_1, x_2, \ldots, x_{n-1}, x_n\}.$$

We'll adopt either notation as convenient; the context should make it clear which one is being used. There is always one more endpoint than interval.

Definition . A bounded function $f : [a, b] \to \mathbb{R}$ is Riemann integrable on $[a, b]$ if its upper integral $U(f)$ and lower integral $L(f)$ are equal. In that case, the Riemann integral of f on $[a, b]$, denoted by

$$\int_a^b f(x)\, dx, \qquad \int_a^b f, \qquad \int_{[a,b]} f$$

or similar notations, is the common value of $U(f)$ and $L(f)$.

An unbounded function is not Riemann integrable. In the following, "integrable" will mean "Riemann integrable, and "integral" will mean "Riemann integral" unless stated explicitly otherwise.

Example 1 The constant function $f(x) = 1$ on $[0, 1]$ is Riemann integrable, and

$$\int_0^1 1\,dx = 1.$$

To show this, let $P = \{I_1, I_2, \ldots, I_n\}$ be any partition of $[0, 1]$ with endpoints

$$\{0, x_1, x_2, \ldots, x_{n-1}, 1\}.$$

Since f is constant,

$$M_k = \sup_{I_k} f = 1, \quad m_k = \inf_{I_k} f = 1 \qquad \text{for } k = 1, \ldots, n,$$

and therefore

$$U(f; P) = L(f; P) = \sum_{k=1}^{n} (x_k - x_{k-1}) = x_n - x_0 = 1.$$

Geometrically, this equation is the obvious fact that the sum of the areas of the rectangles over (or, equivalently, under) the graph of a constant function is exactly equal to the area under the graph. Thus, every upper and lower sum of f on $[0, 1]$ is equal to 1, which implies that the upper and lower integrals

$$U(f) = \inf_{P \in \Pi} U(f; P) = \inf\{1\} = 1, \qquad L(f) = \sup_{P \in \Pi} L(f; P) = \sup\{1\} = 1$$

are equal, and the integral of f is 1.

More generally, the same argument shows that every constant function $f(x) = c$ is integrable and

$$\int_a^b c\,dx = c(b - a).$$

The following is an example of a discontinuous function that is Riemann integrable.

Example The function

$$f(x) = \begin{cases} 0 & \text{if } 0 < x \leq 1 \\ 1 & \text{if } x = 0 \end{cases}$$

is Riemann integrable, and

$$\int_0^1 f\,dx = 0.$$

To show this, let $P = \{I_1, I_2, \ldots, I_n\}$ be a partition of $[0, 1]$. Then, since $f(x) = 0$ for $x > 0$,

$$M_k = \sup_{I_k} f = 0, \quad m_k = \inf_{I_k} f = 0 \qquad \text{for } k = 2, \ldots, n.$$

The first interval in the partition is $I_1 = [0, x_1]$, where $0 < x_1 \leq 1$, and

$$M_1 = 1, \quad m_1 = 0,$$

since $f(0) = 1$ and $f(x) = 0$ for $0 < x \leq x_1$. It follows that

$$U(f; P) = x_1, \qquad L(f; P) = 0.$$

Thus, $L(f) = 0$ and

$$U(f) = \inf\{x_1 : 0 < x_1 \leq 1\} = 0,$$

so $U(f) = L(f) = 0$ are equal, and the integral of f is 0. In this example, the infimum of the upper Riemann sums is not attained and $U(f; P) > U(f)$ for every partition P.

A similar argument shows that a function $f : [a, b] \to \mathbb{R}$ that is zero except at finitely many points in $[a, b]$ is Riemann integrable with integral 0.

The next example is a bounded function on a compact interval whose Riemann integral doesn't exist.

Theorem 1 A monotonic function $f : [a, b] \to \mathbb{R}$ on a compact interval is Riemann integrable.

Proof. Suppose that f is monotonic increasing, meaning that $f(x) \leq f(y)$ for $x \leq y$. Let $P_n = \{I_1, I_2, \dots, I_n\}$ be a partition of $[a, b]$ into n intervals $I_k = [x_{k-1}, x_k]$, of equal length $(b - a)/n$, with endpoints

$$x_k = a + (b - a)\frac{k}{n}, \qquad k = 0, 1, \dots, n - 1, n.$$

Since f is increasing,

$$M_k = \sup_{I_k} f = f(x_k), \qquad m_k = \inf_{I_k} f = f(x_{k-1}).$$

Hence, summing a telescoping series, we get

Hence, summing a telescoping series, we get

$$\begin{aligned} U(f; P_n) - L(U; P_n) &= \sum_{k=1}^{n} (M_k - m_k)(x_k - x_{k-1}) \\ &= \frac{b-a}{n} \sum_{k=1}^{n} [f(x_k) - f(x_{k-1})] \\ &= \frac{b-a}{n} [f(b) - f(a)]. \end{aligned}$$

It follows that $U(f; P_n) - L(U; P_n) \to 0$ as $n \to \infty$, and implies that f is integrable.

The proof for a monotonic decreasing function f is similar, with

$$\sup_{I_k} f = f(x_{k-1}), \qquad \inf_{I_k} f = f(x_k),$$

or we can apply the result for increasing functions to $-f$ and use below.

Monotonic functions needn't be continuous, and they may be discontinuous at a countably infinite number of points.

Properties of the Riemann integral

The integral has the following three basic properties.

(1) Linearity:

$$\int_a^b cf = c\int_a^b f, \qquad \int_a^b (f+g) = \int_a^b f + \int_a^b g.$$

(2) Monotonicity: if $f \leq g$, then

$$\int_a^b f \leq \int_a^b g.$$

(3) Additivity: if $a < c < b$, then

$$\int_a^c f + \int_c^b f = \int_a^b f.$$

In this section, we prove these properties and derive a few of their consequences.

These properties are analogous to the corresponding properties of sums (or convergent series):

$$\sum_{k=1}^n ca_k = c\sum_{k=1}^n a_k, \qquad \sum_{k=1}^n (a_k+b_k) = \sum_{k=1}^n a_k + \sum_{k=1}^n b_k;$$

$$\sum_{k=1}^n a_k \leq \sum_{k=1}^n b_k \qquad \text{if } a_k \leq b_k;$$

$$\sum_{k=1}^m a_k + \sum_{k=m+1}^n a_k = \sum_{k=1}^n a_k.$$

Linearity. We begin by proving the linearity. First we prove linearity with respect to scalar multiplication and then linearity with respect to sums.

Theorem If $f : [a,b] \to \mathbb{R}$ is integrable and $c \in \mathbb{R}$, then cf is integrable and

$$\int_a^b cf = c\int_a^b f.$$

Proof. Suppose that $c \geq 0$. Then for any set $A \subset [a,b]$, we have

$$\sup_A cf = c\sup_A f, \qquad \inf_A cf = c\inf_A f,$$

so $U(cf;P) = cU(f;P)$ for every partition P. Taking the infimum over the set Π of all partitions of $[a,b]$, we get

$$U(cf) = \inf_{P\in\Pi} U(cf;P) = \inf_{P\in\Pi} cU(f;P) = c\inf_{P\in\Pi} U(f;P) = cU(f).$$

Similarly, $L(cf;P) = cL(f;P)$ and $L(cf) = cL(f)$. If f is integrable, then

$$U(cf) = cU(f) = cL(f) = L(cf),$$

which shows that cf is integrable and

$$\int_a^b cf = c\int_a^b f.$$

Now consider $-f$. Since

$$\sup_A(-f) = -\inf_A f, \qquad \inf_A(-f) = -\sup_A f,$$

we have

$$U(-f;P) = -L(f;P), \qquad L(-f;P) = -U(f;P).$$

Therefore

$$U(-f) = \inf_{P\in\Pi} U(-f;P) = \inf_{P\in\Pi} [-L(f;P)] = -\sup_{P\in\Pi} L(f;P) = -L(f),$$
$$L(-f) = \sup_{P\in\Pi} L(-f;P) = \sup_{P\in\Pi} [-U(f;P)] = -\inf_{P\in\Pi} U(f;P) = -U(f).$$

Hence, $-f$ is integrable if f is integrable and

$$\int_a^b (-f) = -\int_a^b f.$$

Finally, if $c < 0$, then $c = -|c|$, and a successive application of the previous results shows that cf is integrable with $\int_a^b cf = c\int_a^b f$. □

Next, we prove the linearity of the integral with respect to sums. If f, g are bounded, then $f+g$ is bounded and

$$\sup_I (f+g) \le \sup_I f + \sup_I g, \qquad \inf_I (f+g) \ge \inf_I f + \inf_I g.$$

It follows that

$$\operatorname{osc}_I (f+g) \le \operatorname{osc}_I f + \operatorname{osc}_I g,$$

so $f+g$ is integrable if f, g are integrable. In general, however, the upper (or lower) sum of $f+g$ needn't be the sum of the corresponding upper (or lower) sums of f and g. As a result, we don't get

$$\int_a^b (f+g) = \int_a^b f + \int_a^b g$$

simply by adding upper and lower sums. Instead, we prove this equality by estimating the upper and lower integrals of $f+g$ from above and below by those of f and g.

Monotonicity. Next, we prove the monotonicity of the integral.

Theorem Suppose that $f, g : [a,b] \to \mathbb{R}$ are integrable and $f \le g$. Then

$$\int_a^b f \le \int_a^b g.$$

Proof. First suppose that $f \ge 0$ is integrable. Let P be the partition consisting of the single interval $[a,b]$. Then

$$L(f;P) = \inf_{[a,b]} f \cdot (b-a) \ge 0,$$

so

$$\int_a^b f \ge L(f;P) \ge 0.$$

If $f \ge g$, then $h = f - g \ge 0$, and the linearity of the integral implies that

$$\int_a^b f - \int_a^b g = \int_a^b h \ge 0,$$

which proves the theorem.

Additivity. Finally, we prove additivity. This property refers to additivity with respect to the interval of integration, rather than linearity with respect to the function being integrated.

Theorem Suppose that $f : [a, b] \to \mathbb{R}$ and $a < c < b$. Then f is Riemann integrable on $[a, b]$ if and only if it is Riemann integrable on $[a, c]$ and $[c, b]$. In that case,

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

Proof. Suppose that f is integrable on $[a, b]$. Then, given $\epsilon > 0$, there is a partition P of $[a, b]$ such that $U(f; P) - L(f; P) < \epsilon$. Let $P' = P \cup \{c\}$ be the refinement of P obtained by adding c to the endpoints of P. (If $c \in P$, then $P' = P$.) Then $P' = Q \cup R$ where $Q = P' \cap [a, c]$ and $R = P' \cap [c, b]$ are partitions of $[a, c]$ and $[c, b]$ respectively. Moreover,

$$U(f; P') = U(f; Q) + U(f; R), \qquad L(f; P') = L(f; Q) + L(f; R).$$

It follows that

$$\begin{aligned} U(f; Q) - L(f; Q) &= U(f; P') - L(f; P') - [U(f; R) - L(f; R)] \\ &\leq U(f; P) - L(f; P) < \epsilon, \end{aligned}$$

which proves that f is integrable on $[a, c]$. Exchanging Q and R, we get the proof for $[c, b]$.

Conversely, if f is integrable on $[a, c]$ and $[c, b]$, then there are partitions Q of $[a, c]$ and R of $[c, b]$ such that

$$U(f; Q) - L(f; Q) < \frac{\epsilon}{2}, \qquad U(f; R) - L(f; R) < \frac{\epsilon}{2}.$$

Let $P = Q \cup R$. Then

$$U(f; P) - L(f; P) = U(f; Q) - L(f; Q) + U(f; R) - L(f; R) < \epsilon,$$

which proves that f is integrable on $[a, b]$.

Finally, with the partitions P, Q, R as above, we have

$$\begin{aligned}\int_a^b f \le U(f;P) &= U(f;Q) + U(f;R)\\ &< L(f;Q) + L(f;R) + \epsilon\\ &< \int_a^c f + \int_c^b f + \epsilon.\end{aligned}$$

Similarly,

$$\begin{aligned}\int_a^b f \ge L(f;P) &= L(f;Q) + L(f;R)\\ &> U(f;Q) + U(f;R) - \epsilon\\ &> \int_a^c f + \int_c^b f - \epsilon.\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we see that $\int_a^b f = \int_a^c f + \int_c^b f$.

We can extend the additivity property of the integral by defining an oriented Riemann integral.

Proposition 1 Suppose that $f : [a, b] \to \mathbb{R}$ is bounded and integrable on $[a, r]$ for every $a < r < b$. Then f is integrable on $[a, b]$ and

$$\int_a^b f = \lim_{r \to b^-} \int_a^r f.$$

Proof. Since f is bounded, $|f| \le M$ on $[a, b]$ for some $M > 0$. Given $\epsilon > 0$, let

$$r = b - \frac{\epsilon}{4M}$$

(where we assume ϵ is sufficiently small that $r > a$). Since f is integrable on $[a, r]$, there is a partition Q of $[a, r]$ such that

$$U(f;Q) - L(f;Q) < \frac{\epsilon}{2}.$$

Then $P = Q \cup \{b\}$ is a partition of $[a, b]$ whose last interval is $[r, b]$. The boundedness of f implies that

$$\sup_{[r,b]} f - \inf_{[r,b]} f \leq 2M.$$

Therefore

$$U(f; P) - L(f; P) = U(f; Q) - L(f; Q) + \left(\sup_{[r,b]} f - \inf_{[r,b]} f\right) \cdot (b - r)$$
$$< \frac{\epsilon}{2} + 2M \cdot (b - r) = \epsilon,$$

so f is integrable on $[a, b]$ by Theorem 1.14. Moreover, using the additivity of the integral, we get

$$\left| \int_a^b f - \int_a^r f \right| = \left| \int_r^b f \right| \leq M \cdot (b - r) \to 0 \qquad \text{as } r \to b^-.$$

An obvious analogous result holds for the left endpoint.

Fundamental theorem I. First we prove the statement about the integral of a derivative.

Theorem 1 (Fundamental theorem of calculus I). If $F : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable in (a, b) with $F' = f$ where $f : [a, b] \to \mathbb{R}$ is Riemann integrable, then

$$\int_a^b f(x)\, dx = F(b) - F(a).$$

Proof. Let

$$P = \{a = x_0, x_1, x_2, \ldots, x_{n-1}, x_n = b\}$$

be a partition of $[a, b]$. Then

$$F(b) - F(a) = \sum_{k=1}^{n} \left[F(x_k) - F(x_{k-1})\right].$$

The function F is continuous on the closed interval $[x_{k-1}, x_k]$ and differentiable in the open interval (x_{k-1}, x_k) with $F' = f$. By the mean value theorem, there exists $x_{k-1} < c_k < x_k$ such that

$$F(x_k) - F(x_{k-1}) = f(c_k)(x_k - x_{k-1}).$$

Since f is Riemann integrable, it is bounded, and it follows that

$$m_k(x_k - x_{k-1}) \leq F(x_k) - F(x_{k-1}) \leq M_k(x_k - x_{k-1}),$$

where

$$M_k = \sup_{[x_{k-1},x_k]} f, \qquad m_k = \inf_{[x_{k-1},x_k]} f.$$

Hence, $L(f;P) \leq F(b) - F(a) \leq U(f;P)$ for every partition P of $[a,b]$, which implies that $L(f) \leq F(b) - F(a) \leq U(f)$. Since f is integrable, $L(f) = U(f)$ and $F(b) - F(a) = \int_a^b f$.

Improper integrals. First, we define the improper integral of a function that fails to be integrable at one endpoint of a bounded interval.

Definition Suppose that $f : (a,b] \to \mathbb{R}$ is integrable on $[c,b]$ for every $a < c < b$. Then the improper integral of f on $[a,b]$ is

$$\int_a^b f = \lim_{\epsilon \to 0^+} \int_{a+\epsilon}^b f.$$

The improper integral converges if this limit exists (as a finite real number), otherwise it diverges. Similarly, if $f : [a,b) \to \mathbb{R}$ is integrable on $[a,c]$ for every $a < c < b$, then

$$\int_a^b f = \lim_{\epsilon \to 0^+} \int_a^{b-\epsilon} f.$$

We use the same notation to denote proper and improper integrals; it should be clear from the context which integrals are proper Riemann integrals (i.e., ones given by Definition 1.3) and which are improper. If f is Riemann integrable on $[a,b]$, then Proposition 1.39 shows that its improper and proper integrals agree, but an improper integral may exist even if f isn't integrable.

Example 1 If $p > 0$, the integral

$$\int_0^1 \frac{1}{x^p}\,dx$$

isn't defined as a Riemann integral since $1/x^p$ is unbounded on $(0,1]$. The corresponding improper integral is

$$\int_0^1 \frac{1}{x^p}\,dx = \lim_{\epsilon \to 0^+} \int_\epsilon^1 \frac{1}{x^p}\,dx.$$

For $p \neq 1$, we have

$$\int_\epsilon^1 \frac{1}{x^p}\, dx = \frac{1-\epsilon^{1-p}}{1-p},$$

so the improper integral converges if $0 < p < 1$, with

$$\int_0^1 \frac{1}{x^p}\, dx = \frac{1}{p-1},$$

and diverges to ∞ if $p > 1$. The integral also diverges (more slowly) to ∞ if $p = 1$ since

$$\int_\epsilon^1 \frac{1}{x}\, dx = \ln \frac{1}{\epsilon}.$$

Thus, we get a convergent improper integral if the integrand $1/x^p$ does not grow too rapidly as $x \to 0^+$ (slower than $1/x$).

We define improper integrals on an unbounded interval as limits of integrals on bounded intervals.

Example Suppose $p > 0$. The improper integral

$$\int_1^\infty \frac{1}{x^p}\, dx = \lim_{r\to\infty} \int_1^r \frac{1}{x^p}\, dx = \lim_{r\to\infty} \left(\frac{r^{1-p}-1}{1-p}\right)$$

converges to $1/(p-1)$ if $p > 1$ and diverges to ∞ if $0 < p < 1$. It also diverges (more slowly) if $p = 1$ since

$$\int_1^\infty \frac{1}{x}\, dx = \lim_{r\to\infty} \int_1^r \frac{1}{x}\, dx = \lim_{r\to\infty} \ln r = \infty.$$

Thus, we get a convergent improper integral if the integrand $1/x^p$ decays sufficiently rapidly as $x \to \infty$ (faster than $1/x$).

A divergent improper integral may diverge to ∞ (or $-\infty$) as in the previous examples, or — if the integrand changes sign — it may oscillate.

Theorem *A sequence $\{f_n\}$ converges to f with respect to the metric of $\mathcal{C}(X)$ if and only if $f_n \to f$ uniformly on X.*

Proof. *Necessary condition:* Suppose $f_n \to f$ with respect to the metric d of $\mathcal{C}(X)$.
Therefore, for every $\epsilon > 0$ there exist $n \in \mathbb{N}$ such that

$$d(f_n, f) < \epsilon, \quad \forall\ n \geq N.$$

That is

$$||f_n - f|| < \epsilon, \quad \forall n \geq N,$$

or,

$$\sup_{x \in X} |f_n(x) - f(x)| < \epsilon, \quad \forall n \geq N,$$

which shows

$$|f_n(x) - f(x)| < \epsilon, \quad \forall n \geq N \text{ and } x \in X.$$

Therefore, $f_n \to f$ uniformly on X.

Sufficient condition: Suppose $f_n \to f$ uniformly.

For a given $\epsilon > 0$, there exist a k such that

$$|f_n(x) - f(x)| \leq \epsilon, \quad \forall x \in X \text{ and } n \geq k$$

Therefore,

$$\sup_{x \in X} |f_n(x) - f(x)| \leq \epsilon, \quad \forall n \geq k$$

Hence,

$$||f_n - f|| \quad \leq \quad \epsilon, \quad \forall n \geq k,$$

that is

$$d(f_n, f) \quad \leq \quad \epsilon, \quad \forall n \geq k.$$

Hence, $f_n \to f$ with respect to the metric of $\mathcal{C}(X)$ and follows the desired result.

Darboux's Theorem:

Darboux's Theorem states that if a function is differentiable on an interval, its derivative takes on every value between the derivatives at the endpoints of any subinterval. The proof involves defining a helper function g(t) = u(t) - f(t), where u is a constant between f'(c) and f'(d). This function is shown to achieve a maximum value at some point x in the interval [c, d], and by Fermat's theorem on stationary points, its derivative must be zero there, which leads to the conclusion that f'(x) = u.

Let I be an open interval and let $f: I \to \mathbb{R}$ be a differentiable function. If a and b are points of I with $a \leq b$ and if y lies between $f'(a)$ and $f'(b)$ then there exists a number x in [a, b] such that $f'(x) = y$

Proof:-

We may clearly assume that y lies strictly between $f'(a)$ and $f'(b)$. Continuous functions $f_a, f_b : I \to \mathbb{R}$ by

$$f_a(t) = \begin{cases} f'(a) & \text{for } t = a \\ \dfrac{f(a) - f(t)}{a - t} & \text{for } t \neq a \end{cases},$$

and

$$f_b(t) = \begin{cases} f'(b) & \text{for } t = b \\ \dfrac{f(t) - f(b)}{t - b} & \text{for } t \neq b \end{cases}$$

that

$f_a(a) = f'(a)$, $f_a(b) = f_b(a)$ and

$f_a(a) = f'(a)$, $f_a(b) = f_b(a)$ and

$f_b(b) = f'(b)$

hence

y lies between $f_a(a)$ and $f_a(b)$ or y lies b/w $f_b(a)$ and $f_b(b)$

If y lies b/w $f_a(a)$ and $f_a(b)$, then there exists s in (a, b) with

$$y = f_a(s) = \frac{f(s) - f(a)}{s-a}$$

The mean value theorem ensures that there exists x in $[a,s]$ such that

$$y = \frac{f(s) - f(a)}{s-a} = f'(x)$$

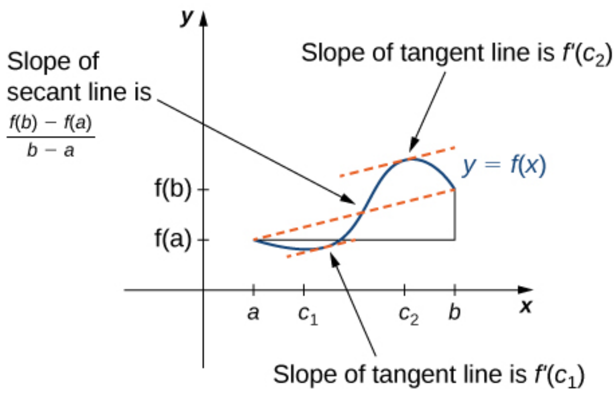
If y lies b/w $f_b(a)$ and $f_b(b)$ then an analogous argument (exploiting the continuity of f_b) shows that there exists s in $[a,b]$ and x in $[s,b]$ such that

$$y = \frac{f(b) - f(s)}{b-s} = f'(x)$$

This completes the proof.

The Mean Value Theorem and Its Meaning

Rolle's theorem is a special case of the Mean Value Theorem. In Rolle's theorem, we consider differentiable functions f that are zero at the endpoints. The Mean Value Theorem generalizes Rolle's theorem by considering functions that are not necessarily zero at the endpoints. Consequently, we can view the Mean Value Theorem as a slanted version of Rolle's theorem The Mean Value Theorem states that if f is continuous over the closed interval $[a, b]$ and differentiable over the open interval (a, b), then there exists a point $c \in (a, b)$ such that the tangent line to the graph of f at c is parallel to the secant line connecting $(a, f(a))$ and $(b, f(b))$.



Let f be continuous over the closed interval $[a, b]$ and differentiable over the open interval (a, b). Then, there exists at least one point $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

The proof follows from Rolle's theorem by introducing an appropriate function that satisfies the criteria of Rolle's theorem. Consider the line connecting $(a, f(a))$ and $(b, f(b))$. Since the slope of that line is

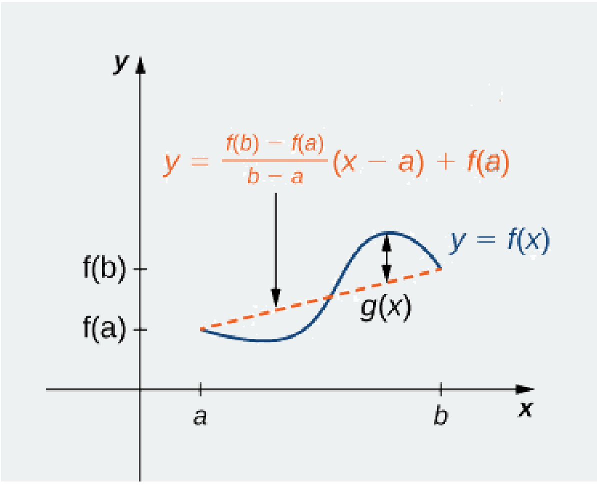
$$\frac{f(b) - f(a)}{b - a}$$

and the line passes through the point $(a, f(a))$, the equation of that line can be written as

$$y = \frac{f(b) - f(a)}{b - a}(x - a) + f(a).$$

Let $g(x)$ denote the vertical difference between the point $(x, f(x))$ and the point (x, y) on that line. Therefore,

$$g(x) = f(x) - \left[\frac{f(b) - f(a)}{b - a}(x - a) + f(a)\right].$$



Since the graph of f intersects the secant line when $x = a$ and $x = b$, we see that $g(a) = 0 = g(b)$. Since f is a differentiable function over (a, b), g is also a differentiable function over (a, b). Furthermore, since f is continuous over $[a, b]$, g is also continuous over $[a, b]$. Therefore, g satisfies the criteria of Rolle's theorem. Consequently, there exists a point $c \in (a, b)$ such that $g'(c) = 0$. Since

$$g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a},$$

we see that

$$g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}.$$

Since $g'(c) = 0$, we conclude that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Hence the proof.