UNIT- II

Continuity Definition

A function is said to be continuous in a given interval if there is no break in the graph of the function in the entire interval range. Assume that "f" is a real function on a subset of the real numbers and "c" is a point in the domain of f. Then f is continuous at c if.

$$\lim_{x \to c} f(x) = f(c)$$

In other words, if the left-hand limit, right-hand limit and the value of the function at x = c exist and are equal to each other, i.e.

$$\lim_{x \to c^-} f(x) = f(c) = \lim_{x \to c^+} f(x)$$

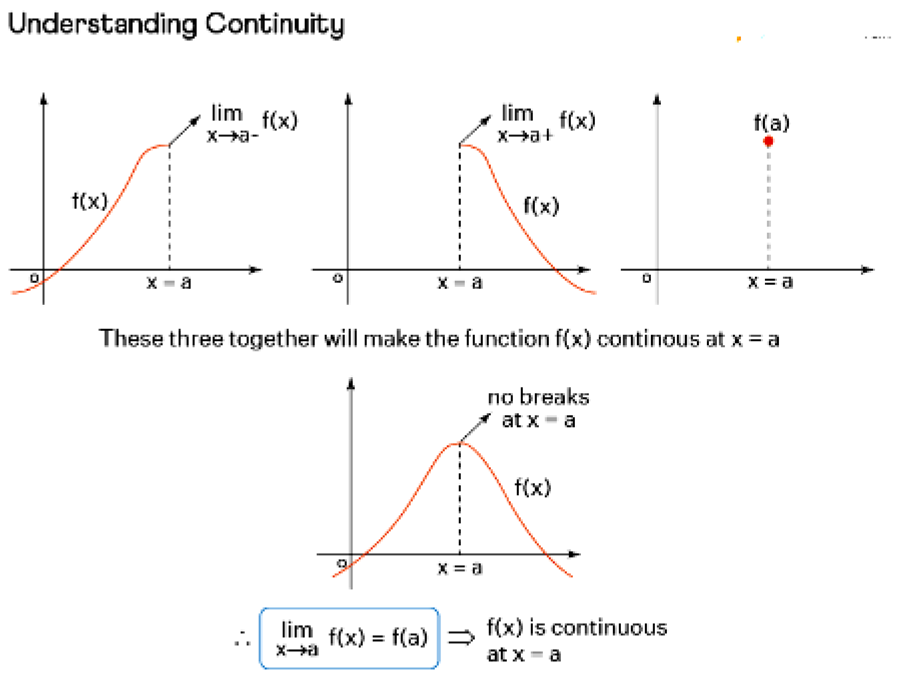
Then f is said to be continuous at x = c

What is Continuous Function?

A function f(x) is said to be a **continuous function** in calculus at a point x = a if the curve of the function does NOT break at the point x = a. The mathematical definition of the continuity of a function is as follows. A function f(x) is continuous at a point x = a if

- f(a) exists;
- $\lim_{x \to a} f(x)$ exists;
  [i.e., $\lim_{x \to a-} f(x) = \lim_{x \to a+} f(x)$] and
- Both of the above values are equal. i.e., $\mathbf{\lim_{x \to a} f(x) = f(a)}$.

Is this definition really giving the meaning that the function shouldn't have a break at x = a? Let's see. "$\lim_{x \to a} f(x)$ exists" means, the function should approach the same value both from the left side and right side of the value x = a and "$\lim_{x \to a} f(x) = f(a)$" means the limit of the function at x = a is same as f(a). These two conditions together will make the function to be continuous (without a break) at that point. You can understand this from the following figure.



A function is said to be continuous over an interval if it is continuous at each and every point on the interval. i.e., over that interval, the graph of the function shouldn't break or jump.

Conditions for Continuity

- A function "f" is said to be continuous in an open interval (a, b) if it is continuous at every point in this interval.
- A function "f" is said to be continuous in a closed interval [a, b] if
    - f is continuous in (a, b)
    - $\lim_{x \to a^+} f(x) = f(a)$
    - $\lim_{x \to b^-} f(x) = f(b)$

Discontinuity Definition

The function "f" will be discontinuous at x = a in any of the following cases:

- f (a) is not defined.
- $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ exist but are not equal.
- $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ exist and are equal but not equal to f (a).

Types of Discontinuity

The four different types of discontinuities are:

- Removable Discontinuity
- Jump Discontinuity

- Infinite Discontinuity

Let's discuss the different types of discontinuity in detail.

Jump Discontinuity

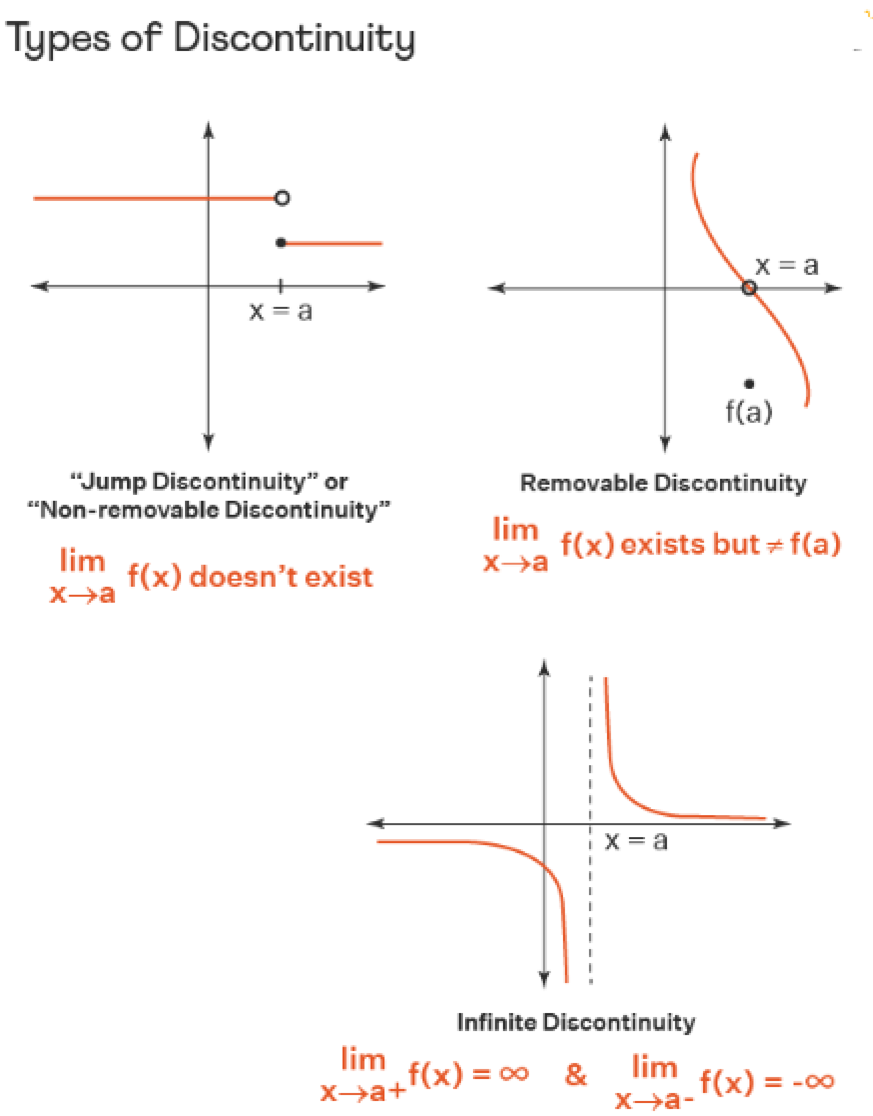
$\lim_{x \to a-} f(x)$ and $\lim_{x \to a+} f(x)$ exist but they are NOT equal. It is called "jump discontinuity" (or) "non-removable discontinuity".

Removable Discontinuity

$\lim_{x \to a} f(x)$ exists (i.e., $\lim_{x \to a-} f(x) = \lim_{x \to a+} f(x)$) but it is NOT equal to f(a). It is called "removable discontinuity".

Infinite Discontinuity

The values of one or both of the limits $\lim_{x \to a-} f(x)$ and $\lim_{x \to a+} f(x)$ is $\pm \infty$. It is called "infinite discontinuity".



Properties of Continuity

Here are some properties of continuity of a function. If two functions f(x) and g(x) are continuous at x = a then

- f + g, f - g, and fg are continuous at x = a.
- f/g is also continuous at x = a provided $g(a) \neq 0$.
- If f is continuous at g(a), then the composition function (f o g) is also continuous at x = a.
- All polynomial functions are continuous over the set of all real numbers.
- The absolute value function |x| is continuous over the set of all real numbers.
- Exponential functions are continuous at all real numbers.
- The functions sin x and cos x are continuous at all real numbers.
- The functions tan x, cosec x, sec x, and cot x are continuous on their respective domains.
- The functions like log x, ln x, √x, etc are continuous on their respective domains.

Continuity of Composite Functions

If the function u = f(x) is continuous at the point x = a, and the function y = g(u) is continuous at the point u = f(a), then the composite function y = g(x) = g(f(x)) is continuous at the point x = a.

Continuity

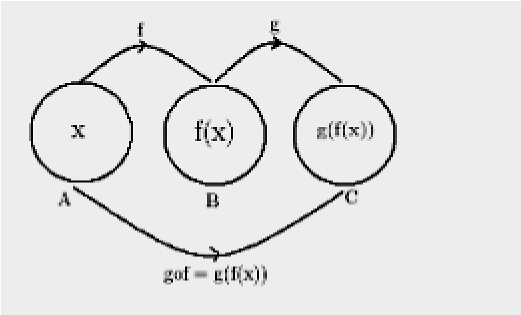
A function is continuous at a point if its graph can be drawn without lifting the pen. Formally, a function f(x) is continuous at a point 'c' if the limit of f(x) as x approaches 'c' exists, is equal to f(c), and is finite.

Composition

The composition of two functions, f(g(x)), means applying the function g first to an input x, and then applying the function f to the result of g(x).

Composition of function

Let $f : A \rightarrow B$ and $g : B \rightarrow C$ be two functions. Then the composition of f and g is denoted by gof and defined as the function gof : $A \rightarrow C$ given by $g \circ f(x) = g(f(x))$



If the function $f(x)$ is continuous at the point $x = a$ and the function $y = g(x)$ is continuous at the point $x = f(a)$, then the composite function $y = (g \circ f)(x) = g(f(x))$ is continuous at the point $x = a$.
Consider the function $f(x) = \frac{1}{1-x}$, which is discontinuous at $x = 1$
If $g(x) = f(f(x))$
$g(x)$ will not be defined when $f(x)$ is not defined, so $g(x)$ is discontinuous at $x = 1$
Also $g(x) = f(f(x))$ is discontinuous when $f(x) = 1$
i.e. $\frac{1}{1-x} = 1 \Rightarrow x = 0$

We ca check it by finding $g(x)$, $g(x) = \frac{1}{1-f(x)} = \frac{1}{1-\frac{1}{1-x}} = \frac{x-1}{x}$
It is discontinuous at $x = 0$
So, $g(x) = f(f(x))$ is discontinuous at $x = 0$ and $x = 1$

Now consider,

$$h(x) = f(f(f(x))) = f\left(\frac{x-1}{x}\right)$$
$$= \frac{1}{1 - \frac{x-1}{x}} = x$$

seems to be continuous, but it is discontinuous at $x = 1$ and $x = 0$ where $f(x)$ and $f(f(x))$ respectively are not defined.

Equivalent Conditions for Continuity

1. **Epsilon-Delta Definition**

   This is the standard definition. A function f is continuous at a point 'c' if for every ε > 0, there exists a δ > 0 such that if |x - c| < δ, then |f(x) - f(c)| < ε.

2. **Sequential Continuity**

   A function f is continuous at a point 'c' if for every sequence {x_n} converging to 'c', the sequence {f(x_n)} converges to f(c).

3. **Open Set Definition**

   A function f is continuous at a point 'c' if for every open set V containing f(c), there exists an open set U containing 'c' such that f(U) is a subset of V.

4. **Closed Set Definition**
   A function f is continuous at a point 'c' if for every closed set F containing f(c), there exists a closed set C containing 'c' such that f(C) is a subset of F.

Example

Consider the function $f(x) = x^2$. Let's verify its continuity at $x = 2$ using the sequential definition.

- Let {x_n} be a sequence converging to 2.
- We have lim (n->infinity) x_n = 2.
- Then, lim (n->infinity) f(x_n) = lim (n->infinity) x_n^2 = (lim (n->infinity) $x\_n)^2 = 2^2 = 4$.
- Since $f(2) = 2^2 = 4$, the sequence {f(x_n)} converges to f(2).
- Therefore, $f(x) = x^2$ is sequentially continuous (and thus continuous) at x = 2.

Algebra of Continuous Functions

Algebra of continuous functions is defined for the four arithmetic operations:

- Addition of continuous functions
- Subtraction of continuous functions
- Multiplication of continuous functions
- Division of continuous functions

If two functions are continuous at a point, then the algebraic operations between two functions are also continuous. Let us understand the algebra of continuous functions with the respective theorem and proof. Also, we will solve examples to understand the concept better.

Addition of Continuous Functions

Theorem: Let us say, f and g are two real functions that are continuous at a point 'a', where 'a' is a real number. Then the addition of the two functions f and g is also continuous at 'a'.

f(x) + g(x) is continuous at x = a

Proof: Given,

$\lim_{x\to a} f(x) = f(a)$

$\lim_{x\to a} g(x) = g(a)$

Now as per the theorem,

$\lim_{x\to a} (f+g)(x) \Rightarrow \lim_{x\to c} [f(x) + g(x)]$

$\Rightarrow \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$

$\Rightarrow f(a) + g(a)$

$\Rightarrow (f + g)(a)$

Therefore,

$\lim_{x \to a} (f+g)(x) = (f + g)(c)$

Hence, f+g is continuous at x = a.

Subtraction of Continuous Functions

Theorem: Let us say, f and g are two real functions that are continuous at a point ‘a’, where ‘a’ is a real number. Then the subtraction of the two functions f and g is also continuous at ‘a’.

f(x) – g(x) is continuous at x = a

Proof: Given,

$\lim_{x \to a} f(x) = f(a)$

$\lim_{x \to a} g(x) = g(a)$

Now as per the theorem,

$\lim_{x \to a} (f - g)(x) \Rightarrow \lim_{x \to c} [f(x) - g(x)]$

$\Rightarrow \lim_{x \to c} f(x) - \lim_{x \to c} g(x)$

$\Rightarrow f(a) - g(a)$

$\Rightarrow (f - g)(a)$

Therefore,

$\lim_{x \to a} (f - g)(x) = (f - g)(c)$

Hence, f – g is continuous at x = a.

Multiplication of Continuous Functions

Theorem: If f and g are two real functions that are continuous at a point ‘a’, where ‘a’ is a real number. Then the product of the two functions f and g is also continuous at ‘a’.

f(x) . g(x) is continuous at x = a

Proof: Given,

$\lim_{x \to a} f(x) = f(a)$

$\lim_{x \to a} g(x) = g(a)$

So, the limit of product of two functions, f and g at x is given by:

$\lim_{x \to a} (f \, . \, g)(x) \Rightarrow \lim_{x \to c} [f(x) \, . \, g(x)]$

$\Rightarrow \lim_{x \to c} f(x) \, . \, \lim_{x \to c} g(x)$

$\Rightarrow f(a) \, . \, g(a)$

$\Rightarrow (f \, . \, g)(a)$

Therefore,

$\lim_{x \to a} (f \, . \, g)(x) = (f \, . \, g)(c)$

Hence, f . g is continuous at x = a.

Division of Continuous Function

Theorem: Suppose, f and g are two real functions that are continuous at a point ‘a’, where ‘a’ is a real number. Then the division of the two functions f and g will remain continuous at ‘a’.

f(x) ÷ g(x) is continuous at x = a

Proof: Given,

$\lim_{x \to a} f(x) = f(a)$

$\lim_{x \to a} g(x) = g(a)$

Now as per the theorem,

$\lim_{x \to a} (f+g)(x) \Rightarrow \lim_{x \to c} [f(x) \div g(x)]$

$\Rightarrow \lim_{x \to c} f(x) \div \lim_{x \to c} g(x)$

$\Rightarrow f(a) \div g(a)$

$\Rightarrow (f \div g)(a)$

Therefore,

$\lim_{x \to a} (f \div g)(x) = (f \div g)(c)$

Hence, f ÷ g is continuous at x = a.

Homeomorphism

In real analysis, a homeomorphism is a special type of function between two topological spaces that is continuous, bijective (one-to-one and onto), and has a continuous inverse. Essentially, it's a continuous deformation that preserves the topological properties of the spaces, meaning they are topologically equivalent. This means that if two spaces are homeomorphic, they are indistinguishable from a topological viewpoint.

Examples:

- **x^3:** The function $f(x) = x^3$ is a homeomorphism on the real numbers.
- **$x + \sin(x)$:** The function $g(x) = x + \sin(x)$ is also a homeomorphism according to a Mathematics Stack Exchange post.
- **Circle and Square:** A circle and a square are homeomorphic because you can continuously deform one into the other without tearing or gluing.
- **Torus and Coffee Mug:** A torus (donut shape) and a coffee mug are homeomorphic.

Homeomorphism in mathematics, a correspondence between two figures or surfaces or other geometrical objects, defined by a one-to-one mapping that is continuous in both directions. The vertical projection shown in the figure sets up such a one-to-one correspondence between the straight segment *x* and the curved interval *y*. If *x* and *y* is topologically equivalent, there is a function $h: x \to y$ such that *h* is continuous, *h* is onto (each point of *y* corresponds to a point of *x*), *h* is one-to-one, and the inverse function, h^{-1}, is continuous. Thus *h* is called a homeomorphism.

Definition Let (X,d) and (Y,e) be metric spaces. A bijective mapping f is called a *homeomorphism* if both f and f^{-1} are continuous. If such mapping exists (X,d) and (Y,e) are *homeomorphic*.

Theorem Let (X,d) and (Y,e) be homeomorphic metric spaces, and f a homeomorphism. Then A is nowhere dense in X if and only if $f(A)$ is nowhere dense in Y.

Proof. We first we make a general observation. Since f is a homeomorphism we know that a is a limit point of A if and only if $f(a)$ is a limit point of $f(A)$. Thus

$$\overline{f(A)} = f(\overline{A}).$$

Now, assume that A is nowhere dense and suppose that $f(A)$ is not nowhere dense, then there exists an open ball B, such that

$$B \subset \overline{f(A)} = f(\overline{A}).$$

By applying f^{-1}, we get

$$f^{-1}(B) \subset \overline{A},$$

Since f is continuous, $f^{-1}(B)$ is open. So, A is dense in the open ball $f^{-1}(B)$, in contradiction to A being nowhere dense. Thus, $f(A)$ is nowhere dense.

Conversely, assume that $f(A)$ is nowhere dense. Suppose that A is not nowhere dense, then there exists an open ball B, such that

$$B \subset \overline{A}$$

By applying f, we get

$$f(B) \subset f(\overline{A}) = \overline{f(A)},$$

Since f^{-1} is continuous, $f(B)$ is open. So, $f(A)$ is dense in the open ball $f(B)$, in contradiction to $f(A)$ being nowhere dense. Thus, A is nowhere dense.

Uniform Continuity

For a function to be uniformly continuous on a set, you need to find one delta (that works for all points in the set) for each epsilon. This means the function's behavior is consistently "well-behaved" across the entire set, according to math resources.

Formal Definition:

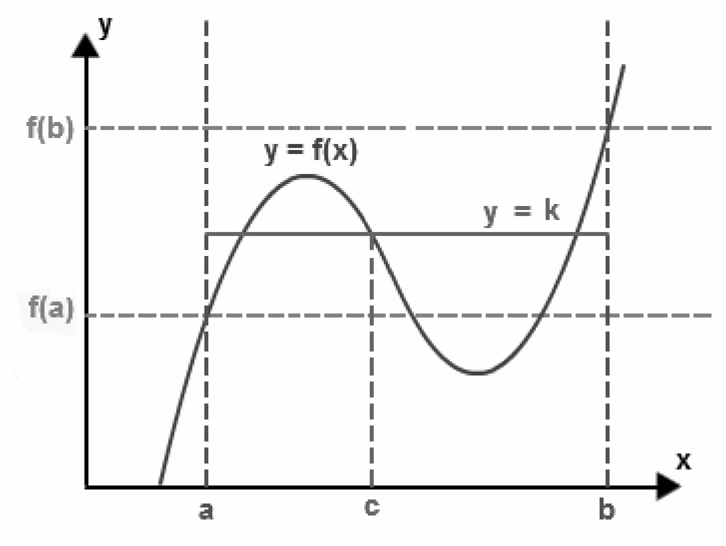
A function f(x) is uniformly continuous on a set A if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for all x, y in A, if $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon$.

Example

- The function $f(x) = x^2$ is uniformly continuous on a closed interval like.
- The function $f(x) = 1/x$ is not uniformly continuous on the interval (0, 1), but it is uniformly continuous on the interval $[2, \infty)$.

Intermediate Value Theorem Statement

Intermediate value theorem states that if "f" be a continuous function over a closed interval [a, b] with its domain having values f(a) and f(b) at the endpoints of the interval, then the function takes any value between the values f(a) and f(b) at a point inside the interval. This theorem is explained in two different ways:



Statement 1:

If k is a value between f(a) and f(b), i.e.

either f(a) < k < f(b) or f(a) > k > f(b)

then there exists at least a number c within a to b i.e. c ∈ (a, b) in such a way that f(c) = k

Statement 2:

The set of images of function in interval [a, b], containing [f(a), f(b)] or [f(b), f(a)], i.e.

Either f([a, b]) ⊇ [f(a), f(b)] or f([a, b]) ⊇ [f(b), f(a)]

Theorem Explanation:

The statement of intermediate value theorem seems to be complicated. But it can be understood in simpler words. Let us consider the above diagram, there is a continuous function f with endpoints a and b, then the height of the point "a" and "b" would be "f(a)" and "f(b)".

If we pick a height k between these heights f(a) and f(b), then according to this theorem, this line must intersect the function f at some point (say c), and this point must lie between a and b.

An intermediate value theorem, if c = 0, then it is referred to as Bolzano's **theorem**.

Intermediate Theorem Proof

We are going to prove the first case of the first statement of the intermediate value theorem since the proof of the second one is similar.

We will prove this theorem by the use of completeness property of real numbers. The proof of "f(a) < k < f(b)" is given below:

Let us assume that A is the set of all the values of x in the interval [a, b], in such a way that $f(x) \leq k$.

Here A is supposed to be a non-empty set as it has an element "a" and also A is bounded above by the value "b".

Thus, by completeness property, we have that, "c" be the lowest value which is greater than or equal to each element of A. Hence, we can say that f(c) = k.

Given that f is continuous. Then let us consider a $\varepsilon > 0$, there exists "a $\delta > 0$" such that

$| f(x) – f(c) | < \varepsilon$ for every $| x – c | < \delta$. This gives us

$f(x) – \varepsilon < f(c) < f(x) + \varepsilon$

For each x lying within $c – \delta$ and $c + \delta$. So, we have values of x lying between c and $c - \delta$, contained in A, such that :

$f(c) < (f(x) + \varepsilon) \leq (k + \varepsilon)$ ——— (1)

Similarly, values of x between c and $c + \delta$ that are not contained in A, such that

$f(c) > (f(x) – \varepsilon) > (k - \varepsilon)$ ———(2)

Combining both the inequality relations, obtain

$k – \varepsilon < f(c) < k + \varepsilon$

For every $\varepsilon > 0$

Hence, the theorem is proved.

Connectedness

A metric space E is said to be *connected* if the only subsets of E that are both open and closed are E and $\emptyset$. A subset of a metric space E is said to be *connected* if it is a connected subspace of E. If a set is not connected, then it is said to be *disconnected*.

The following proposition shows that this definition is consistent with the intuitive approach to connectedness.

Definition *A topological space $(X, \mathcal{T})$ is said to be disconnected if there exist disjoint nonempty subsets $A, B \subseteq X$ such that $X = A \sqcup B$, and $\overline{A} \cap B = A \cap \overline{B} = \emptyset$. If $(X, \mathcal{T})$ is not disconnected, it is said to be connected.*

Connected subsets of R

A subset of R is connected if and only if it is an interval. Proof. Suppose that C is a connected subset of R. To show that C is an interval, suppose $a, b \in C$ with $a<b$, and let x satisfy $a<x<b$.

- **Definition:** A subset of ℝ is connected if it cannot be expressed as the union of two non-empty, disjoint open sets.

  Examples:
  - [0, 1](a closed interval)
  - (0, 1) (an open interval)

Disconnected Subsets of ℝ

Definition

A subset of ℝ is disconnected if it can be written as the union of two non-empty, disjoint open sets.

Examples:

{0, 1}: A set with two distinct points is disconnected because you can find open sets that separate them.

[0, 1]∪ [2, 3]: This is disconnected because the interval [0, 1] and [2, 3] are separated by a gap between 1 and 2.

Connectedness Example

Two open, disjoint intervals cannot cover the set [0, 2]; for example, the open sets (-1, 1) and (1, 2) do not cover [0, 2] since the point x = 1 is not in their union. Therefore, [0, 2] is connected.

However, the set {0, 2} can be covered by the union of (-1, 1) and (1, 3). In this case, {0, 2} is not connected.

Properties of Connectedness

Some of the important properties of the connectedness of a set are listed below.

- A subset of a topological space is said to be connected if it is connected in the subspace topology.
- The interval $(0, 1) \subset R$ with its usual topology is connected.
- Intervals are the only connected subsets of R with the usual topology.
- The continuous image of a connected space is connected. That means if $f : [a, b] \to R$ is continuous, the image of f is connected.