REAL ANALYSIS AND LINEAR ALGEBRA (23ST01CC)

UNIT-I

INTRODUCTION

Real analysis and linear algebra are fundamental mathematical disciplines. Real analysis focuses on the rigorous study of real numbers, sequences, series, and their limits, while linear algebra deals with vector spaces, linear transformations, and matrices. These areas are interconnected and often studied together, especially in advanced mathematics courses, as they provide a strong foundation for various scientific and engineering fields.

Sets

A set is a collection of objects, called the elements or members of the set. The objects could be anything (planets, squirrels, characters in Shakespeare's plays, or other sets) but for us they will be mathematical objects such as numbers, or sets of numbers. We write $x \in X$ if x is an element of the set X and $x \in X$ if x is not an element of X.

Subsets. A set A is a subset of a set X, written $A \subset X$ or $X \supset A$, if every element of A belongs to X; that is, if

$$x \in A \text{ implies that } x \in X.$$

We also say that A is included in X.[1] For example, if P is the set of prime numbers, then $P \subset \mathbb{N}$, and $\mathbb{N} \subset \mathbb{R}$. The empty set $\varnothing$ and the whole set X are subsets of any set X. Note that $X = Y$ if and only if $X \subset Y$ and $Y \subset X$; we often prove the equality of two sets by showing that each one includes the other.

In our notation, $A \subset X$ does not imply that A is a proper subset of X (that is, a subset of X not equal to X itself), and we may have $A = X$. This notation for non-strict inclusion is not universal; some authors use $A \subset X$ to denote strict inclusion, in which $A \neq X$, and $A \subseteq X$ to denote non-strict inclusion, in which $A = X$ is allowed.

Set operations. The intersection $A \cap B$ of two sets A, B is the set of all elements that belong to both A and B; that is

$$x \in A \cap B \text{ if and only if } x \in A \text{ and } x \in B.$$

Two sets A, B are said to be disjoint if $A \cap B = \varnothing$; that is, if A and B have no elements in common.

The union $A \cup B$ is the set of all elements that belong to A or B; that is

$$x \in A \cup B \text{ if and only if } x \in A \text{ or } x \in B.$$

Note that we always use 'or' in an inclusive sense, so that $x \in A \cup B$ if x is an element of A or B, or both A and B. (Thus, $A \cap B \subset A \cup B$.)

The set-difference of two sets B and A is the set of elements of B that do not belong to A,

$$B \setminus A = \{x \in B : x \notin A\}.$$

If we consider sets that are subsets of a fixed set X that is understood from the context, then we write $A^c = X \setminus A$ to denote the complement of $A \subset X$ in X. Note that $(A^c)^c = A$.

Functions

A function $f : X \to Y$ between sets X, Y assigns to each $x \in X$ a unique element $f(x) \in Y$. Functions are also called maps, mappings, or transformations. The set X on which f is defined is called the domain of f and the set Y in which it takes its values is called the codomain. We write $f : x \mapsto f(x)$ to indicate that f is the function that maps x to $f(x)$.

Example The identity function $\mathrm{id}_X : X \to X$ on a set X is the function $\mathrm{id}_X : x \mapsto x$ that maps every element to itself.

Numbers

The infinite sets we use are derived from the natural and real numbers, about which we have a direct intuitive understanding.

Ourunderstandingofthenaturalnumbers1,2,3,...derives from counting. We denote the set of natural numbers by

N={1,2,3,...}.

Sequences of Real Numbers

A sequence of real numbers is a real-valued function whose domain is the set of natural numbers. Rather than denoting a sequence with standard functional notation such as

f : N--> R, it is customary to use subscripts, replace f (n) with a_n, and denote a sequence by $\{a_n\}$. A natural number n is called an index for the sequence, and the number a_n corresponding to the index n is called the n^{th} term of the sequence. Just as we say that areal-valued function is bounded provided its image is a bounded set of real numbers, we say a sequence $\{a_n)$ is bounded provided there is some c >0 such that $|a_n| < c$ for all n. A sequence is said to be increasing provided $a_n < a_{n+1}$ for all n, is said to be decreasing provided $\{-a_n\}$ is increasing, and said to be monotone provided it is either increasing or decreasing.

Definition *A sequence $\{a_n\}$ is said to* **converge** *to the number a provided for every $\epsilon > 0$, there is an index N for which*

$$\text{if } n \geq N, \text{ then } |a - a_n| < \epsilon.$$

We call a the **limit** *of the sequence and denote the convergence of $\{a_n\}$ by writing*

$$\{a_n\} \to a \text{ or } \lim_{n \to \infty} a_n = a.$$

We leave the proof of the following proposition as an exercise.

Definition *Let $\{a_n\}$ be a sequence of real numbers. The limit superior of $\{a_n\}$, denoted by* $\limsup\{a_n\}$, *is defined by*

$$\limsup\{a_n\} = \lim_{n \to \infty} [\sup \{a_k \mid k \geq n\}].$$

The limit inferior of $\{a_n\}$, denoted by $\liminf\{a_n\}$, *is defined by*

$$\liminf\{a_n\} = \lim_{n \to \infty} [\inf \{a_k \mid k \geq n\}].$$

We leave the proof of the following proposition as an exercise.

Composition and inverses of functions

The successive application of mappings leads to the notion of the composition of functions.

Definition The composition of functions $f : X \to Y$ and $g : Y \to Z$ is the function $g \circ f : X \to Z$ defined by

$$(g \circ f)(x) = g(f(x)).$$

The order of application of the functions in a composition is crucial and is read from from right to left. The composition $g \circ f$ can only be defined if the domain of g includes the range of f, and the existence of $g \circ f$ does not imply that $f \circ g$ even makes sense.

Example If $f : \mathbb{R} \to \mathbb{R}$ is the function $f(x) = x^3$, which is one-to-one and onto, then the inverse function $f^{-1} : \mathbb{R} \to \mathbb{R}$ is given by

$$f^{-1}(x) = x^{1/3}.$$

On the other hand, the reciprocal function $g = 1/f$ is given by

$$g(x) = \frac{1}{x^3}, \qquad g : \mathbb{R} \setminus \{0\} \to \mathbb{R}.$$

The reciprocal function is not defined at $x = 0$ where $f(x) = 0$.

If $f : X \to Y$ and $A \subset X$, then we let

$$f(A) = \{y \in Y : y = f(x) \text{ for some } x \in A\}$$

denote the set of values of f on points in A. Similarly, if $B \subset Y$, we let

$$f^{-1}(B) = \{x \in X : f(x) \in B\}$$

denote the set of points in X whose values belong to B. Note that $f^{-1}(B)$ makes sense as a set even if the inverse function $f^{-1} : Y \to X$ does not exist.

Indexed sets

We say that a set X is indexed by a set I, or X is an indexed set, if there is an onto function $f : I \to X$. We then write

$$X = \{x_i : i \in I\}$$

where $x_i = f(i)$. For example,

$$\{1, 4, 9, 16, \dots\} = \left\{n^2 : n \in \mathbb{N}\right\}.$$

The set X itself is the range of the indexing function f, and it doesn't depend on how we index it. If f isn't one-to-one, then some elements are repeated, but this doesn't affect the definition of the set X. For example,

$$\{-1, 1\} = \{(-1)^n : n \in \mathbb{N}\} = \left\{(-1)^{n+1} : n \in \mathbb{N}\right\}.$$

If $\mathcal{C} = \{X_i : i \in I\}$ is an indexed collection of sets X_i, then we denote the union and intersection of the sets in $\mathcal{C}$ by

$$\bigcup_{i \in I} X_i = \{x : x \in X_i \text{ for some } i \in I\}, \qquad \bigcap_{i \in I} X_i = \{x : x \in X_i \text{ for every } i \in I\},$$

or similar notation.

Relations

A binary relation R on sets X and Y is a definite relation between elements of X and elements of Y. We write xRy if $x \in X$ and $y \in Y$ are related. One can also define relations on more than two sets, but we shall consider only binary relations and refer to them simply as relations. If $X = Y$, then we call R a relation on X.

Countable and uncountable sets

One way to show that two sets have the same "size" is to pair off their elements. For example, if we can match up every left shoe in a closet with a right shoe, with no right shoes left over, then we know that we have the same number of left and right shoes. That is, we have the same number of left and right shoes if there is a one-to-one, onto map $f : L \to R$, or one-to-one correspondence, from the set L of left shoes to the set R of right shoes.

Definition Two sets X, Y have equal cardinality, written $X \approx Y$, if there is a one-to-one, onto map $f : X \to Y$. The cardinality of X is less than or equal to the cardinality of Y, written $X \lesssim Y$, if there is a one-to-one (but not necessarily onto) map $g : X \to Y$.

Real numbers: algebraic properties

The algebraic properties of $\mathbb{R}$ are summarized in the following axioms, which state that $(\mathbb{R}, +, \cdot)$ is a field.

There exist binary operations

$$a, m : \mathbb{R} \times \mathbb{R} \to \mathbb{R},$$

written $a(x, y) = x + y$ and $m(x, y) = x \cdot y = xy$, and elements $0, 1 \in \mathbb{R}$ such that for all $x, y, z \in \mathbb{R}$:

(a) $x + 0 = x$ (existence of an additive identity 0);

(b) for every $x \in \mathbb{R}$ there exists $y \in \mathbb{R}$ such that $x + y = 0$ (existence of an additive inverse $y = -x$);

(c) $x + (y + z) = (x + y) + z$ (addition is associative);

(d) $x + y = y + x$ (addition is commutative);

(e) $x1 = x$ (existence of a multiplicative identity 1);

(f) for every $x \in \mathbb{R} \setminus \{0\}$, there exists $y \in \mathbb{R}$ such that $xy = 1$ (existence of a multiplicative inverse $y = x^{-1}$);

(g) $x(yz) = (xy)z$ (multiplication is associative);

(h) $xy = yx$ (multiplication is commutative);

(i) $(x + y)z = xz + yz$ (multiplication is distributive over addition).

Real numbers: ordering properties

The real numbers have a natural order relation that is compatible with their algebraic structure. We visualize the ordered real numbers as the real line, with smaller numbers to the left and larger numbers to the right.

There is a strict linear order $<$ on $\mathbb{R}$ such that for all $x, y, z \in \mathbb{R}$:

(a) either $x < y$, $x = y$, or $x > y$;
(b) if $x < y$ then $x + z < y + z$;
(c) if $x < y$ and $z > 0$, then $xz < yz$.

For any $a, b \in \mathbb{R}$ with $a \leq b$, we define the open intervals

$$\begin{aligned}(-\infty, b) &= \{x \in \mathbb{R} : x < b\}, \\ (a, b) &= \{x \in \mathbb{R} : a < x < b\}, \\ (a, \infty) &= \{x \in \mathbb{R} : a < x\},\end{aligned}$$

the closed intervals

$$\begin{aligned}(-\infty, b] &= \{x \in \mathbb{R} : x \leq b\}, \\ [a, b] &= \{x \in \mathbb{R} : a \leq x \leq b\}, \\ [a, \infty) &= \{x \in \mathbb{R} : a \leq x\},\end{aligned}$$

and the half-open intervals

$$\begin{aligned}(a, b] &= \{x \in \mathbb{R} : a < x \leq b\}, \\ [a, b) &= \{x \in \mathbb{R} : a \leq x < b\}.\end{aligned}$$

All standard properties of inequalities follow from Axiom 2.6 and Axiom 2.7. For example: if $x < y$ and $z < 0$, then $xz > yz$, meaning that the direction of an inequality is reversed when it is multiplied by a negative number; and $x^2 > 0$ for every $x \neq 0$. In future, when we write an inequality such as $x < y$, we will implicitly require that $x, y \in \mathbb{R}$.

The supremum and infimum

Next, we use the ordering properties of $\mathbb{R}$ to define the supremum and infimum of a set of real numbers. These concepts are of central importance in analysis. In particular, in the next section we use them to state the completeness property of $\mathbb{R}$.

First, we define upper and lower bounds.

Definition A set $A \subset \mathbb{R}$ of real numbers is bounded from above if there exists a real number $M \in \mathbb{R}$, called an upper bound of A, such that $x \leq M$ for every $x \in A$. Similarly, A is bounded from below if there exists $m \in \mathbb{R}$, called a lower bound of A, such that $x \geq m$ for every $x \in A$. A set is bounded if it is bounded both from above and below.

Equivalently, a set A is bounded if $A \subset I$ for some bounded interval $I = [m, M]$.

Theorem: 1

If A, B are nonempty sets, then

$$\sup(A+B) = \sup A + \sup B, \qquad \inf(A+B) = \inf A + \inf B,$$
$$\sup(A-B) = \sup A - \inf B, \qquad \inf(A-B) = \inf A - \sup B.$$

Proof. The set $A+B$ is bounded from above if and only if A and B are bounded from above, so $\sup(A+B)$ exists if and only if both $\sup A$ and $\sup B$ exist. In that case, if $x \in A$ and $y \in B$, then

$$x + y \leq \sup A + \sup B,$$

so $\sup A + \sup B$ is an upper bound of $A+B$, and therefore

$$\sup(A+B) \leq \sup A + \sup B.$$

To get the inequality in the opposite direction, suppose that $\epsilon > 0$. Then there exist $x \in A$ and $y \in B$ such that

$$x > \sup A - \frac{\epsilon}{2}, \qquad y > \sup B - \frac{\epsilon}{2}.$$

It follows that

$$x + y > \sup A + \sup B - \epsilon$$

for every $\epsilon > 0$, which implies that

$$\sup(A+B) \geq \sup A + \sup B.$$

Thus, $\sup(A+B) = \sup A + \sup B$. It follows from this result and Proposition 2.23 that

$$\sup(A-B) = \sup A + \sup(-B) = \sup A - \inf B.$$

The proof of the results for $\inf(A+B)$ and $\inf(A-B)$ is similar, or we can apply the results for the supremum to $-A$ and $-B$. □

Finally, we prove that taking the supremum over a pair of indices gives the same result as taking successive suprema over each index separately.

Theorem: 2

Suppose that

$$\{x_{ij} : i \in I, j \in J\}$$

is a doubly-indexed set of real numbers. Then

$$\sup_{(i,j)\in I\times J} x_{ij} = \sup_{i\in I}\left(\sup_{j\in J} x_{ij}\right)$$

Proof. For each $a \in I$, we have $\{a\} \times J \subset I \times J$, so

$$\sup_{j\in J} x_{aj} \leq \sup_{(i,j)\in I\times J} x_{ij}.$$

Taking the supremum of this inequality over $a \in I$, and replacing 'a' by 'i', we get that

$$\sup_{i \in I}\left(\sup_{j \in J} x_{ij}\right) \leq \sup_{(i,j) \in I \times J} x_{ij}.$$

To prove the reverse inequality, first note that if

$$\sup_{(i,j) \in I \times J} x_{ij}$$

is finite, then given $\epsilon > 0$ there exists $a \in I$, $b \in J$ such that

$$x_{ab} > \sup_{(i,j) \in I \times J} x_{ij} - \epsilon.$$

It follows that

$$\sup_{j \in J} x_{aj} > \sup_{(i,j) \in I \times J} x_{ij} - \epsilon,$$

and therefore that

$$\sup_{i \in I}\left(\sup_{j \in J} x_{ij}\right) > \sup_{(i,j) \in I \times J} x_{ij} - \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we have

$$\sup_{i \in I}\left(\sup_{j \in J} x_{ij}\right) \geq \sup_{(i,j) \in I \times J} x_{ij}.$$

Similarly, if

$$\sup_{(i,j) \in I \times J} x_{ij} = \infty,$$

then given $M \in \mathbb{R}$ there exists $a \in I$, $b \in J$ such that $x_{ab} > M$, and it follows that

$$\sup_{i \in I}\left(\sup_{j \in J} x_{ij}\right) > M.$$

Since M is arbitrary, we have

$$\sup_{i \in I}\left(\sup_{j \in J} x_{ij}\right) = \infty,$$

which completes the proof.

Convergence and limits

Roughly speaking, a sequence (x_n) converges to a limit x if its terms x_n get arbitrarily close to x for all sufficiently large n.

Definition A sequence (x_n) of real numbers converges to a limit $x \in \mathbb{R}$, written

$$x = \lim_{n\to\infty} x_n, \quad \text{or} \quad x_n \to x \text{ as } n \to \infty,$$

if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$|x_n - x| < \epsilon \qquad \text{for all } n > N.$$

A sequence converges if it converges to some limit $x \in \mathbb{R}$, otherwise it diverges.

Although we don't show it explicitly in the definition, N is allowed to depend on ϵ. Typically, the smaller we choose ϵ, the larger we have to make N. One way to view a proof of convergence is as a game: If I give you an $\epsilon > 0$, you have to come up with an N that "works." Also note that $x_n \to x$ as $n \to \infty$ means the same thing as $|x_n - x| \to 0$ as $n \to \infty$.

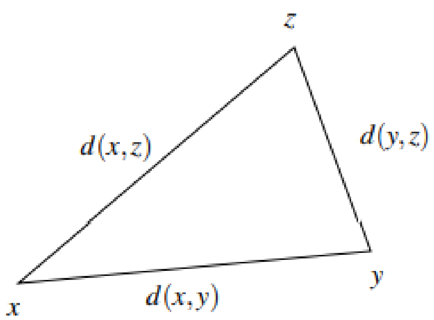
Metric space

When we measure the distance between two points x and y, what we are really doing is taking x and y as inputs to a "distance function" whose corresponding output is a real number — namely, the distance between x and y. This is the idea underlying the following definition.

Definition (Metric Space). Let X be any nonempty set. A function $d : X \times X \to \mathbb{R}$ is called a *metric* (on X) if it satisfies the following properties:

- Positivity: $d(x, y) \geq 0$ for all $(x, y) \in X \times X$, and $d(x, y) = 0$ if and only if $x = y$;
- Symmetry: $d(x, y) = d(y, x)$ for all $(x, y) \in X \times X$;
- Triangle Inequality: Given any three elements $x, y, z \in X$, we have
$$d(x, z) \leq d(x, y) + d(y, z).$$

A set X for which a metric $d : X \times X \to \mathbb{R}$ is defined is called a *metric space*.



Example (The discrete metric). Let X be any nonempty set. The *discrete metric* is the map $\sigma : X \times X \to \mathbb{R}$ given by

$$\sigma(x,y) = \begin{cases} 0, & \text{if } x = y; \\ 1, & \text{otherwise.} \end{cases}$$

It is good practice to show that σ is in fact a metric; see Exercise 2.2.

The triangle inequality allows us to deduce another fact with a familiar geometric interpretation: the difference in the lengths of any two legs of a triangle is no more than the length of the third side. This fact, suitably generalized, is called the *second triangle inequality*.

Sequences

The notion of a sequence in a metric space is very similar to a sequence of real numbers.

Definition A *sequence* in a metric space (X,d) is a function $x \colon \mathbb{N} \to X$. As before we write x_n for the nth element in the sequence and use the notation $\{x_n\}$, or more precisely

$$\{x_n\}_{n=1}^{\infty}.$$

A sequence $\{x_n\}$ is *bounded* if there exists a point $p \in X$ and $B \in \mathbb{R}$ such that

$$d(p,x_n) \leq B \qquad \text{for all } n \in \mathbb{N}.$$

In other words, the sequence $\{x_n\}$ is bounded whenever the set $\{x_n : n \in \mathbb{N}\}$ is bounded.

If $\{n_j\}_{j=1}^{\infty}$ is a sequence of natural numbers such that $n_{j+1} > n_j$ for all j then the sequence $\{x_{n_j}\}_{j=1}^{\infty}$ is said to be a *subsequence* of $\{x_n\}$.

Definition *A mapping from a metric space* (X, ρ) *to a metric space* (Y, σ) *is said to be* **uniformly continuous**, *provided for every* $\epsilon > 0$, *there is a* $\delta > 0$ *such that for* $u, v \in X$,

$$\text{if } \rho(u,v) < \delta, \text{ then } \sigma(f(u), f(v)) < \epsilon.$$

We infer from the ϵ–δ criterion for continuity at a point that a uniformly continuous mapping is continuous. The converse is not true.

Definition (Pointwise convergence). Let $\{f_n\}$ be a sequence of functions defined on a common domain D. If the limit

$$\lim_{n\to\infty} f_n(x)$$

exists for all $x \in D$, we say that $\{f_n\}$ *converge pointwise* on D. This limit defines a function on D

$$f(x) = \lim_{n\to\infty} f_n(x),$$

we write $f_n \to f$, to denote pointwise convergence.

It is well known that the pointwise limit of a sequence of continuous functions need not be continuous. But, the continuity of the limit function, can be guaranteed under additional assumptions on the character of convergence.

Definition (Uniform convergence). Let $\{f_n\}$ be a sequence of functions defined on a common domain D. We say that $\{f_n\}$ *converge uniformly* to f on D, if for every $\varepsilon > 0$ there exists N such that

$$n \geq N \Rightarrow |f_n(x) - f(x)| < \varepsilon \text{ for all } x \in D.$$

A classical theorem states that the limit of a uniformly convergent sequence of continuous functions is continuous. However, pointwise limits also have some traces of continuity, as we will see in this chapter.

1. Arithmetic Sequences

- In an arithmetic sequence, each term is found by adding a constant value (the common difference) to the previous term.
- Example: 2, 4, 6, 8, 10... (Common difference is 2).

2. **Geometric Sequences**

- In a geometric sequence, each term is found by multiplying the previous term by a constant value (the common ratio).
- Example: 2, 4, 8, 16, 32... (common ratio is 2).

3. Fibonacci Sequence

- The Fibonacci sequence is a unique sequence where each number is the sum of the two preceding ones, typically starting with 0 and 1.
- Example: 0, 1, 1, 2, 3, 5, 8, 13...

4. Quadratic Sequences

- These sequences have a constant second difference (the difference between consecutive differences).
- They often involve square numbers (1, 4, 9, 16, etc.) or combinations of them.

5. Harmonic Sequences

- A harmonic sequence is a sequence whose reciprocals form an arithmetic sequence.
- Example: If the arithmetic sequence is 1/2, 1/4, 1/6, 1/8, then the harmonic sequence is 2, 4, 6, 8.

6. Finite and Infinite Sequences

- A finite sequence has a definite end, with a specific number of terms.
- An infinite sequence continues without end, having an unlimited number of terms.

Open and closed sets

Topology

It is useful to define a so-called *topology*. That is we define closed and open sets in a metric space. Before doing so, let us define two special sets.

Definition Let (X,d) be a metric space, $x \in X$ and $\delta > 0$. Then define the *open ball* or simply *ball* of radius δ around x as

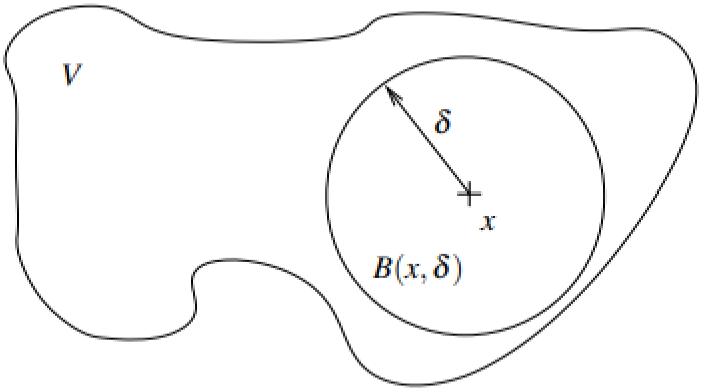
$$B(x,\delta) := \{y \in X : d(x,y) < \delta\}.$$

Similarly we define the *closed ball* as

$$C(x,\delta) := \{y \in X : d(x,y) \leq \delta\}.$$

When we are dealing with different metric spaces, it is sometimes convenient to emphasize which metric space the ball is in. We do this by writing $B_X(x,\delta) := B(x,\delta)$ or $C_X(x,\delta) := C(x,\delta)$.

OPEN AND CLOSED SETS



Proposition *Let (X,d) be a metric space.*

(i) $\emptyset$ and X are open in X.

(ii) If $V_1, V_2, \ldots, V_k$ are open then

$$\bigcap_{j=1}^{k} V_j$$

is also open. That is, finite intersection of open sets is open.

(iii) If $\{V_\lambda\}_{\lambda \in I}$ is an arbitrary collection of open sets, then

$$\bigcup_{\lambda \in I} V_\lambda$$

is also open. That is, union of open sets is open.

Proof. The set X and $\emptyset$ are obviously open in X.

Let us prove (ii). If $x \in \bigcap_{j=1}^{k} V_j$, then $x \in V_j$ for all j. As V_j are all open, there exists a $\delta_j > 0$ for every j such that $B(x,\delta_j) \subset V_j$. Take $\delta := \min\{\delta_1,\ldots,\delta_k\}$ and note that $\delta > 0$. We have $B(x,\delta) \subset B(x,\delta_j) \subset V_j$ for every j and thus $B(x,\delta) \subset \bigcap_{j=1}^{k} V_j$. Thus the intersection is open.

Let us prove (iii). If $x \in \bigcup_{\lambda \in I} V_\lambda$, then $x \in V_\lambda$ for some $\lambda \in I$. As V_λ is open then there exists a $\delta > 0$ such that $B(x,\delta) \subset V_\lambda$. But then $B(x,\delta) \subset \bigcup_{\lambda \in I} V_\lambda$ and so the union is open.

Connected sets

Definition . A nonempty metric space (X,d) is *connected* if the only subsets that are both open and closed are $\emptyset$ and X itself.

When we apply the term *connected* to a nonempty subset $A \subset X$, we simply mean that A with the subspace topology is connected.

In other words, a nonempty X is connected if whenever we write $X = X_1 \cup X_2$ where $X_1 \cap X_2 = \emptyset$ and X_1 and X_2 are open, then either $X_1 = \emptyset$ or $X_2 = \emptyset$. So to test for disconnectedness, we need to find nonempty disjoint open sets X_1 and X_2 whose union is X. For subsets, we state this idea as a proposition.

Closure and boundary

Sometime we wish to take a set and throw in everything that we can approach from the set. This concept is called the closure.

Definition . Let (X,d) be a metric space and $A \subset X$. Then the *closure* of A is the set

$$\overline{A} := \bigcap\{E \subset X : E \text{ is closed and } A \subset E\}.$$

That is, $\overline{A}$ is the intersection of all closed sets that contain A.

Completeness and compactness

Interior point

In real analysis, an interior point of a set is a point that has an open neighborhood entirely contained within the set. A point x is an interior point of a set S if there exists a neighborhood of x that is completely contained within S. In other words, there's a small "bubble" around x consisting only of points in S.

Example

If you consider the open interval (0, 1) in the real numbers, every point within this interval is an interior point because you can always find a smaller open interval around any point that still lies entirely within (0, 1). However, 0 and 1 are not interior points because any neighborhood of 0 or 1 will contain points outside the interval (0, 1).

Interior set

Let (X,τ) be a topological space and A be a subset of X, then the interior of A is denoted by Int (A)Int or A^0 is defined to be the union of all open sets contained in A.

In other words let (X,τ) be a topological space and A be a subset of X. The interior of A is the union of all open subsets of A, and a point in the interior of A is called an interior point of A.

In real analysis, a closed set is a set that contains all of its limit points. Equivalently, a set is closed if its complement is an open set. Closed sets are fundamental in topology and analysis, and are used to define compactness, continuity, and other important concepts.

Closed Sets

In real analysis, a closed set is a set that contains all of its limit points. Equivalently, a set is closed if its complement is an open set. Another way to define it is that a closed set includes all its boundary points.

Limit Points and Closed Sets

- A limit point of a set is a point where every neighborhood (an open interval around the point) contains at least one point from the set, other than the point itself.
- A set is closed if it includes all of its limit points.

  For example, the interval is closed because it includes both 0 and 1, which are its limit points. The interval (0, 1) is not closed because it does not include 0 and 1.
  Complement and Closed Sets
- The complement of a set A (denoted as A^c or R \ A) is the set of all elements in the universal set (in this case, the real numbers R) that are not in A.
- A set is closed if its complement is open.
- An open set is a set where every point has a neighborhood entirely contained within the set.
- For example, the interval (0, 1) is open, and its complement, which is (-∞, ∪ [1, ∞), is closed.

Properties of Closed Sets

- The intersection of any collection of closed sets is closed.
- The union of a finite number of closed sets is closed.
- The empty set (∅) and the entire set of real numbers (R) are both closed.

Relationship to Open Sets

Open and closed sets are related but not opposites.

- A set can be both open and closed (like the empty set and the entire real number line in standard topology) or neither open nor closed (like the interval [0, 1)).
- The complement of an open set is closed, and the complement of a closed set is open.

Boundary Points

Boundary points are points where every neighborhood contains both points in the set and points not in the set.

Examples

- **Closed Interval:** The interval [a, b] (including both endpoints a and b) is a closed set.
- **Finite Sets:** Any finite set of real numbers is also closed.
- **The Set of Real Numbers (R):** The entire set of real numbers is considered closed.
- **The Empty Set (∅):** The empty set is also considered closed.

Closures

Definition *Let $(X, \mathcal{T})$ be a topological space, and let $A \subseteq X$. We define the closure of A in $(X, \mathcal{T})$, which we denote with $\overline{A}$, by:*

$$x \in \overline{A} \text{ if and only if for every open set } U \text{ containing } x,\ U \cap A \neq \emptyset.$$

Or, in symbols:

$$\overline{A} = \{ x \in X \ : \ \forall U \in \mathcal{T} \text{ such that } x \in U,\ U \cap A \neq \emptyset \}.$$

When there can be no confusion about the topological space with respect to which a closure is being considered, we will simply write $\overline{A}$ without specifying "… in $(X, \mathcal{T})$."

In words, $x \in \overline{A}$ if and only if any open set containing x also contains an element of A. After we define what it means for a set to be closed, we will be able to present an alternate way of defining the closure of a set.

Before going any further with examples we examine some elementary properties of closures, the proofs of which use only the definition of closure and absolutely no cleverness or new ideas.

Cantor's Intersection Theorem

You are already familiar with 'Nested Interval Theorem' which states that, "If a sequence of closed intervals $[a_n, b_n]$ in the real line is such that

(i) $[a_{n+1}, b_{n+1}]$ $[a_n, b_n]$ for all $n \in$ N.

(ii) $\lim_{n \to \infty} (b_n - a_n) = 0$,

then there exists precisely one point common to all the intervals."

Now, we are going to discuss a theorem which gives a generalisation of 'nested interval theorem' from the real line to an arbitrary complete metric space.

Nested Sequence of Sets. In a metric space (X, d) by a nested sequence of sets we mean a sequence $\{F_n\}$ of sets satisfying

$$F_1 \supset F_2 \supset F_3 \supset \ldots . \supset F_n \supset F_{n+1} \supset \ldots .$$

i.e., $F_{n+1} \subset F_n$ for all $n \in$ N. In this case the sequence $\{F_n\}$ of sets is said to form a nest.

Theorem . (Cantor's Intersection Theorem)

Let (X, d) be a metric space and $\{F_n\}$ is any nested sequence of non-empty closed sets such that $\delta(F_n) \to x$ as $n \to \infty$. Then the set $F = \bigcap_{n=1}^{\infty} F_n$ contains one and only one point of X if and only if the metric space (X, d) is complete.

Proof. First we assume that (X, d) is a complete metric space and $\{F_n\}$ is a nested sequence of non-empty closed sets closed such that $\delta(F_n) \to x$ as $n \to \infty$. We are to prove that the set $F = \bigcap_{n=1}^{\infty} F_n$ contains exactly one point.

By proposition $F_n \supset F_{n+1}$ for all and $\delta(F_n) \to x$ as $n \to \infty$.

We consider an arbitrary element x_n from the set F_n, where n is any natural number. We shall show that the sequence $\{x_n\}$ thus generated is a Cauchy sequence.

Since for all $p \in$ N, $F_{n+p} \subset F_n$, it follows that $x_{n+p} \in F_n$ for all $p \in$ N. Thus, x_n, x_{n+1}, ...F_n for all $n \in$ N.

Let ε (>0) be arbitrarily chosen. Since $\delta(F_n) \to x$ as $n \to \infty$, there exists a positive integer $N = N(\varepsilon)$ such that

$n > N \Rightarrow \delta(F_n) < \varepsilon$

$\Rightarrow \sup \{d(y, z) : y, z\ F_n\} < \varepsilon$

$\Rightarrow d(x_{n+p}, x_n) < \varepsilon$ since $x_{n+p}, x_n \in F_n$ for all $n \in$ N.

So, $\{x_n\}$ is a Cauchy sequence in (X, d). Since the metric space (X, d) is complete, the sequence $\{x_n\}$ converges in (X, d). Let $x_n \to x \in X$ as $n \to \infty$.

For any $n \in$ N, all the elements x_n, x_{n+1}, x_{n+2}, ... belong to F_n. Since F_n is closed, it contains all the limit points of the above sequence of elements. Therefore,

$x \in F_n$ for all $n \in$ N

$$\Rightarrow x \in \bigcap_{n=1}^{\infty} F_n = F.$$

Now, we shall prove that x is unique. Let us assume, if possible, there is a point $x' \in X$ such that

$$x' \in F = \bigcap_{n=1}^{\infty} F_n.$$

Then, $\delta(F_n) \geq d(x, x') \geq 0$ for all $n \in \mathrm{N}$

$\Rightarrow d(x, x') \to 0$ as $n \to \infty$, since $\delta(F_n) \to 0$ as $n \to \infty$.

This, implies $d(x, x') = 0$, i.e., $x = x'$. Therefore, the set F contains precisely one point.

Conversely we assume that for every nested sequence $\{F_n\}$ of closed sets in (X, d) with $\delta(F_n) \to 0$ as $n \to \infty$ the set $F = \bigcap_{n=1}^{\infty} F_n$ contains precisely one point. We are to prove that the metric space (X, d) is complete.

Let $\{x_n\}$ be an arbitrary Cauchy sequence in (X, d). Let us consider the sets

$$G_n = \{x_n, x_{n+1}, x_{n+2}, \ldots\}$$

for all $n \in \mathrm{N}$. Since $\{x_n\}$ is a cauchy sequence in (X, d), corresponding to ε (>0), chosen arbitrarily, there exists a positive integer $n = N(\varepsilon)$ such that

$n > N \Rightarrow d(x_{n+p}, x_n) < \varepsilon$ for all $p \in \mathrm{N}$

$\Rightarrow S(G_n) \leq \varepsilon.$

This implies, $\delta(G_n) \to 0$ as $n \to \infty$.

Let $H_n = \overline{G_n}$, for all $n \in \mathrm{N}$, where $\overline{G_n}$ denotes the closure of the set G_n. Then, by theorem 22 (vii) of Chapter II, $\delta(H_n) = \delta(G_n)$. Hence $\delta(H_n) \to 0$ as $n \to \infty$.

Moreover, we have, for all $n \in \mathrm{N}$

$G_{n+1} \subset G_n$

$\Rightarrow \overline{G_{n+1}} \subset \overline{G_n}$,

i.e. $\quad H_{n+1} \subset H_n.$

Therefore, the sequence $\{H_n\}$ is a nested sequence of closed sets in (X, d) with (H_n) 0. So, by hypothesis, there exists a unique element

$$x \in H = \bigcap_{n=1}^{\infty} H_n \subset X.$$

Also, for each $n \in \mathrm{N}$.

$x \in G_{n+1} \subset H_n$

$\Rightarrow x \in H_n$

$\Rightarrow d(x_n, x) \leq \delta(H_n)$ since $x_n \in H_n$

Therefore, $d(x_n, x) \to 0$ as $n \to \infty$ i.e., $x_n \to x$.

Thus, every Cauchy sequence of (X, d) converges to some element of X. This implies that (X, d) is complete.

Baire theorem

We started this chapter, by stating that a function of the first class has traces of continuity. The previous examples have indicated that the set of discontinuity points may be dense on the domain of the function. The next lemma, and the proceeding theorem, due to Baire, shows that a first class function cannot be everywhere discontinuous

Let f be a function of the first class on $[a,b]$. Then, for any $\gamma > 0$, the set

$$E = \{x : \omega_f(x) \geq \gamma \quad \text{for every } x \in [a,b]\},$$

Is closed and nowhere dense

Baire's Theorem

Baire's theorem is a result on complete metric spaces which will be used in this chapter to prove some very important results on Banach spaces.

Theorem (**Baire's Theorem**) *Let (X, d) be a complete metric space. Let $\{V_n\}_{n=1}^{\infty}$ be a collection of open dense sets. Then*

$$\cap_{n=1}^{\infty} V_n$$

is also dense.

Proof: Let W be any non-empty open set in X. We need to show that the intersection $\cap_n V_n$ has a point in W.

Since V_1 is dense, it follows that $W \cap V_1 \neq \emptyset$. Thus, we can find a point x_1 in this intersection (which is also an open set). Hence, there exists $r_1 > 0$ such that the open ball $B(x_1; r_1)$, with centre at x_1 and radius r_1, is contained in $W \cap V_1$. By shrinking r if necessary, we may also assume that the closure of this ball $\overline{B(x_1; r_1)}$ is also a subset of $W \cap V_1$ and that $0 < r_1 < 1$.

If $n \geq 2$, assume that we have chosen x_{n-1} and r_{n-1} suitably. The denseness of V_n shows that $V_n \cap B(x_{n-1}; r_{n-1}) \neq \emptyset$ and so we can choose x_n and r_n such that $0 < r_n < 1/n$ and

$$\overline{B(x_n; r_n)} \subset V_n \cap B(x_{n-1}; r_{n-1}).$$

Thus we now have a sequence of points $\{x_n\}$ in X. If $i > n$ and $j > n$, it is clear that both x_i and x_j both lie in $B(x_n; r_n)$ and so $d(x_i, x_j) < 2r_n < 2/n$ and thus, the sequence $\{x_n\}$ is Cauchy. Since X is complete, it follows that there exists $x \in X$ such that $x_n \to x$.

Now, for $i > n$, it follows that $x_i \in B(x_n; r_n)$ and so $x \in \overline{B(x_n; r_n)}$ for all n and so $x \in V_n$ for all n. Also $x \in \overline{B(x_1; r_1)}$ and so $x \in W$ as well. This completes the proof.