UNIT-IV

Introduction to Matrices

A rectangular array of m × n numbers (real or complex) in the form of m horizontal lines (called rows) and n vertical lines (called columns) is called a matrix of order m by n, written as m × n matrix. Such an array is enclosed by [ ] or ( ). In this article, we will learn the meaning of **matrices**, types of matrices, important formulas, etc.

An m × n matrix is usually written as:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & \cdots\cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots\cdots & a_{mn} \end{bmatrix}$$

In brief, the above matrix is represented by $A = [a_{ij}]_{mxn}$. The numbers a_{11}, a_{12}, ….. etc., are known as the elements of the matrix A, where a_{ij} belongs to the i^{th} row and j^{th} column and is called the $(i, j)^{th}$ element of the matrix $A = [a_{ij}]$.

Matrices are key concepts in mathematics, widely used in solving equations and problems in fields like physics and computer science. A matrix is simply a grid of numbers, and a determinant is a value calculated from a square matrix.

$$\textit{Example:} \begin{bmatrix} 6 & 9 \\ 5 & -4 \end{bmatrix}_{2\times 2}, \begin{bmatrix} 3 & -4 & 5 \\ 1 & 7 & 6 \\ 6 & -2 & 9 \end{bmatrix}_{3\times 3}$$

The different types of matrices are given below:

Type of Matrix	Details
Row Matrix	$A = [a_{ij}]_{1\times n}$
Column Matrix	$A = [aij]_{m\times 1}$
Zero or Null Matrix	$A = [a_{ij}]_{mxn}$ where, $a_{ij} = 0$
Singleton Matrix	$A = [a_{ij}]_{mxn}$ where, m = n =1
Horizontal Matrix	$[a_{ij}]_{mxn}$ where n > m
Vertical Matrix	$[a_{ij}]_{mxn}$ where, m > n

Type of Matrix	Details
Square Matrix	$[a_{ij}]_{mxn}$ where, m = n
Diagonal Matrix	A = $[a_{ij}]$ when $i \neq j$
Scalar Matrix	A = $[a_{ij}]_{mxn}$ where, $a_{ij} = \begin{Bmatrix} 0, & i \neq j \\ k, & i = j \end{Bmatrix}$ where k is a constant.
Identity (Unit) Matrix	A = $[a_{ij}]_{m\times n}$ where, $a_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$
Equal Matrix	A = $[a_{ij}]_{mxn}$ and B = $[b_{ij}]_{rxs}$ where, $a_{ij} = b_{ij}$, m = r, and n = s
Triangular Matrices	Can be either upper triangular ($a_{ij} = 0$, when $i > j$) or lower triangular ($a_{ij} = 0$ when $i < j$)
Singular Matrix	$\|A\| = 0$
Non-Singular Matrix	$\|A\| \neq 0$
Symmetric Matrices	A = $[a_{ij}]$ where, $a_{ij} = a_{ji}$
Skew-Symmetric Matrices	A = $[a_{ij}]$ where, $a_{ij} = a_{ji}$
Hermitian Matrix	$A = A^{\theta}$
Skew – Hermitian Matrix	$A^{\theta} = -A$
Orthogonal Matrix	$A A^T = I_n = A^T A$
Idempotent Matrix	$A^2 = A$
Involuntary Matrix	$A^2 = I$, $A^{-1} = A$
Nilpotent Matrix	$\exists$ p $\in$ N such that $A^P = 0$

Definition of a Matrix

Definition . **(Matrix)** A rectangular array of numbers is called a matrix.

We shall mostly be concerned with matrices having real numbers as entries.

The horizontal arrays of a matrix are called its ROWS and the vertical arrays are called its COLUMNS. A matrix having m rows and n columns is said to have the order $m \times n$.

A matrix A of ORDER $m \times n$ can be represented in the following form:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix},$$

where a_{ij} is the entry at the intersection of the i^{th} row and j^{th} column.

In a more concise manner, we also denote the matrix A by $[a_{ij}]$ by suppressing its order.

Definition **(Equality of two Matrices)** Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ having the same order $m \times n$ are equal if $a_{ij} = b_{ij}$ for each $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

In other words, two matrices are said to be equal if they have the same order and their corresponding entries are equal.

Special Matrices

Definition 1. A matrix in which each entry is zero is called a zero-matrix, denoted by $\mathbf{0}$. For example,

$$\mathbf{0}_{2\times 2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } \mathbf{0}_{2\times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

2. A matrix having the number of rows equal to the number of columns is called a square matrix. Thus, its order is $m \times m$ (for some m) and is represented by m only.

3. In a square matrix, $A = [a_{ij}]$, of order n, the entries $a_{11}, a_{22}, \ldots, a_{nn}$ are called the diagonal entries and form the principal diagonal of A.

4. A square matrix $A = [a_{ij}]$ is said to be a diagonal matrix if $a_{ij} = 0$ for $i \neq j$. In other words, the non-zero entries appear only on the principal diagonal. For example, the zero matrix $\mathbf{0}_n$ and $\begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$ are a few diagonal matrices.

A diagonal matrix D of order n with the diagonal entries $d_1, d_2, \ldots, d_n$ is denoted by $D = \text{diag}(d_1, \ldots, d_n)$. If $d_i = d$ for all $i = 1, 2, \ldots, n$ then the diagonal matrix D is called a **scalar matrix**.

5. A square matrix $A = [a_{ij}]$ with $a_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$ is called the identity matrix, denoted by I_n.

   For example, $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, and $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

   The subscript n is suppressed in case the order is clear from the context or if no confusion arises.

6. A square matrix $A = [a_{ij}]$ is said to be an upper triangular matrix if $a_{ij} = 0$ for $i > j$.

   A square matrix $A = [a_{ij}]$ is said to be an lower triangular matrix if $a_{ij} = 0$ for $i < j$.

   A square matrix A is said to be triangular if it is an upper or a lower triangular matrix.

   For example $\begin{bmatrix} 2 & 1 & 4 \\ 0 & 3 & -1 \\ 0 & 0 & -2 \end{bmatrix}$ is an upper triangular matrix. An upper triangular matrix will be represented by $\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$.

Operations on Matrices

Definition **(Transpose of a Matrix)** The transpose of an $m \times n$ matrix $A = [a_{ij}]$ is defined as the $n \times m$ matrix $B = [b_{ij}]$, with $b_{ij} = a_{ji}$ for $1 \leq i \leq m$ and $1 \leq j \leq n$. The transpose of A is denoted by A^t.

That is, by the transpose of an $m \times n$ matrix A, we mean a matrix of order $n \times m$ having the rows of A as its columns and the columns of A as its rows.

For example, if $A = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 2 \end{bmatrix}$ then $A^t = \begin{bmatrix} 1 & 0 \\ 4 & 1 \\ 5 & 2 \end{bmatrix}$.

Thus, the transpose of a row vector is a column vector and vice-versa.

Definition **(Addition of Matrices)** let $A = [a_{ij}]$ and $B = [b_{ij}]$ be are two $m \times n$ matrices. Then the sum $A + B$ is defined to be the matrix $C = [c_{ij}]$ with $c_{ij} = a_{ij} + b_{ij}$.

Note that, we define the sum of two matrices only when the order of the two matrices are same.

Definition **(Multiplying a Scalar to a Matrix)** Let $A = [a_{ij}]$ be an $m \times n$ matrix. Then for any element $k \in \mathbb{R}$, we define $kA = [ka_{ij}]$.

For example, if $A = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 2 \end{bmatrix}$ and $k = 5$, then $5A = \begin{bmatrix} 5 & 20 & 25 \\ 0 & 5 & 10 \end{bmatrix}$.

Definition **(Additive Inverse)** Let A be an $m \times n$ matrix.

1. Then there exists a matrix B with $A + B = \mathbf{0}$. This matrix B is called the additive inverse of A, and is denoted by $-A = (-1)A$.
2. Also, for the matrix $\mathbf{0}_{m \times n}$, $A + \mathbf{0} = \mathbf{0} + A = A$. Hence, the matrix $\mathbf{0}_{m \times n}$ is called the additive identity.

Multiplication of Matrices

Definition **(Matrix Multiplication / Product)** Let $A = [a_{ij}]$ be an $m \times n$ matrix and $B = [b_{ij}]$ be an $n \times r$ matrix. The product AB is a matrix $C = [c_{ij}]$ of order $m \times r$, with

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} = a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots + a_{in} b_{nj}.$$

Observe that the product AB is defined if and only if
THE NUMBER OF COLUMNS OF $A =$ THE NUMBER OF ROWS OF B.

For example, if $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 3 \\ 1 & 0 & 4 \end{bmatrix}$ then

$$AB = \begin{bmatrix} 1+0+3 & 2+0+0 & 1+6+12 \\ 2+0+1 & 4+0+0 & 2+12+4 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 19 \\ 3 & 4 & 18 \end{bmatrix}.$$

Note that in this example, while AB is defined, the product BA is not defined. However, for square matrices A and B of the same order, both the product AB and BA are defined.

Submatrix of a Matrix

Definition A matrix obtained by deleting some of the rows and/or columns of a matrix is said to be a submatrix of the given matrix.

For example, if $A = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 2 \end{bmatrix}$, a few submatrices of A are

$$[1], [2], \begin{bmatrix} 1 \\ 0 \end{bmatrix}, [1\ 5], \begin{bmatrix} 1 & 5 \\ 0 & 2 \end{bmatrix}, A.$$

But the matrices $\begin{bmatrix} 1 & 4 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 4 \\ 0 & 2 \end{bmatrix}$ are not submatrices of A. (The reader is advised to give reasons.)

Definition **(Conjugate Transpose of a Matrix)** 1. Let A be an $m \times n$ matrix over $\mathbb{C}$. If $A = [a_{ij}]$ then the Conjugate of A, denoted by $\overline{A}$, is the matrix $B = [b_{ij}]$ with $b_{ij} = \overline{a_{ij}}$.

For example, Let $A = \begin{bmatrix} 1 & 4+3i & i \\ 0 & 1 & i-2 \end{bmatrix}$. Then

$$\overline{A} = \begin{bmatrix} 1 & 4-3i & -i \\ 0 & 1 & -i-2 \end{bmatrix}.$$

2. Let A be an $m \times n$ matrix over $\mathbb{C}$. If $A = [a_{ij}]$ then the Conjugate Transpose of A, denoted by A^*, is the matrix $B = [b_{ij}]$ with $b_{ij} = \overline{a_{ji}}$.

For example, Let $A = \begin{bmatrix} 1 & 4+3i & i \\ 0 & 1 & i-2 \end{bmatrix}$. Then

$$A^* = \begin{bmatrix} 1 & 0 \\ 4-3i & 1 \\ -i & -i-2 \end{bmatrix}.$$

3. A square matrix A over $\mathbb{C}$ is called Hermitian if $A^* = A$.
4. A square matrix A over $\mathbb{C}$ is called skew-Hermitian if $A^* = -A$.
5. A square matrix A over $\mathbb{C}$ is called unitary if $A^*A = AA^* = I$.
6. A square matrix A over $\mathbb{C}$ is called Normal if $AA^* = A^*A$.

Definition and a Solution Method

Definition **(Linear System)** A linear system of m equations in n unknowns $x_1, x_2, \ldots, x_n$ is a set of equations of the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad\qquad &\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

where for $1 \le i \le n$, and $1 \le j \le m$; $a_{ij}, b_i \in \mathbb{R}$. Linear System (2.2.1) is called HOMOGENEOUS if $b_1 = 0 = b_2 = \cdots = b_m$ and NON-HOMOGENEOUS otherwise.

We rewrite the above equations in the form $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \text{ and } \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

The matrix A is called the COEFFICIENT matrix and the block matrix $[A \ \ \mathbf{b}]$, is the AUGMENTED matrix of the linear system

Definition **(Solution of a Linear System)** A solution of the linear system $A\mathbf{x} = \mathbf{b}$ is a column vector $\mathbf{y}$ with entries $y_1, y_2, \ldots, y_n$ such that the linear system (2.2.1) is satisfied by substituting y_i in place of x_i.

That is, if $\mathbf{y}^t = [y_1, y_2, \ldots, y_n]$ then $A\mathbf{y} = \mathbf{b}$ holds.

Note: The zero n-tuple $\mathbf{x} = \mathbf{0}$ is always a solution of the system $A\mathbf{x} = \mathbf{0}$, and is called the TRIVIAL solution. A non-zero n-tuple $\mathbf{x}$, if it satisfies $A\mathbf{x} = \mathbf{0}$, is called a NON-TRIVIAL solution.

Definition **(Equivalent Linear Systems)** Two linear systems are said to be equivalent if one can be obtained from the other by a finite number of elementary operations.

The linear systems at each step in Example 2.2.4 are equivalent to each other and also to the original linear system.

Definition **(Elementary Row Operations)** The elementary row operations are defined as:

1. interchange of two rows, say "interchange the i^{th} and j^{th} rows", denoted R_{ij};
2. multiply a non-zero constant throughout a row, say "multiply the k^{th} row by $c \neq 0$", denoted $R_k(c)$;
3. replace a row by itself plus a constant multiple of another row, say "replace the k^{th} row by k^{th} row plus c times the j^{th} row", denoted $R_{kj}(c)$.

Elementary Matrices

Definition A square matrix E of order n is called an **elementary matrix** if it is obtained by applying exactly one elementary row operation to the identity matrix, I_n.

Remark *There are three types of elementary matrices.*

1. *E_{ij}, which is obtained by the application of the elementary row operation R_{ij} to the identity matrix, I_n. Thus, the $(k,\ell)^{th}$ entry of E_{ij} is* $(E_{ij})_{(k,\ell)} = \begin{cases} 1 & \text{if } k = \ell \text{ and } \ell \neq i, j \\ 1 & \text{if } (k,\ell) = (i,j) \text{ or } (k,\ell) = (j,i) \\ 0 & \text{otherwise} \end{cases}$.
2. *$E_k(c)$, which is obtained by the application of the elementary row operation $R_k(c)$ to the identity matrix, I_n. The $(i,j)^{th}$ entry of $E_k(c)$ is* $(E_k(c))_{(i,j)} = \begin{cases} 1 & \text{if } i = j \text{ and } i \neq k \\ c & \text{if } i = j = k \\ 0 & \text{otherwise} \end{cases}$.
3. *$E_{ij}(c)$, which is obtained by the application of the elementary row operation $R_{ij}(c)$ to the identity matrix, I_n. The $(k,\ell)^{th}$ entry of $E_{ij}(c)$ is* $(E_{ij})_{(k,\ell)} \begin{cases} 1 & \text{if } k = \ell \\ c & \text{if } (k,\ell) = (i,j) \\ 0 & \text{otherwise} \end{cases}$.

In particular,

$$E_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \; E_1(c) = \begin{bmatrix} c & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ and } E_{23}(c) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}.$$

Rank of a Matrix

Definition . (Consistent, Inconsistent) A linear system is called CONSISTENT if it admits a solution and is called INCONSISTENT if it admits no solution.

The question arises, as to whether there are conditions under which the linear system $A\mathbf{x} = \mathbf{b}$ is consistent. The answer to this question is in the affirmative. To proceed further, we need a few definitions and remarks.

Recall that the row reduced echelon form of a matrix is unique and therefore, the number of non-zero rows is a unique number. Also, note that the number of non-zero rows in either the row reduced form or the row reduced echelon form of a matrix are same.

Definition . (Row rank of a Matrix) The number of non-zero rows in the row reduced form of a matrix is called the row-rank of the matrix.

By the very definition, it is clear that row-equivalent matrices have the same row-rank. For a matrix A, we write 'row-rank (A)' to denote the row-rank of A.

Rank of a Matrix :-

$V_C(A)$: Column space of A (Vector space generated by the column of A).

$V_R(A)$: Row-space of A.

Column Rank of A is defined as $\dim\{V_C(A)\}$ or number of LIN columns of A.

Row rank of A = Column rank of A = $R(A) \le \min(m, n)$, where $A_{mn} = A$.

Definition :- Rank of the matrix A is the order of highest order non-vanishing minor of A.

Some Useful Results:-

1. $\text{Rank}(AB) \leq \min[\text{Rank}(A), \text{Rank}(B)]$

Proof:- Let A be a matrix of order $m \times n$ and B be a matrix of order $n \times n$.

Suppose $A = ((a_{ij}))$

$B = ((b_{ij}))$

$$A = (\underset{\sim}{\alpha}_1, \underset{\sim}{\alpha}_2, \ldots, \underset{\sim}{\alpha}_n)$$

$$AB = (\underset{\sim}{\alpha}_1, \underset{\sim}{\alpha}_2, \ldots, \underset{\sim}{\alpha}_n)\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix}$$

$$= \left(\sum_{K=1}^{n} b_{K1}\underset{\sim}{\alpha}_K, \sum_{K=1}^{n} b_{K2}\underset{\sim}{\alpha}_K, \ldots, \sum b_{Kn}\underset{\sim}{\alpha}_K\right)$$

Columns of AB are linear combination of columns of A.

$$V_c(AB) \subseteq V_c(A)$$

$$\Rightarrow \dim V_c(AB) \leq \dim V_c(A)$$

$$\Rightarrow R(AB) \leq R(A) \quad \ldots\ldots (i)$$

Similarly we can show that $R(AB) \leq R(B) \quad \ldots\ldots (ii)$

Combining (i) & (ii) we have, $R(AB) \leq \min[R(A), R(B)]$.

2. If A be an idempotent matrix then Rank (A) = Trace(A).

Proof:- Let A be a matrix of order $n \times n$, such that $A^2 = A$ and rank (A) = r.

By rank-factorisation theorem, we have

$$A = B_{n \times r} C_{r \times n} \text{ where } \operatorname{rank}(B) = \operatorname{rank}(C) = r.$$

$$A^2 = A$$

$$\Rightarrow BCBC = BC$$

$$\Rightarrow B'BCBCC' = B'BCC'$$

We know, $R(B'B) = \operatorname{Rank}(B) = r$, where BB' is of order r.

i.e. BB' is non-singular $\Rightarrow (BB')^{-1}$ exists.

Similarly, $(CC')^{-1}$ exists.

$$(B'B)^{-1} B'BCBCC'(CC')^{-1}$$
$$= (B'B)^{-1} B'BCC'(CC')^{-1}$$

$$\Rightarrow CB = I_r$$

$\therefore$ Trace (CB) = trace(BC) = trace(A) = trace(I_r) = r = rank(A).

Definition 2.5.6 The number of non-zero rows in the row reduced form of a matrix A is called the **rank** of A, denoted rank (A).

Theorem 2.5.7 Let A be a matrix of rank r. Then there exist elementary matrices $E_1, E_2, \ldots, E_s$ and $F_1, F_2, \ldots, F_\ell$ such that

$$E_1 E_2 \ldots E_s \; A \; F_1 F_2 \ldots F_\ell = \begin{bmatrix} I_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

Proof. Let C be the row reduced echelon matrix obtained by applying elementary row operations to the given matrix A. As $\operatorname{rank}(A) = r$, the matrix C will have the first r rows as the non-zero rows. So by Remark 2.4.5, C will have r leading columns, say $i_1, i_2, \ldots, i_r$. Note that, for $1 \le s \le r$, the i_s^{th} column will have 1 in the s^{th} row and zero elsewhere.

We now apply column operations to the matrix C. Let D be the matrix obtained from C by successively interchanging the s^{th} and i_s^{th} column of C for $1 \le s \le r$. Then the matrix D can be written in the form $\begin{bmatrix} I_r & B \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$, where B is a matrix of appropriate size. As the $(1,1)$ block of D is an identity matrix, the block $(1,2)$ can be made the zero matrix by application of column operations to D. This gives the required result. □

Inverse of a Matrix

Definition **(Inverse of a Matrix)** Let A be a square matrix of order n.

1. A square matrix B is said to be a LEFT INVERSE of A if $BA = I_n$.
2. A square matrix C is called a RIGHT INVERSE of A, if $AC = I_n$.
3. A matrix A is said to be INVERTIBLE (or is said to have an INVERSE) if there exists a matrix B such that $AB = BA = I_n$.

Adjoint of a Matrix

Recall that for a square matrix A, the notations A_{ij} and $C_{ij} = (-1)^{i+j} A_{ij}$ were respectively used to denote the $(i,j)^{\text{th}}$ minor and the $(i,j)^{\text{th}}$ cofactor of A.

Definition **(Adjoint of a Matrix)** Let A be an $n \times n$ matrix. The matrix $B = [b_{ij}]$ with $b_{ij} = C_{ji}$, for $1 \le i,j \le n$ is called the Adjoint of A, denoted $Adj(A)$.

Example Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$. Then $Adj(A) = \begin{bmatrix} 4 & 2 & -7 \\ -3 & -1 & 5 \\ 1 & 0 & -1 \end{bmatrix}$;
as $C_{11} = (-1)^{1+1} A_{11} = 4, C_{12} = (-1)^{1+2} A_{12} = -3, C_{13} = (-1)^{1+3} A_{13} = 1$, and so on.

Definition **(Inverse Linear Transformation)** Let $T : V \longrightarrow W$ be a linear transformation. If the map T is one-one and onto, then the map $T^{-1} : W \longrightarrow V$ defined by

$$T^{-1}(\mathbf{w}) = \mathbf{v} \quad \text{whenever } T(\mathbf{v}) = \mathbf{w}$$

is called the inverse of the linear transformation T.

A **symmetric matrix**

Let's begin with the symmetric matrix definition:

A symmetric matrix is a square matrix that is equal to its transpose. That is, $A = A^T$

This means that the elements are symmetric with respect to the main diagonal. Mathematically, for a matrix A = [a_{ij}], it is symmetric if:

$a_{ij} = a_{ji}$ for all i,j

Explanation

1. **Definition of a Symmetric Matrix:** A matrix A is symmetric if it equals its transpose, meaning $A = A^T$.
2. **Property of Transpose of an Inverse:** For any invertible matrix A, the transpose of its inverse is equal to the inverse of its transpose, i.e., $(A^T)^{-1} = (A^{-1})^T$.
3. **Combining the Properties:**
   - If A is symmetric, then $A = A^T$.
   - Using the property of the transpose of an inverse, substitute A^T with A: $(A^T)^{-1} = (A^{-1})^T$ becomes $A^{-1} = (A^{-1})^T$.
   - Since $A^{-1} = (A^{-1})^T$, by definition, A^{-1} is also a symmetric matrix.

A **symmetric matrix** in linear algebra is a square matrix that remains unaltered when its transpose is calculated. That means, a matrix whose transpose is equal to the matrix itself, is called a symmetric matrix. It is mathematically defined as follows:

A square matrix B which of size n × n is considered to be symmetric if and only if $B^T = B$. Consider the given matrix B, that is, a square matrix that is equal to the transposed form of that matrix, called a symmetric matrix.

Symmetric and Skew-symmetric Matrix:- A square matrix A is said to be a symmetric matrix if $A' = A$ and will be skew-symmetric if $A' = -A$.

$A' = A \Leftrightarrow a_{ij} = a_{ji} \ \forall (i,j)$

$A' = -A \Leftrightarrow a_{ij} = -a_{ji} \ \forall (i,j)$ and $a_{ii} = 0 \ \forall i$.

Here, AA', $A'A$ and $(A \pm A')$ are also symmetric matrices. (Check)

Note :- Any square matrix can uniquely be written as a sum of symmetric and skew-symmetric matrix.

$$A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A').$$

DETERMINANTS :-

$$A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

$$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{23}a_{31}$$

$$= \sum_{1 \le i_1 \ne i_2 \ne i_3 \le 3} (-1)^{N(i_1, i_2, i_3)} a_{1i_1} a_{2i_2} a_{3i_3}$$

$$= \sum_{1 \le i_1 \ne i_2 \ne i_3 \le 3} (-1)^{N(i_1, i_2, i_3)} a_{i_1 1} a_{i_2 2} a_{i_3 3}$$

$$= ((a_{ij}))_{n \times n} .$$

$$\therefore \; |A| = \det(A) = \sum_{1 \le i_1 \ne i_2 \ne \cdots \ne i_n \le n} (-1)^{N(i_1, i_2, i_3, \ldots, i_n)} \prod_{j'=1}^{n} a_{i_{j'} j},$$

where $N(i_1, \ldots, i_n)$ = No. of inversions in $(i_1, \ldots, i_n)$.

Summation is taken over all possible permutations of $(1, 2, \ldots, n)$, i.e. no. of terms under the summation is $n!$.

Properties : 1. $\text{Det}(A) = \text{Det}(A')$

Sol. Let $A = ((a_{ij}))_{n\times n}$, where $B = ((b_{ij}))_{n\times n}$, $B = A'$.

Clearly, $a_{ji} = b_{ij} \ \forall (i,j)$

$$|A'| = |B| = \sum_{1\le i_1 \ne i_2 \ne \cdots \ne i_n \le n} (-1)^{N(i_1,\ldots,i_n)} a_{i_1 1} a_{i_2 2} \cdots a_{i_n n}$$

$$= |A| .$$

P:2. $\det(\lambda A) = \lambda^n \det(A)$, where A is of order n.

Sol. Let $A = ((a_{ij}))$, $B = ((b_{ij}))$

$B = \lambda A$, clearly, $b_{ij} = \lambda a_{ij} \ \forall (i,j)$

$$|B| = \sum_{1\le i_1 \ne i_2 \ne \cdots \ne i_n \le n} (-1)^{N(i_1,\ldots,i_n)} b_{1i_1} b_{2i_2} \cdots b_{ni_n}$$

$$= \sum_{1\le i_1 \ne i_2 \ne \cdots \ne i_n \le n} (-1)^{N(i_1,\ldots,i_n)} \lambda a_{1i_1} \lambda a_{2i_2} \cdots \lambda a_{ni_n} .$$

$$= \lambda^n |A| .$$

P.3. Let $A = \begin{pmatrix} \underset{\sim}{\alpha}_1' \\ \vdots \\ \underset{\sim}{\alpha}_m' \end{pmatrix} = ((a_{ij}))$ and $B = \begin{pmatrix} \underset{\sim}{\alpha}_1' + \sum_{j\ne 1} \lambda_j \underset{\sim}{\alpha}_j' \\ \underset{\sim}{\alpha}_2' \\ \vdots \\ \underset{\sim}{\alpha}_m' \end{pmatrix}$ then $|B| = |A|$

Sol. Let $B = ((b_{ij}))_{m\times m}$

$b_{1j} = a_{1j} + \sum_{k\ne 1} \lambda_k a_{kj}$, $b_{ij} = a_{ij} + \sum_{k \ne 1} \lambda_k a_{kj}$

$$|B| = \sum_{1\le i_1 \ne i_2 \ne \cdots \ne i_m \le m} (-1)^{N(i_1,\ldots,i_m)} b_{1i_1} b_{2i_2} \cdots b_{mi_m}$$

$$= \sum_{1\le i_1 \ne i_2 \ne \cdots \ne i_m \le m} (-1)^{N(i_1,\ldots,i_m)} \left(a_{1i_1} + \sum_{k=2}^{m} \lambda_k a_{ki_1}\right) a_{2i_2} \cdots a_{mi_m}$$

$$= \sum_{1\le i_1 \ne i_2 \ne \cdots \ne i_m \le m} (-1)^{N(i_1,\ldots,i_m)} a_{1i_1} a_{2i_2} \cdots a_{mi_m} + 0 = |A| .$$

P.A :- $|AB| = |A||B|$

Let $A = ((a_{ij}))_{n\times n}$ $\quad AB = ((\sum a_{ik} b_{kj}))_{n \times n}$

$B = ((b_{ij}))_{n\times n}$

$$|AB| = \sum_{1 \le i_1 \ne i_2 \ne \cdots \ne i_n \le n} (-1)^{N(i_1,\dots,i_n)} \sum_{k=1}^{n} a_{1k} b_{k i_1} \sum_{k=1}^{n} a_{2k} b_{k i_2} \cdots \sum_{k=1}^{n} a_{nk} b_{k i_n},$$

$$= \sum_{1 \le k_1, k_2, \dots, k_n \le n} a_{1k_1} a_{2k_2} \cdots a_{nk_n} \sum_{1 \le i_1 \ne i_2 \ne \cdots \ne i_n \le n} (-1)^{N(i_1,\dots,i_n)} b_{k_1 i_1} b_{k_2 i_2} \cdots b_{k_n i_n}$$

Rewrite B as $B = B(1,2,\dots,n)$

$B(k_1,\dots,k_n)$ = matrix obtained from B replacing the 1st row by k_1th row and 2nd row by k_2th row and so on, $\forall i = 1,\dots,n \; \forall i$.

$$|AB| = \sum_{1 \le k_1, k_2, \dots, k_n \le n} a_{1k_1} a_{2k_2} \cdots a_{nk_n} |B(k_1,\dots,k_n)|$$

$$= \sum_{1 \le k_1 \ne k_2 \ne \cdots \ne k_n \le n} (-1)^{N(k_1,\dots,k_n)} |B(1,2,\dots,n)| a_{1k_1} a_{2k_2} \cdots a_{nk_n}$$

, if at least two k_i's are equal then $|B(k_1,\dots,k_n)|$ vanishes.

$$= |B| \sum_{1 \le k_1 \ne \cdots \ne k_n \le n} (-1)^{N(k_1,\dots,k_n)} a_{1k_1} a_{2k_2} \cdots a_{nk_n}$$

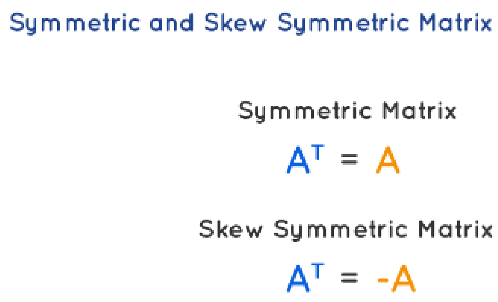
$$= |A||B|.$$

Difference between Skew Symmetric and Symmetric Matrix

The symmetric and skew-symmetric matrices share a close relationship with each other. There is one major difference between symmetric matrix and skew-symmetric matrix. The differences between symmetric and skew-symmetric matrices are explained in the below-given table:

Symmetric Matrix	Skew Symmetric Matrix
A square matrix B which is of size n × n, is considered to be symmetric if and only if $B^T = B$.	A square matrix B which is of size n × n, is considered to be symmetric if and only if $B^T = -B$.

Symmetric Matrix	**Skew Symmetric Matrix**
Here, b_{ij} = b_{ji}.	Here, b_{ij} = - b_{ji}.
There is nothing specific about the determinant of symmetric matrix.	The determinant of a skew-symmetric matrix of odd order is 0.
The eigenvalues of the symmetric matrix are real.	The eigenvalues of the skew-symmetric matrix are purely imaginary.
The elements of the principal diagonal may be any elements.	The elements of the principal diagonal are always zeros.



Properties of Symmetric Matrix

Here are some of the important properties of symmetric matrices.

- The sum and difference of two symmetric matrices give the resultant as a symmetric matrix.
- The property stated above is not always true for the product: Given the symmetric matrices A and B, then AB is symmetric if and only if A and B follow commutative property of multiplication, i.e., if AB = BA.
- For integer n, if A is symmetric, ⇒ A^n is symmetric.
- The eigenvalues of a symmetric matrix are always real and positive.
- The determinant of a matrix and its transpose are same for a symmetric matrix.
- The adjoint of a symmetric matrix is symmetric.
- The inverse of symmetric matrix is symmetric.

Definition of Determinant

A determinant can be defined in many ways for a square matrix.

The first and most simple way is to formulate the determinant by taking into account the top row elements and the corresponding minors. Take the first element of the top row and multiply it by it's minor, then subtract the product of the second element and its minor. Continue to alternately add and subtract the product of each element of the top row with its respective minor until all the elements of the top row have been considered.

Determinant of a Matrix

To solve the system of linear equations and to find the inverse of a matrix, the determinants play an important role. Now, let us discuss how to find the determinant of 2×2 matrix and 3×3 matrix. If A is a matrix, then the determinant of a matrix A is generally represented using det (A) or |A|.

Finding Determinants for 2×2 matrix

Let us assume a 2×2 square matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \, then$$

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$|A| = a_{11}a_{22} - a_{21}a_{12}$$

Finding Determinants for 3×3 Matrix

Now, assume the 3×3 matrix, say

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \, then$$

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$|A| = a_{11}\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12}\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13}\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

Properties of Determinant

- If I_n is the identity matrix of the order nxn, then det(I) = 1
- If the matrix M^T is the transpose of matrix M, then det (M^T) = det (M)
- If matrix M^{-1} is the inverse of matrix M, then det (M^{-1}) = 1/det (M) = det $(M)^{-1}$
- If two square matrices M and N have the same size, then det (MN) = det (M) det (N)
- If matrix M has a size axa and C is a constant, then det (CM) = C^a det (M)
- If X, Y, and Z are three positive semidefinite matrices of equal size, then the following holds true along with the corollary det (X+Y) ≥ det(X) + det (Y) for X,Y, Z ≥ 0 det (X+Y+Z) + det C ≥ det (X+Y) + det (Y+Z)
- In a triangular matrix, the determinant is equal to the product of the diagonal elements.
- The determinant of a matrix is zero if all the elements of the matrix are zero.
- Laplace’s Formula and the Adjugate Matrix.

Determinants and Matrices

Matrices are grids of numbers organized in rows and columns. They're used to solve equations and describe transformations in math.

Determinants are special numbers calculated from square matrices. They help find areas, volumes, and solve equations in geometry and algebra.

Aspect	Matrices	Determinants
Definition	A rectangular array of numbers arranged in rows and columns.	A scalar value derived from a square matrix.
Structure	Consists of multiple rows and columns.	A single numerical value.
Dimensions	Can be any size (m x n), where m and n are positive integers.	Only defined for square matrices (n x n).
Purpose	Used to represent and solve systems of linear equations, transformations, and data structures.	Determines properties of a matrix, such as invertibility and volume calculations.
Applications	Widely used in fields like physics, engineering, computer science, and economics for various computations.	Used in solving linear equations, finding areas, volumes, and in various geometric and algebraic computations.
Invertibility	A matrix itself may be invertible if it has a non-zero determinant.	Indicates if a matrix is invertible (non-zero) or not (zero).

Example	$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$	For the matrix $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, the determinant is 1(4) – 2(3) = −2.

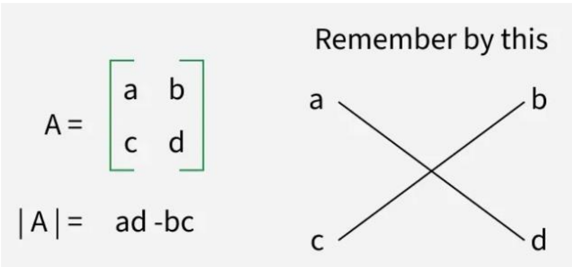
Determinants of Different Types of Matrices
To understand how determinants are evaluated, let us go through the process step by step, starting from the simplest 1×1 matrix and gradually moving to more complex and special cases.

1×1 Matrix

Let X = [a] be the matrix of order one, then its determinant is given by **det(X) = a.**

2×2 Matrix

The determinant of any 2×2 square matrix A = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}_{2\times 2}$ is calculated using the formula **|A| = ad - bc.**



Example: Find the Determinant of A = $\begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$.

Solution:

Determinant of A = $\begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}_{2\times 2}$ *is calculated as,*

$|A| = \begin{vmatrix} 3 & 2 \\ 2 & 3 \end{vmatrix}$

$|A| = 3 \times 3 - 2 \times 2$
$= 9 - 4$
$= 5$

3×3 Matrix

The determinant of a 3x3 Matrix is determined by expressing it in terms of 2nd-order determinants. It can be expanded either along rows(R1, R2 or R3) or column(C1 , C2 or C3).
Consider a matrix A of order 3×3.

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$|A| = a(ei - fh) - b(di - gf) + c(dh - eg)$$

In terms of Cofactor:

$$\begin{bmatrix} a_x & & \\ & e & f \\ & h & i \end{bmatrix} - \begin{bmatrix} & b_x & \\ d & & f \\ g & & i \end{bmatrix} + \begin{bmatrix} & & X^c \\ d & e & \\ g & h & \end{bmatrix}$$

Similarly, in this way, we can expand it along any row and any column.

Example: Evaluate the determinant det(A) = $\begin{vmatrix} 1 & 3 & 0 \\ 4 & 1 & 0 \\ 2 & 0 & 1 \end{vmatrix}$

Solution:

We see that the third column has most number of zeros, so it will be easier to expand along that column.

$$det(A) = (-1)^{1+3}0\begin{vmatrix} 4 & 1 \\ 2 & 0 \end{vmatrix} + (-1)^{2+3}0\begin{vmatrix} 1 & 3 \\ 2 & 0 \end{vmatrix} + (-1)^{1+3}1\begin{vmatrix} 1 & 3 \\ 4 & 1 \end{vmatrix}$$

$$= -11$$

4×4 Matrix

Determining the determinant of a 4 × 4 matrix involves more complex methods, such as expansion by minors or Gaussian elimination. These techniques require breaking down the matrix into smaller submatrices and recursively finding their determinants. While there isn't a direct formula like Sarrus' Rule for 3x3 matrices, the process involves systematic calculations based on the properties of determinants.

Determinant of 4 × 4 Matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

$$[A] = a_{11}[A_{11}] - a_{12}[A_{12}] + a_{13}[A_{13}] - a_{14}[A_{14}]$$

Where Aij denotes the submatrix by deleting i^{th} row and j^{th} column.

$$[A] = a_{11}\begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} & a_{24} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{vmatrix} - a_{14}\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{vmatrix}$$

Identity Matrix

An identity matrix is a square matrix in which all the elements of the main diagonal are ones and all other elements are zeros. For example, a 3x3 identity matrix looks like this:

$$I = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

Given below is the determinant of an identity matrix:

Determinant of an identity matrix is 1

$\det(\mathbf{I}) = 1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The determinant of an identity matrix of any size is always 1. This property can be understood intuitively by considering that the identity matrix represents a transformation that leaves vectors unchanged when multiplied by it. Since the determinant measures how a matrix scales the space, the determinant of an identity matrix, which doesn't scale the space at all, is 1.

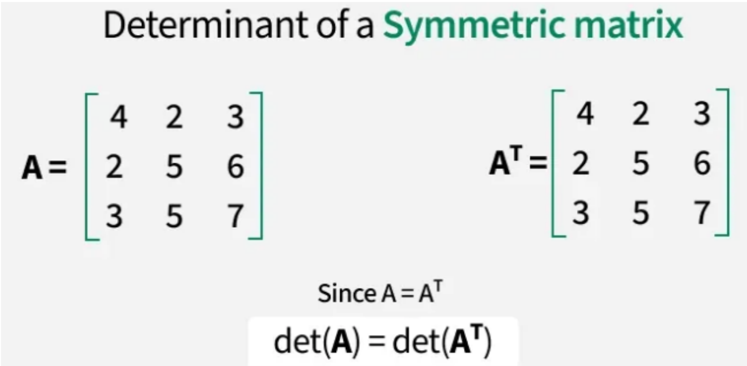
Mathematically, we can express this as:

det (I) = 1

Symmetric Matrix

A symmetric matrix is a square matrix that is equal to its transpose. In other words, if A is a symmetric matrix, then $A = A^T$. Symmetric matrices have several interesting properties, one of which is that their determinants remain unchanged under transpose.

Given below is the determinant of a Symmetric matrix:



Hence, for a <u>symmetric matrix</u> A , we have:

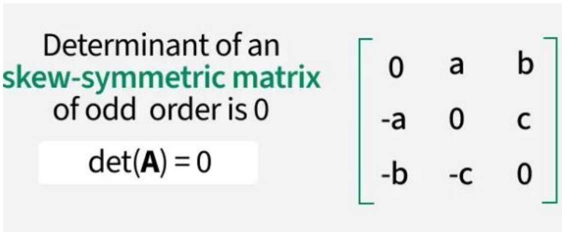
$\det(A) = \det(A^T)$

This property simplifies the computation of determinants for symmetric matrices since you can work with either the original matrix or its transpose, whichever is more convenient.

Skew-Symmetric Matrix

A skew-symmetric (or antisymmetric) matrix is a square matrix whose transpose is equal to its negative. In other words, if A is a skew-symmetric matrix, then $A = -A^T$. <u>Skew-symmetric matrices</u> have interesting properties, one of which is that their determinants have specific values based on the order of the matrix.

Given below is the determinant of a Skew-Symmetric matrix:



For skew-symmetric matrices of odd order, the determinant is always 0. This is because the determinant of a skew-symmetric matrix is always the square of its <u>eigenvalues</u>, and a non-zero square is always positive. Since the order of the matrix is odd, at least one eigenvalue must be zero, resulting in a determinant of 0.

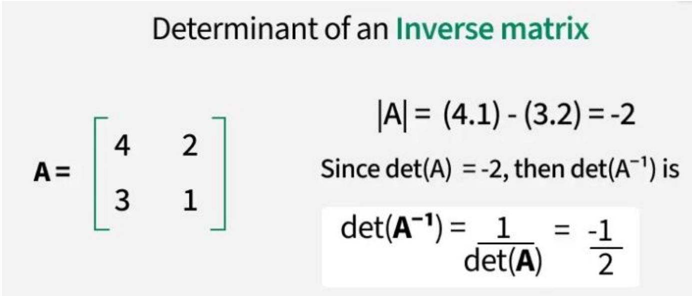
For skew-symmetric matrices of even order, the determinant is a non-zero value, which can be calculated based on the elements of the matrix. However, determining the exact value typically involves more complex methods such as cofactor expansion or using properties of determinants.

Inverse Matrix

To understand the determinant of the inverse matrix, let's first define w as the <u>inverse of a matrix</u>

The inverse of a <u>square matrix</u> A, denoted as A^{-1}, is a matrix such that when it's multiplied by A, the result is the identity matrix $\boldsymbol{I}$. Mathematically, if $\mathbf{A \cdot A^{-1} = I}$, then $\mathbf{A^{-1}}$ is the inverse of A.

Given below is the determinant of an Inverse matrix:



Now, the determinant of the inverse matrix, denoted as det($A-1$), is related to the determinant of the original matrix A. Specifically, it can be expressed by the formula:
det(A −1) = 1/det(A)

This formula illustrates an important relationship between the determinants of a matrix and its inverse. If the determinant of A is non-zero, meaning d**et(A) ≠ 0** , then the inverse matrix exists, and its determinant is the reciprocal of the determinant of A . Conversely, **(*A*) = 0**an , the matrix A is said to be singular, and it does not have an inverse.

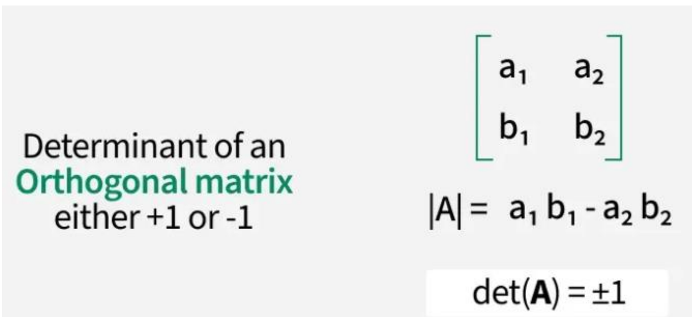
Here are some key points about the determinant of the inverse matrix:

- **Non-Singular Matrices**: For non-singular matrices (those with non-zero determinants), their inverses exist, and the determinant of the inverse is the reciprocal of the determinant of the original matrix.
- **Singular Matrices**: Singular matrices (those with zero determinants) do not have inverses. Attempting to find the inverse of a singular matrix results in an undefined or non-existent inverse.
- **Geometric Interpretation**: The Determinant of the Matrix measures how it scales the space. Similarly, the determinant of the inverse matrix measures the scaling effect of the inverse transformation. If the original transformation expands the space, its inverse contraction will be inversely proportional, and vice versa.

Orthogonal Matrix

An orthogonal matrix is a square matrix whose rows and columns are orthonormal vectors, meaning that the dot product of any two distinct rows or columns equals zero, and the dot product of each row or column with itself equals one. Mathematically, if A is an orthogonal matrix , then $A\ T \cdot A = I$, where $A\ T$ denotes the transpose of A and I represents the identity matrix.

Given below is the determinant of an Orthogonal matrix:



The determinant of an orthogonal matrix has a special property:
det (A) = ±1

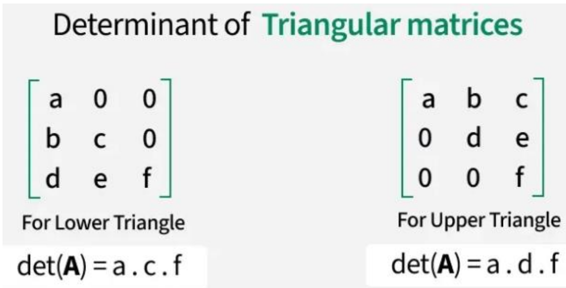
The determinant of an orthogonal matrix is either +1 or −1. This property arises from the fact that the determinant represents the scaling factor of the matrix transformation. Since orthogonal transformations preserve lengths, the determinant must be either positive (for preserving orientation) or negative (for reversing orientation).

The determinant of an orthogonal matrix being +1+1 implies that the transformation preserves orientation, while a determinant of −1 indicates a transformation that reverses orientation.

Triangular Matrix

A **triangular matrix** is a special type of **square matrix** in which all the elements **above** or **below** the main diagonal are zero.

Given below is the determinant of Triangular matrices:



There are two main types:

1) Lower Triangular Matrix:

- A square matrix in which all elements above the main diagonal are zero.
  - Example: $\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

2) Upper Triangular Matrix:

- A square matrix in which all elements below the main diagonal are zero.
  - Example: $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$

Properties of Determinants of a Matrix

Various Properties of the Determinants of the square matrix are discussed below:

- **Reflection Property**: Value of the determinant remains unchanged even after rows and columns are interchanged. The Determinant of a Matrix and its transpose remains the same.
- **Switching Property:** If any two rows or columns of a determinant are interchanged, then the sign of the determinant changes.
- **Scalar Multiplication Property**: If each element in a row or column of a matrix A is multiplied by a scalar k, then the determinant of the resulting matrix is k times the

determinant of A. Mathematically, if B is the matrix obtained by multiplying each element of a row or column of A by **det**k, then **det (*B*) = *k* · det(*A*)**.

- **Multiplicative Property**: The determinant of the product of two matrices A and B is equal to the product of their determinants. Symbolically, **det(*AB*) = det(*A*)·det(*B*)**. However, this property holds true only for square matrices.
- **Determinant of Transpose**: The Determinant of Matrix A is equal to the determinant of its transpose A T Mathematically, **det(*A*) = det(*A T*)** .

Conjugate of a Matrix

It is possible to find the conjugate for a given matrix by replacing each element of the matrix with its complex conjugate.

Mathematically, a **conjugate matrix** is a matrix $\overline{A}$

Obtained by replacing the complex conjugate of all the elements of the matrix A.
Let's have a look at the example given below.

Example: Find the conjugate matrix of the matrix

$$A = \begin{bmatrix} 1-5i & 0 \\ 0 & 6+7i \end{bmatrix}$$

.

Solution:

Given,

$$A = \begin{bmatrix} 1-5i & 0 \\ 0 & 6+7i \end{bmatrix}$$

Now, the conjugates of each of the elements of matrix A are:

Conjugate of 0 is 0.

Conjugate of 1 – 5i is 1 + 5i.

Conjugate of 6 + 7i is 6 – 7i.

Therefore, the conjugate of A is

$$\overline{A} = \begin{bmatrix} 1+5i & 0 \\ 0 & 6-7i \end{bmatrix}$$

Conjugate Matrix

A conjugate matrix is a matrix $\bar{A}$ obtained from a given matrix A by taking the complex conjugate of each element of A.

$$\overline{(a_{ij})} = (\bar{a}_{ij}).$$

The notation A is sometimes also used, which can lead to confusion since this symbol is also used to denote the conjugate transpose.

Using a matrix X in a similarity transformation $X^{-1}AX$ of a given matrix A is also known as conjugating A by X. In this case, $B = X^{-1}AX$ and A are known as similar matrices.

Elementary matrix

An elementary matrix is a square matrix that has been obtained by performing an elementary row or column operation on an identity matrix.

Definition

Remember that there are three types of elementary row operations:

1. interchange two rows;
2. multiply a row by a non-zero constant;
3. add a multiple of one row to another row.

Elementary column operations are defined similarly (interchange, addition and multiplication are performed on columns).

When elementary operations are carried out on identity matrices they give rise to so-called elementary matrices.

Definition A $K \times K$ matrix is said to be an elementary matrix if and only if it is obtained by performing an elementary (row or column) operation on the $K \times K$ identity matrix I.

Elementary transformation in matrices

Elementary transformations are operations done on the rows and columns of matrices to change their shape so that the computations become easier. It is also used to discover the inverse of a matrix, the determinants of a matrix, and to solve a system of linear equations.

Elementary Row Transformations

Row transformations are performed only on the basis of a few sets of rules. An individual cannot perform any other kind of row operation apart from the below-stated rules. There are three kinds of elementary row transformations.

1. **Interchanging the rows within the matrix:** In this operation, the entire row in a matrix is swapped with another row. It is symbolically represented as Ri ↔ Rj, where i and j are two different row numbers.
2. **Scaling the entire row with a non zero number:** The entire row is multiplied with the same non zero number. It is symbolically represented as Ri → k Ri which indicates that each element of the row is scaled by a factor 'k'.
3. **Add one row to another row multiplied by a non zero number:** Each element of a row is replaced by a number obtained by adding it to the scaled element of another row. It is symbolically represented as Ri → Ri + k Rj.

Two matrices are said to be row equivalent if and only if one matrix can be obtained from the other by performing any of the above elementary row transformations.

Example for Row Equivalent Matrices

1. Show that matrices A and B are row equivalent if

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \text{and B} = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 3 & 1 \end{bmatrix}$$

Solution:

Consider the matrix A. Apply row transformation such that R1 → R1 + R2

Applying row transformations to the first row, A11 = 1 + 2, A12 = -1 + 1 and A13 = 0 + 1
So matrix A will be equal to

$$\begin{bmatrix} 3 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

Now let us retain the first row and apply row transformation to the second row such that R2 → 3 R2 - R1

So the elements of second row in A will be given as follows:

A21 = 2 x 3 - 3 = 3

A22 = 1 x 3 - 0 = 3

A23 = 1 x 3 - 1 = 2

So matrix A will be equal to

$$\begin{bmatrix} 3 & 0 & 1 \\ 3 & 3 & 2 \end{bmatrix}$$

Retain R1 and apply row transformation to R2 such that $R2 \rightarrow R2 - R1$.

A21 = 3 - 3 = 0

A22 = 3 - 0 = 3

A23 = 2 - 1 = 1

So the matrix A will be equal to matrix B.

$$\begin{bmatrix} 3 & 0 & 1 \\ 0 & 3 & 1 \end{bmatrix}$$

From this, we can conclude that A and B are row equivalent matrices.

Elementary Column Transformations

There are also a few sets of rules to be followed while performing column transformations. There are three different forms of elementary column transformations. No other column transformations are allowed apart from these three.

1. **Interchanging the columns within the matrix:** In this operation, the entire column in a matrix is swapped with another column. It is symbolically represented as $C_i \leftrightarrow C_j$, where i and j are two different column numbers.
2. **Multiplying the entire column with a non zero number:** The entire column is multiplied or divided by the same non zero number. It is symbolically represented as $C_i \rightarrow k\, C_i$ which indicates that each element of the column is multiplied by a scaling factor 'k'.

3. **Add one column to another column scaled by a non zero number:** Each element of a column is replaced by a number obtained by adding it to the scaled element of another column. It is symbolically represented as $C_i \rightarrow C_i + k\, C_j$.

Two matrices are said to be column equivalent if and only if one matrix can be obtained from the other by performing any of the above elementary column transformations.

Elementary Matrix Operations

Generally, there are three known elementary matrix operations performed on rows and columns of matrices. The operations performed on the rows are known as **elementary matrix row operations**. Whereas, the operations performed on columns are known as **elementary matrix column operations.**

The three different elementary matrix operations for rows are:

- Interchange Two Rows
- Multipling a Row by a Number
- Adding one Row to Another Row

And, the three elementary matrix operations for columns are:

- Interchange Columns
- Multiply a Column by a Mumber
- Adding one Column to Another

Now, let's look into how are these operations performed.

Elementary Matrix Row Operations

To perform the elementary row operations let suppose a matrix $A_{r\times c}$ that will be $A_{3\times 3}$

$$\text{Let } A = \begin{bmatrix} 2 & 4 & 5 \\ 4 & 8 & 3 \\ 7 & 1 & 2 \end{bmatrix}$$

Interchanging two Rows

This operation can be carried out by interchanging the position of any two rows of the matrix. It is indicated by **R1<=>R2.**

Interchanging the rows of Matrix

$$A = \begin{bmatrix} 2 & 4 & 5 \\ 4 & 8 & 3 \\ 7 & 1 & 2 \end{bmatrix}$$

Hence, R_1<=>R_2 will be

$$\begin{bmatrix} 2 & 4 & 5 \\ 4 & 8 & 3 \\ 7 & 1 & 2 \end{bmatrix} \leftrightarrow \begin{bmatrix} 4 & 8 & 3 \\ 2 & 4 & 5 \\ 7 & 1 & 2 \end{bmatrix}$$

Here, the row 1 is replaced by row 2 and row 2 is replaced by 1. Whereas, the row 3 remains unchanged.

Multiplying a Row by a Number

This operation can be carried out by multiplying a row with a non-zero constant that will replace the elements of the row.

Lets multiply row 2 of the given

$$\text{Matrix A} = \begin{bmatrix} 2 & 4 & 5 \\ 4 & 8 & 3 \\ 7 & 1 & 2 \end{bmatrix} \text{ by 2.}$$

Hence, R_2 <=>$2R_2$ will be

$$\begin{bmatrix} 2 & 4 & 5 \\ 4 \times 2 & 8 \times 2 & 3 \times 2 \\ 7 & 1 & 2 \end{bmatrix} \leftrightarrow \begin{bmatrix} 2 & 4 & 5 \\ 8 & 16 & 6 \\ 7 & 1 & 2 \end{bmatrix}$$

Here, the 2nd row is replaced by 2 times of itself.

Adding one Row to Another

This operation can be performed by summing up anyone row with another one in the matrix. The remaining rows of the matrix remain unchanged. It can be indicated by **R1+R2<=>R2**
Let's sum up rows 1 and 3 to replace the elements of row 3 in the given matrix.

$$\begin{bmatrix} 2 & 4 & 5 \\ 4 & 8 & 3 \\ 7+2 & 1+4 & 2+5 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 4 & 5 \\ 4 & 8 & 3 \\ 9 & 5 & 7 \end{bmatrix}$$

Here, row 3 is replaced by the sum of rows 1 and 3. Whereas, row 1 and 2 remains unchanged.

Elementary Matrix Column Operations

To perform the elementary matrix column operation let us suppose a matrix $A_{r\times c}$ that will be $A_{3\times 3}$.

$$\text{Let} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 4 \\ 0 & 3 & 5 \end{bmatrix}$$

Interchanging Two Columns

This operation can be carried out by interchanging the position of any two columns of the matrix. It is indicated by **C1 <=> C2**.
Interchanging the columns of the matrix

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 4 \\ 0 & 3 & 5 \end{bmatrix}$$

Hence, C_1 <=> C_2 will be

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 4 \\ 0 & 3 & 5 \end{bmatrix} \leftrightarrow \begin{bmatrix} 2 & 1 & 4 \\ 2 & 0 & 4 \\ 3 & 0 & 5 \end{bmatrix}$$

Here, the column 1 is replaced by column 2 and column 2 is replaced by 1. Whereas, the column 3 remains unchanged.

Multiplying a Column by a Number

This operation can be carried out by multiplying a column with a non-zero constant that will replace the elements of the column.
Let's multiply column 2 of the given matrix

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 4 \\ 0 & 3 & 5 \end{bmatrix}$$

Hence, $2C_2 \Rightarrow C_2$ will be

$$\begin{bmatrix} 1 & 2 \times 2 & 4 \\ 0 & 2 \times 2 & 4 \\ 0 & 3 \times 2 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 4 & 4 \\ 0 & 4 & 4 \\ 0 & 6 & 5 \end{bmatrix}$$

Here, the column 2 is replaced by 2 times of itself.

Adding one Column to Another

This operation can be performed by summing up anyone column with another one in the matrix. The remaining columns of the matrix remain unchanged. It can be indicated by **C1 + C2 = C2**
Let's sum up columns 1 and 2 to replace the elements of column 2 in the given matrix.

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 4 \\ 0 & 3 & 5 \end{bmatrix}$$

Hence, $C_1 + C_2 = C_2$ will be

$$\begin{bmatrix} 1 & 2+1 & 4 \\ 0 & 2+0 & 4 \\ 0 & 3+0 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 4 \\ 0 & 2 & 4 \\ 0 & 3 & 5 \end{bmatrix}$$

Here, column 2 is replaced by the sum of columns 1 and 2. Whereas, column 1 and 3 remains unchanged.

Reducing a matrix into Normal form:—

Let A be a matrix of order $m \times n$, suppose $R(A) = r < \min(m,n)$. Then $\exists$ non-singular matrices P and Q such that

$$PAQ = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}.$$

Proof:- Since $R(A) = r < \min(m,n)$

$\exists$ a non-singular matrix P_1 such that $P_1 A = E$, an echelon matrix with r non-null rows.

Clearly, $E = \begin{pmatrix} E_1 \\ 0 \end{pmatrix}_{m\times n}$, E_1 is also an echelon matrix.

Now $\text{Rank}(E_1) = r =$ Column rank of E_1.

E_1 has $n-r$ columns each of which is LD on the rest r LIN columns. Hence through column operations those dependent columns can be reduced to null columns.

i.e. $E_1 \sim \begin{pmatrix} E_2 & 0 \\ {\scriptstyle r\times r} & {\scriptstyle r\times n} \end{pmatrix}$, where E_2 has full rank.

$\therefore \exists$ a non-singular matrix $P_2 \ni$

$$P_1 A P_2 = \begin{pmatrix} E_2 & 0 \\ 0 & 0 \end{pmatrix}$$

As E_2 is a square matrix of full rank through row column operation it can be reduced to I_r.

Hence $\exists$ non-singular matrices P_3 and P_4

$$P_3 P_1 A P_2 P_4 = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}_{m\times n}$$

Let $P_3 P_1 = P$ and $P_2 P_4 = Q$

Clearly, P and Q are non-singular matrices as they are product of non-singular matrices.

A square matrix of full rank is always non-singular.

Let A be a square matrix of order n possessing full rank then $\exists$ non-singular matrices P and Q $\ni$

$$PAQ = I_n.$$

$$|P||A||Q| = 1$$

$$\therefore |A| \neq 0.$$

Reducing a matrix to its normal form

Multiply Row by a Non-Zero Scalar Reducing a matrix to its normal form involves applying elementary row and column operations to transform the matrix into a form where the leading non-zero entries are 1s and all other entries in those columns are 0s. The resulting matrix resembles an identity matrix in its top-left corner, with zeros elsewhere. This normal form is unique and is often used to find the [rank]() of the original matrix, which is equal to the number of non-zero rows in the normal form.

Steps to Reduce a Matrix to its Normal Form

1. **1. Apply Elementary Row Operations:**

   Use row operations to get the matrix into [row echelon form]().

- **Swap Rows:** Interchange two rows ($R_i \leftrightarrow R_j$).
- **:** Multiply a row by a non-zero constant ($kR_i \rightarrow R_i$).
- **Add Multiple of one Row to Another:** Add a multiple of one row to another row ($R_i + kR_j \rightarrow R_i$).

2. Apply Elementary Column Operations:

After reaching row echelon form, apply column operations to introduce zeros above the leading 1s.

- **Swap Columns:** Interchange two columns ($C_i \leftrightarrow C_j$).
- **Multiply Column by a Non-Zero Scalar:** Multiply a column by a non-zero constant ($kC_i \rightarrow C_i$).
- **Add Multiple of one Column to Another:** Add a multiple of one column to another column ($C_i + kC_j \rightarrow C_i$).

3. Achieve Normal Form:

Continue applying these operations until the matrix has the following structure:

Steps to Reduce a Matrix to Normal Form:

1. Start with the original matrix:

Write out the matrix you want to reduce.

2. Use elementary row operations to get a pivot:

- If the top-left element is not 1, divide the first row by that element to make it 1 (if it's not already 1 and non-zero).

- If the first row is all zeros, swap it with a row that has a non-zero element in the first column.
- If there are zeros in the first column and you need to get a 1 in the top-left position, swap rows.

3. Eliminate entries below the first pivot:

- Subtract multiples of the first row from the rows below it to make the first column entries below the pivot zero.

4. Move to the second row and pivot:

- Focus on the submatrix to the right of the first column.
- Repeat the process: make the leading entry of the second row a 1 (if possible) and then use it to make the entries below it zero.

5. Continue the process:

Repeat this for each subsequent row, moving down and to the right, until the entire matrix is in row-reduced echelon form.

6. Use elementary column operations (if necessary):

In some cases, you may also need to perform column operations to get the normal form, which involves a specific structure of ones and zeros in the "diagonal" area.

7. Verify the normal form:

The final form will have leading 1s in each non-zero row, with zeros in the columns of these leading 1s, and zero rows at the bottom.

- **Normal form**: is also known as row-reduced echelon form or canonical form.
- **Elementary row operations**: allowed are multiplying a row by a non-zero scalar, adding a multiple of one row to another, and swapping two rows.
- The rank of the matrix is determined by counting the number of non-zero rows in its normal form.

Equivalent matrices are matrices that can be transformed into one another through a sequence of elementary row operations, such as multiplying a row by a scalar, swapping rows, or adding a multiple of one row to another. This means they represent the same linear transformation and have the same rank, but they are not necessarily identical in their elements or dimensions.

Example of Equivalent Matrices

Let's consider two matrices, A and B:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

We can perform an elementary row operation on matrix A to obtain matrix B. For example, let's swap the first and second rows of A to get B:

$$B = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$$

Why are they equivalent?

- **Same Rank:**

  Both matrices have the same rank, which is a fundamental property of equivalent matrices.

- **Same Underlying Linear Transformation:**

  Although the matrices themselves look different, they represent the same underlying linear transformation. The specific arrangement of elements (rows and columns) is just a different representation of that same transformation.

- **Obtained by Elementary Row Operations:**

  Matrix B was obtained from matrix A by a single, valid elementary row operation (swapping rows). This is the key criterion for two matrices to be equivalent.

Conditions for Equivalence
Two matrices, A and B, are considered equivalent if:

- They have the same dimension (same number of rows and columns).
- One can be transformed into the other using a finite sequence of elementary row or column operations.
- They share the same rank.