UNIT-V

Characteristic roots and vector

Characteristic roots, or Eigen values, and characteristic vectors, or eigenvectors, are a pair of a scalar and a non-zero vector related to a square matrix

Characteristic roots (or eigen values) and their associated characteristic vectors (or eigenvectors) are fundamental concepts in linear algebra that describe how a matrix scales and transforms vectors.

The relationship is defined by the equation $Av = \lambda v$, where A is the matrix, v is the eigenvector, and λ is the eigenvalue. Eigenvalues represent how the matrix stretches or compresses a vector, while eigenvectors are the directions that remain unchanged, only scaled by the eigenvalue.

Characteristic roots (eigenvalues)

- A scalar value associated with a linear transformation and a specific eigenvector.
- Found by solving the characteristic equation, which is the determinant of the matrix A minus the eigenvalue (λ) times the identity matrix (I) set to zero: $det(A - \lambda I) = 0$.

Characteristic vectors (eigenvectors)

- A non-zero vector that, when a linear transformation is applied, is only scaled by a single factor (the eigenvalue) and does not change its direction.
- For each eigenvalue λ, the corresponding eigenvector v is found by solving the equation $(A - \lambda I)v = 0$.
- **Equation:** $Av = \lambda v$
- Characteristic Root (λ): The scalar that multiplies the eigenvector.
- Characteristic Vector (v): The non-zero vector that is only scaled by the eigenvalue.
- **How to find them:**
  1. Solve the characteristic equation $det(A - \lambda I) = 0$ to find the eigenvalues (λ).
  2. For each eigenvalue, solve the system of equations $(A - \lambda I)v = 0$ to find the corresponding eigenvector (v).

Important Properties

- For an $n \times n$ matrix, the characteristic equation will be an n-th degree polynomial, yielding n characteristic roots (counting multiplicities).
- If a characteristic root has multiple linearly independent characteristic vectors associated with it, those vectors must also be linearly independent.
- The characteristic vectors are not unique; if x is a characteristic vector, then any scalar multiple cx (where c is a non-zero constant) is also a characteristic vector corresponding to the same root.

1.1. **Statement of the characteristic root problem.** Find values of a scalar λ for which there exist vectors $x \neq 0$ satisfying

$$Ax = \lambda x \tag{1}$$

where A is a given nth order matrix. The values of λ that solve the equation are called the characteristic roots or eigenvalues of the matrix A. To solve the problem rewrite the equation as

$$\begin{aligned} Ax &= \lambda x = \lambda I x \\ \Rightarrow (\lambda I - A)x &= 0 \qquad x \neq 0 \end{aligned} \tag{2}$$

For a given λ, any x which satisfies 1 will satisfy 2. This gives a set of n homogeneous equations in n unknowns. The set of x's for which the equation is true is called the null space of the matrix $(\lambda I - A)$. This equation can have a non-trivial solution iff the matrix $(\lambda I - A)$ is singular. This equation is called the characteristic or the determinantal equation of the matrix A. To see why the matrix must be singular consider a simple 2x2 case. First solve the system for x_1

$$\begin{aligned} Ax &= 0 \\ \Rightarrow a_{11} x_1 + a_{12} x_2 &= 0 \\ a_{21} x_1 + a_{22} x_2 &= 0 \\ \Rightarrow x_1 &= -\frac{a_{12} x_2}{a_{11}} \end{aligned} \tag{3}$$

Now substitute x_1 in the second equation

$$\begin{aligned} -a_{21} \frac{a_{12} x_2}{a_{11}} + a_{22} x_2 &= 0 \\ \Rightarrow x_2 \left(a_{22} - \frac{a_{21} a_{12}}{a_{11}} \right) &= 0 \\ \Rightarrow x_2 = 0 \text{ or } \left(a_{22} - \frac{a_{21} a_{12}}{a_{11}} \right) &= 0 \\ If\ x_2 \neq 0\ then \left(a_{22} - \frac{a_{21} a_{12}}{a_{11}} \right) &= 0 \\ \Rightarrow |A| &= 0 \end{aligned} \tag{4}$$

Determinantal equation used in solving the characteristic root problem. Now consider the singularity condition in more detail

$$
\begin{aligned}
(\lambda I - A)x &= 0 \\
\Rightarrow |\lambda I - A| &= 0 \\
\Rightarrow \begin{vmatrix} \lambda - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \lambda - a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & \lambda - a_{nn} \end{vmatrix} &= 0
\end{aligned} \tag{5}
$$

This equation is a polynomial in λ since the formula for the determinant is a sum containing n! terms, each of which is a product of n elements, one element from each column of A. The fundamental polynomials are given as

$$
|\lambda I - A| = \lambda^n + b_{n-1}\lambda^{n-1} + b_{n-2}\lambda^{n-2} + \cdots + b_1\lambda + b_0 \tag{6}
$$

This is obvious since each row of $|\lambda I - A|$ contributes one and only one power of λ as the determinant is expanded. Only when the permutation is such that column included for each row is the same one will each term contain λ, giving λ^n. Other permutations will give lesser powers and b_0 comes from the product of the terms on the diagonal (not containing λ) of A with other members of the matrix. The fact that b_0 comes from all the terms not involving λ implies that it is equal to $|-A|$.

Consider a 2x2 example

$$
\begin{aligned}
|\lambda I - A| &= \begin{vmatrix} \lambda - a_{11} & -a_{12} \\ -a_{21} & \lambda - a_{22} \end{vmatrix} \\
&= (\lambda - a_{11})(\lambda - a_{22}) - (a_{12}a_{21}) \\
&= \lambda^2 - a_{11}\lambda - a_{22}\lambda + a_{11}a_{22} - a_{12}a_{21} \\
&= \lambda^2 + (-a_{11} - a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21} \\
&= \lambda^2 - \lambda(a_{11} + a_{22}) + (a_{11}a_{22} - a_{12}a_{21}) \\
&= \lambda^2 + b_1\lambda + b_0 \\
b_0 &= |-A|
\end{aligned} \tag{7}
$$

Consider also a 3x3 example where we find the determinant using the expansion of the first row

$$
\begin{aligned}
|\lambda I - A| &= \begin{vmatrix} \lambda - a_{11} & -a_{12} & -a_{13} \\ -a_{21} & \lambda - a_{22} & -a_{23} \\ -a_{31} & -a_{32} & \lambda - a_{33} \end{vmatrix} \\
&= (\lambda - a_{11}) \begin{vmatrix} \lambda - a_{22} & -a_{23} \\ -a_{32} & \lambda - a_{33} \end{vmatrix} + a_{12} \begin{vmatrix} -a_{21} & -a_{23} \\ -a_{31} & \lambda - a_{33} \end{vmatrix} - a_{13} \begin{vmatrix} -a_{21} & \lambda - a_{22} \\ -a_{31} & -a_{32} \end{vmatrix}
\end{aligned} \tag{8}
$$

Now expand each of the three determinants in equation 8. We start with the first term

$$(\lambda - a_{11})\begin{vmatrix} \lambda - a_{22} & -a_{23} \\ -a_{32} & \lambda - a_{33} \end{vmatrix} = (\lambda - a_{11})\left[\lambda^2 - \lambda a_{33} - \lambda a_{22} + a_{22}a_{33} - a_{23}a_{32}\right]$$
$$= (\lambda - a_{11})\left[\lambda^2 - \lambda\,(a_{33} + a_{22}) + a_{22}a_{33} - a_{23}a_{32})\right]$$
$$= \lambda^3 - \lambda^2\,(a_{33} + a_{22}) + \lambda\,(a_{22}a_{33} - a_{23}a_{32}) - \lambda^2 a_{11} + \lambda a_{11}\,(a_{33} + a_{22}) - a_{11}\,(a_{22}a_{33} - a_{23}a_{32})$$
$$= \lambda^3 - \lambda^2\,(a_{11} + a_{22} + a_{33}) + \lambda\,(a_{11}a_{33} + a_{11}a_{22} + a_{22}a_{33} - a_{23}a_{32}) - a_{11}a_{22}a_{33} + a_{11}a_{23}a_{32} \tag{9}$$

Now the second term

$$a_{12}\begin{vmatrix} -a_{21} & -a_{23} \\ -a_{31} & \lambda - a_{33} \end{vmatrix} = a_{12}\left[-\lambda a_{21} + a_{21}a_{33} - a_{23}a_{31}\right] \tag{10}$$
$$= -\lambda a_{12}a_{21} + a_{12}a_{21}a_{33} - a_{12}a_{23}a_{31}$$

Now the third term

$$-a_{13}\begin{vmatrix} -a_{21} & \lambda - a_{22} \\ -a_{31} & -a_{32} \end{vmatrix} = -a_{13}\left[a_{21}a_{32} + \lambda a_{31} - a_{22}a_{31}\right] \tag{11}$$
$$= -a_{13}a_{21}a_{32} - \lambda a_{13}a_{31} + a_{13}a_{22}a_{31}$$

We can then combine the three expressions to obtain the determinant. The first term will be λ^3, the others will give polynomials in λ^2, and λ. Note that the constant term is the negative of the determinant of A. expressions to obtain

$$|\lambda I - A| = \begin{vmatrix} \lambda - a_{11} & -a_{12} & -a_{13} \\ -a_{21} & \lambda - a_{22} & -a_{23} \\ -a_{31} & -a_{32} & \lambda - a_{33} \end{vmatrix}$$

$$\begin{aligned} &= \lambda^3 - \lambda^2\,(a_{11} + a_{22} + a_{33}) + \lambda\,(a_{11}a_{33} + a_{11}a_{22} + a_{22}a_{33} - a_{23}a_{32}) - a_{11}a_{22}a_{33} + a_{11}a_{23}a_{32} \\ &- \lambda a_{12}a_{21} + a_{12}a_{21}a_{33} - a_{12}a_{23}a_{31} \\ &- a_{13}a_{21}a_{32} - \lambda a_{13}a_{31} + a_{13}a_{22}a_{31} \\ &= \lambda^3 - \lambda^2\,(a_{11} + a_{22} + a_{33}) + \lambda\,(a_{11}a_{33} + a_{11}a_{22} + a_{22}a_{33} - a_{23}a_{32} - a_{12}a_{21} - a_{13}a_{31}) \\ &- a_{11}a_{22}a_{33} + a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33} - a_{12}a_{23}a_{31} - a_{13}a_{21}a_{32} + a_{13}a_{22}a_{31} \end{aligned} \tag{12}$$

Cayley – Hamilton Theorem

The Cayley–Hamilton Theorem is a fundamental result in linear algebra that connects a matrix with its characteristic polynomial. Simply put, it states that every square matrix satisfies its own characteristic equation.

For an (n × n) matrix A, the characteristic polynomial P(λ) is given as:

$P(\lambda) = det(\lambda I_n - A)$

where,

- λ is a variable
- I_n is the identity matrix of order n.

This polynomial is of degree n and can be written in general form as:

$$p(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$$

The roots of $p(\lambda)$ are the **eigenvalues** of A.

Theorem Statement

The Cayley–Hamilton Theorem says:

$$p(A) = A^n + a_{n-1} A^{n-1} + \cdots + a_1 A + a_0 I_n = 0$$

That is, if you substitute the matrix A into its own characteristic polynomial, the result is always the **zero matrix**.

The roots of this polynomial are the eigenvalues of the matrix.

This theorem is useful because it allows any power of the matrix A (such as A^k for $k>n$) to be expressed as a linear combination of the lower powers of A($I, A, A^2, \ldots A^{n-1}$) .

The theorem is applied in various mathematical domains, assisting in matrix-related operations like inversion, exponentiation, and control theory.

General Form

This polynomial can be broken down into a simpler form, written as

$$p(\lambda) = a_n\lambda_n + a_{n-1}\lambda_{n-1} + \ldots.. + a_1\lambda_1 + a_0\lambda_0.$$

The coeffiecient of highest degree variable(λ_n), and in this case, a_n is always 1.

Variables are in decreasing order of degree, like $\lambda_{n-1}, \ldots, \lambda_1, \lambda_0$.

$$p(A) = A^n + a_{n-1} A^{n-1} + \ldots.. + a_1A + a_0I_n = 0$$

OR

$$p(A) = 0, \text{ where } A \text{ is an } n\times n \text{ square matrix}$$

Given

$$\mathsf{A}=\begin{vmatrix} a_{11}-x & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22}-x & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mm}-x \end{vmatrix}$$
$$=x^m+c_{m-1}\,x^{m-1}+\ldots+c_0,$$

then

$$\mathsf{A}^m+c_{m-1}\,\mathsf{A}^{m-1}+\ldots+c_0\,\mathsf{I}=0,$$

where I is the identity matrix. Cayley verified this identity for $m=2$ and 3 and postulated that it was true for all m. For $m=2$, direct verification gives

$$\begin{vmatrix} a-x & b \\ c & d-x \end{vmatrix}=(a-x)(d-x)-b\,c$$
$$=x^2-(a+d)\,x+(a\,d-b\,c)$$
$$\equiv x^2+c_1\,x+c_2$$
$$\mathsf{A}=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
$$\mathsf{A}^2=\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
$$=\begin{bmatrix} a^2+b\,c & a\,b+b\,d \\ a\,c+c\,d & b\,c+d^2 \end{bmatrix}$$
$$-(a+d)\,\mathsf{A}=\begin{bmatrix} -a^2-a\,d & -a\,b-b\,d \\ -a\,c-d\,c & -a\,d-d^2 \end{bmatrix}$$
$$(a\,d-b\,c)\,\mathsf{I}=\begin{bmatrix} a\,d-b\,c & 0 \\ 0 & a\,d-b\,c \end{bmatrix},$$

$$\mathsf{A}^2-(a+d)\,\mathsf{A}+(a\,d-b\,c)\,\mathsf{I}=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

The Cayley-Hamilton theorem states that an $n\times n$ matrix A is annihilated by its characteristic polynomial $\det(x\,\mathsf{I}-\mathsf{A})$, which is monic of degree n.

Minimal Equation of Matrix

The "minimal equation" of a matrix is called the minimal polynomial.

Definition:

On a Finite Dimensional Vector Space (FDVS), assume that T is a linear operator. If p(t) is a monic polynomial of least positive degree for which p(T) = 0, i.e. the zero operator, then the polynomial p(t) is called a minimal polynomial of T.

Definition Let A be a K×K matrix. An annihilating polynomial P (i.e., such that P(A)=0) is called a minimal polynomial of A if and only if it is monic and no other monic annihilating polynomial of A has lower degree than P.

Let's say a matrix A has a characteristic polynomial of $(t-2)^2(t-3)$.

- The minimal polynomial must have roots at $t = 2$ and $t = 3$.
- It must be a divisor of $(t-2)^2(t-3)$.
- Possible candidates for the minimal polynomial are $(t-2)(t-3)$ or $(t-2)^2(t-3)$ (among others), depending on the matrix A.
- You would need to test these possibilities by substituting the matrix A into the polynomial equations, e.g., $(A-2I)(A-3I)=0$. The one that results in the zero matrix and has the lowest degree is the minimal polynomial.

Minimal Polynomial Theorem

Assume that p(t) is a minimal polynomial of a linear operator T on a Finite Dimensional Vector Space V.

1. If g(T) = 0, then p(t) divides g(t), for any polynomial g(t). In specific, the minimal polynomial p(t) divides the characteristic polynomial of T.
2. T's minimal polynomial is unique

Minimal Polynomial Proof

(1): Let us consider g(t) is a polynomial, in which g(T) = 0.

Using the division algorithm, there exist polynomials, say, q(t) and r(t) such that

g(t) = q(t) p(t) + r(t)

where r(t) = 0 or deg r(t) < deg p(t).

Now, we can write

g(T) = q(T) p(T) + r(T)

i.e. 0 = q(T). 0 + r(T)

It means that r(T) = 0.

Since deg r(t) < deg p(t) and p(t) is considered to be the minimal polynomial of T.

Thus, r(t) should be zero.

Therefore , p(t) divides g(t).

Hence, proved.

Quadratic Form

QUADRATIC FORMS

Linear Form :— An expression of the type $\sum_{i=1}^{n} a_i x_i$, where a_i's are constant coefficients and x_i are variables, is called a linear form w.r.t. the variables. It may be denoted by $L(x)$, being looked upon as a function of the vector variable $\underset{\sim}{x} = [x_1, x_2, \ldots, x_n]'$. Putting $\underset{\sim}{a} = [a_1, a_2, \ldots, a_n]'$. We may write $L(x) = \underset{\sim}{a}'\underset{\sim}{x}$.

Quadratic Form :— An expression of the type $\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} x_i x_j$, where a_{ij} are constant coefficients while x_i and x_j are variables, is called a quadratic form w.r.t. the variables. It is denoted by $Q(x)$, being looked upon as a function of the vector variable $\underset{\sim}{x}$. Putting $A = (a_{ij})$, we may write $Q(\underset{\sim}{x}) = \underset{\sim}{x}' A \underset{\sim}{x}$.

Def$\underline{n}$. → A quadratic form in $x_1, x_2, \ldots, x_n$ is a second degree homogeneous function in n variables $x_1, x_2, \ldots, x_n$, i.e.

$$Q(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} a_{ii}x_i^2 + \sum_{i\neq j} a_{ij}x_i x_j.$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_i x_j$$

Define, $\underset{\sim}{x} = (x_1, x_2, \ldots x_n)'$, and

$$A = ((a_{ij}))_{n\times n}$$

$$\therefore Q(\underset{\sim}{x}) = \underset{\sim}{x}' A \underset{\sim}{x} = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_i x_j = a_{11}x_1x_1 + a_{12}x_1x_2 + \cdots + a_{1n}x_1x_n + a_{21}x_2x_1 + \cdots + a_{2n}x_2x_n + \cdots + a_{n1}x_nx_1 + \cdots + a_{nn}x_nx_n$$

Ex.1. $n = 2$,

$$Q(x,y) = ax^2 + bxy + cxy + dy^2$$

$$= (x \;\; y)\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

for a quadratic form $\underset{\sim}{x}'A\underset{\sim}{x}$, A is said to be a matrix of the quadratic form. If the matrix of a quadratic form be not symmetric then it can reduced to a symmetric matrix.

Ex.1. $n=2$,

$$Q(x,y) = ax^2+bxy+cxy+dy^2$$
$$= (x \;\; y)\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

for a quadratic form $x'Ax$, A is said to be a matrix of the quadratic form. If the matrix of a quadratic form be not symmetric then it can reduced to a symmetric matrix.

$n=2$,
$$Q(x,y) = ax^2+bxy+cxy+dy^2$$
$$= ax^2+\left(\frac{b+c}{2}\right)xy+\left(\frac{c+b}{2}\right)xy+dy^2$$
$$= (x \;\; y)\begin{pmatrix} a & \frac{b+c}{2} \\ \frac{b+c}{2} & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

Note:→ The square matrix A in a quadratic form $Q(x)$ may, without any loss of generality, be supposed to be a symmetric matrix. For, in case A is not symmetric, we may take another matrix $B=(b_{ij})$ such that
$$b_{ij} = \frac{a_{ij}+a_{ji}}{2} \text{ for all } i,j.$$
which implies that $b_{ii} = a_{ii}$ for all i.

then, $b_{ij} = b_{ji}$ for all ij, so that

$B = B'$, i.e. B is a symmetric matrix.

Now, $$x'Bx = \sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}x_ix_j$$
$$= \sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{a_{ij}+a_{ji}}{2}\right)x_ix_j$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{a_{ij}+a_{ji}}{2}\right)x_ix_j$$
$$= \frac{1}{2}\left[\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_ix_j + \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ji}x_ix_j\right]$$

$$= \frac{1}{2}\left[x'Ax + x'A'x\right]$$
$$= x'Ax.$$

$(x'Ax)' = x'A'x$
But $x'Ax$ is a scalar,
so $\rightarrow (x'Ax)' = x'Ax$
$\therefore x'Ax = x'A'x$.

In our discussion, then, we shall always assume A is symmetric. We shall also assume that each element of $\underset{\sim}{x}$ can take only real values.

The determinant $|A|$ is said to be the <u>discriminant</u> of the quadratic form $\underset{\sim}{x}'A\underset{\sim}{x}$.

<u>Classification of Quadratic forms / Classification of the matrix of the Quadratic forms</u> :—

Every real quadratic form $Q(x)$ can be put into one of the following broad categories, depending on the range of values that it may assume, i.e. depending on the nature of A:

i) <u>Non-negative definite Quadratic form</u> :— If $Q(\underset{\sim}{x}) \geq 0 \ \forall x \in E^n$ then it is said to be <u>n.n.d. quadratic form</u>.

i.e. $Q(\underset{\sim}{x}) = \underset{\sim}{x}'A\underset{\sim}{x}$ is said to be an n.n.d. quadratic form if

$$\underset{\sim}{x}'A\underset{\sim}{x} \geq 0 \ \forall \ \underset{\sim}{x}$$

A is then said to be <u>Non-negative definite matrix</u>.

An n.n.d. matrix will be either positive definite or positive-semi definite matrix.

An n.n.d. matrix is also called a matrix which is atleast p.s.d.q.f.

ii) ** <u>Positive definite Quadratic form</u> :— A quadratic form $Q(\underset{\sim}{x}) = \underset{\sim}{x}'A\underset{\sim}{x}$ is said to be a <u>p.d. quadratic form</u> if

$$\underset{\sim}{x}'A\underset{\sim}{x} > 0 \quad \forall \ \underset{\sim}{x} \neq \underset{\sim}{0}$$
$$= 0 \quad \text{iff } \underset{\sim}{x} = \underset{\sim}{0}$$

$Q(\underset{\sim}{x})$ is a p.d. quadratic form means A is a <u>p.d. matrix</u>.

iii) ** <u>Positive semi-definite Quadratic form</u> :— A quadratic form $Q(\underset{\sim}{x}) = \underset{\sim}{x}'A\underset{\sim}{x}$ is said to be a <u>p.s.d. quadratic form</u> if

$$\underset{\sim}{x}'A\underset{\sim}{x} \geq 0 \quad \forall \ \underset{\sim}{x}$$
$$= 0 \quad \text{for atleast one } \underset{\sim}{x} \neq \underset{\sim}{0}.$$

$Q(\underset{\sim}{x})$ is a p.s.d. quadratic form means A is a <u>p.s.d. matrix</u>.

iii) Positive semi-definite Quadratic form :— A quadratic form $Q(\underset{\sim}{x}) = \underset{\sim}{x}' A \underset{\sim}{x}$ is said to be a p.s.d. quadratic form if

$$\underset{\sim}{x}' A \underset{\sim}{x} \geq 0 \quad \forall \underset{\sim}{x}$$
$$= 0 \quad \text{for atleast one } \underset{\sim}{x} \neq \underset{\sim}{0}.$$

$Q(\underset{\sim}{x})$ is a p.s.d. quadratic form means A is a p.s.d. matrix.

Moreover, An n.n.d. quadratic form which is not p.d. is said to be a positive semi-definite (or p.s.d) quadratic form.

iv) Non-positive Definite Quadratic form :— A quadratic form $Q(\underset{\sim}{x}) = \underset{\sim}{x}' A \underset{\sim}{x}$ is said to be non-positive definite quadratic form if

$$\underset{\sim}{x}' A \underset{\sim}{x} \leq 0 \quad \forall \underset{\sim}{x}.$$

i.e. if $Q(\underset{\sim}{x}) \leq 0 \ \forall \ x \in E^n$, then it is said to be a n.p.d. q.f.

- A n.p.d. $\Longleftrightarrow$ $-A$ n.n.d.

A non-positive definite quadratic form (at least negative semi definite) is either negative definite or negative semi definite quadratic form.

v) ** **Negative definite Quadratic form :** A quadratic form $Q(\underset{\sim}{x}) = \underset{\sim}{x}'A\underset{\sim}{x}$ is said to be a negative definite (or n.d.) quadratic form if

$$\underset{\sim}{x}'A\underset{\sim}{x} < 0 \quad \forall \underset{\sim}{x} \neq \underset{\sim}{0}$$
$$= 0 \text{ iff } \underset{\sim}{x} = \underset{\sim}{0}$$

- $A : nd \Longleftrightarrow -A : pd$ matrix.

vi) **Negative Semi-definite Quadratic Form :** A quadratic form $Q(\underset{\sim}{x}) = \underset{\sim}{x}'A\underset{\sim}{x}$ is said to be an n.s.d. quadratic form if

$$\underset{\sim}{x}'A\underset{\sim}{x} \leq 0 \quad \forall \underset{\sim}{x}$$
$$= 0 \text{ for at least one } \underset{\sim}{x} \neq 0.$$

- $A : nsd \Longleftrightarrow A : psd$

An n.p.d. quadratic form which is not n.d. is said to be negative semi-definite (or n.s.d.).

vii) **Indefinite Quadratic form :** A quadratic form $Q(\underset{\sim}{x})\, \underset{\sim}{x}'A\underset{\sim}{x}$ is said to be indefinite if

$$Q(\underset{\sim}{x}) = \underset{\sim}{x}'A\underset{\sim}{x} \geq 0 \text{ for some } \underset{\sim}{x}$$
$$< 0 \text{ for some } \underset{\sim}{x}$$

A q.f. which is neither p.s.d. nor n.s.d. is called indefinite quadratic form.

Ex.1.
$$Q(x) = 5x_1^2 - 5x_1x_2 + 4x_2^2$$
$$= 5\left(x_1^2 - x_1x_2 + \frac{x_2^2}{4}\right) + \frac{11}{4}x_2^2$$
$$= 5\left(x_1 - \frac{x_2}{2}\right)^2 + \frac{11}{4}x_2^2$$

$\therefore Q(x) > 0 \quad \forall\, x_1, x_2 \neq 0$

$Q(x) = 0 \Rightarrow \left(x_1 - \frac{x_2}{2}\right) = 0$ and $x_2 = 0$

i.e. iff $x_1 = x_2 = 0$.

$\therefore Q(x)$ is a positive definite quadratic form.

Ex. 2. $Q(x) = 3x_1^2 - 6x_1x_2 + 3x_2^2$

$= 3(x_1 - x_2)^2$.

$\therefore Q(x) \geq 0 \quad \forall x_1, x_2$

Here, $Q(x) = 0 \Rightarrow x_1 = x_2$

$Q(x) = 0$ for at least one $x \neq 0$.

$\therefore$ It is positive semidefinite quadratic form.

Theorem :→ If $x'Ax$ is positive definite (p.d.), then $x'(-A)x$ is negative definite (n.d.). Conversely, if $x'Ax$ is negative definite (n.d.) then $x'(-A)x$ is positive definite (p.d.).

Proof :~

We have, $x'(-A)x = \sum_i \sum_j (-a_{ij}) x_i x_j$

$= -\sum_i \sum_j a_{ij} x_i x_j$

$= -x'Ax$ —— (1)

Case-I → Now, let $x'Ax$ is p.d. q.f.,

Then for $x \neq 0$, we have, $x'Ax > 0$.

$\Rightarrow -x'Ax < 0$ [Applying (1)]

$\Rightarrow x'(-A)x < 0$.

if $x = 0$, we have $x'Ax = 0$

$\Rightarrow x'(-A)x = 0$

Hence, $x'(-A)x$ must be n.d. q.f.

Case-II → Again, let $x'Ax$ is n.d. q.f.,

then for $x \neq 0$, we have $x'Ax \leq 0$.

$\Rightarrow x'(-A)x < 0$

for $x = 0$, we have $x'Ax = 0$

$\Rightarrow x'(-A)x = 0$

also hence, $x'(-A)x$ must be p.d. q.f.

Canonical reduction of Quadratic Forms : →

[12.5]

Let A be a p.d. matrix then $\underset{\sim}{x}' A \underset{\sim}{x} = Q(\underset{\sim}{x}) > 0$ $\forall\ \underset{\sim}{x} \neq \underset{\sim}{0}$.

$$Q(\underset{\sim}{x}) \longrightarrow Q(\underset{\sim}{y}) = \underset{\sim}{y}' \Lambda \underset{\sim}{y} = \sum_{i=1}^{n} \lambda_i y_i^2, \text{ where } \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n), \ \lambda_i > 0\ \forall i$$

and $\underset{\sim}{y} = (y_1, \ldots, y_n)'$,

$$Q(\underset{\sim}{y}) \longrightarrow Q(\underset{\sim}{z}) = \sum_{i=1}^{n} z_i^2$$

$\exists$ an n.s. matrix $P \ni P'AP = \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$, $\lambda_i > 0\ \forall i$

Choose $\underset{\sim}{y} \ni \underset{\sim}{x} = P\underset{\sim}{y}$, then we get —

$$Q(\underset{\sim}{x})\Big|_{\underset{\sim}{x} = P\underset{\sim}{y}} = x'Ax = y'P'APy = \underset{\sim}{y}' \Lambda \underset{\sim}{y} = \sum_{i=1}^{n} \lambda_i y_i^2$$

Choose $\underset{\sim}{z} \ni \underset{\sim}{y} = \Lambda^{-1/2} \underset{\sim}{z}$, where $\Lambda^{-1/2} = \text{diag}\left(\frac{1}{\sqrt{\lambda_1}}, \frac{1}{\sqrt{\lambda_2}}, \ldots, \frac{1}{\sqrt{\lambda_n}}\right)$

$$Q(\underset{\sim}{y})\Big|_{\underset{\sim}{y} = \Lambda^{-1/2}\underset{\sim}{z}} = y'\Lambda y = \underset{\sim}{z}' \underbrace{\Lambda^{-1/2} \Lambda^{1/2}}_{I} \underbrace{\Lambda^{1/2} \Lambda^{-1/2}}_{I} \underset{\sim}{z}$$

$$= \underset{\sim}{z}' \underset{\sim}{z}$$

$$= \sum_{i=1}^{n} z_i^2, \text{ where } \Lambda^{1/2} = \text{diag}(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_n})$$

Defn. of canonical form :→ If by any real non-singular linear transformation a real quadratic form be expressed as a sum and difference of the squares of the new variables then this later expression is called the canonical ~~reduced~~ form of the given form.

Result * If A be a p.s.d. matrix of order $n\times n$ with rank$(A)=r<n$ then $Q(\underset{\sim}{x})=\underset{\sim}{x}'A\underset{\sim}{x}\longrightarrow Q(\underset{\sim}{y})=\sum_{i=1}^{r}\lambda_i y_i^2$ (or)

* Every quadratic form $\underset{\sim}{x}'A\underset{\sim}{x}$ can be reduced to a canonical form $\sum_{i=1}^{n}\lambda_i y_i^2=\underset{\sim}{y}'\Delta\underset{\sim}{y}$, where $\Lambda=\text{diag}(\lambda_1,\lambda_2,\ldots,\lambda_n)$ by an n.s. transformation of variables.

Proof :→ Let, $\lambda_1,\lambda_2,\ldots,\lambda_n$ be the eigenvalues and $\underset{\sim}{u}_1,\underset{\sim}{u}_2,\ldots,\underset{\sim}{u}_n$ be the corresponding eigen vectors. We can take these vectors u_i's as orthogonal.

Take $Q=(\underset{\sim}{u}_1,\underset{\sim}{u}_2,\ldots,\underset{\sim}{u}_n)$, obviously Q is non-singular.

Then $\underset{\sim}{x}=Q\underset{\sim}{y}$ is a non-singular transformation.

It reduces the q.f. to,
$$\underset{\sim}{x}'A\underset{\sim}{x}=(Q\underset{\sim}{y})'A(Q\underset{\sim}{y})$$
$$=\underset{\sim}{y}'(Q'AQ)\underset{\sim}{y}$$
$$=\underset{\sim}{y}'\Lambda\underset{\sim}{y} \quad (\#)$$
$$=\sum_{i=1}^{n}\lambda_i y_i^2$$

Thus the theorem is established.

[(#) → As A be a square matrix of order n and $\lambda_1,\ldots,\lambda_n$ be the eigen values, then an real n.s. matrix Q exists, ∋ $Q'AQ=\text{diag}(\lambda_1,\ldots,\lambda_n)=\Lambda$ is true. Using this result there.]

Corollary: We have a connection between the nature of the Eigen values and the nature of the matrix (or, q.f.).

If $\lambda_i > 0$ for each i, then $\underset{\sim}{x}' A \underset{\sim}{x}$ is positive definite (p.d.)

$\lambda_i < 0$ for each i, then $\underset{\sim}{x}' A \underset{\sim}{x}$ is negative definite (n.d.)

$\lambda_i \geq 0$ $\forall$ i and $\lambda_i = 0$ for some i, then $\underset{\sim}{x}' A \underset{\sim}{x}$ is p.s.d.

$\lambda_i \leq 0$ $\forall$ i and $\lambda_i = 0$ for some i, then $\underset{\sim}{x}' A \underset{\sim}{x}$ is n.s.d.

$\lambda_i > 0$ for some i & $\lambda_i < 0$ for some i, then $\underset{\sim}{x}' A \underset{\sim}{x}$ is indefinite

Result The necessary and sufficient condition for a quadratic form $x'Ax$ to be positive definite is the leading principal minors of its matrix A are all positive. i.e.

Let A be a matrix of order $n \times n$, $A = ((a_{ij}))_{n \times n}$, then

$$a_{11} > 0, \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0, \quad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} > 0, \ldots\ldots, |A| > 0.$$

Proof: $\rightarrow$ Necessity part (Only if part):

Let $Q(\underset{\sim}{x}) = \underset{\sim}{x}' A \underset{\sim}{x}$ be a p.d. q.f. in n variable $x_1, x_2, \ldots, x_n$ and let $m (\leq n)$ be a natural number, putting $x_{m+1} = x_{m+2} = \cdots\cdots = x_n = 0$ in the p.d.q.f. $\underset{\sim}{x}' A \underset{\sim}{x}$, we arrive to another p.d.q.f. in m variables $x_1, x_2, \ldots, x_m$, The determinant of whose matrix is the leading principal minor of the matrix A.

Since, the determinant of every p.d.q.f. is positive.

$\therefore$ every leading principal minor of the matrix A is positive.

Sufficiency part (If part): Method of induction will be used to proof this part. For a single variable $\underset{\sim}{x}$, the q.f. $Q(\underset{\sim}{x}) = a x_1^2 > 0$ for $x_1 \neq 0$ if $a > 0$.

Now, suppose that the theorem is true for m variables.

Consider any q.f. in $(m+1)$ variables with the correspondi symmetric matrix $A \ni$ the leading pricipal minors of A is positive.

Let us partition of A as follows:

$$A_{m+1 \times m+1} = \begin{pmatrix} \underset{m \times m}{B} & \underset{m \times 1}{b_1} \\ \underset{1 \times m}{b_1'} & K \end{pmatrix}$$

the leading principal minors of A and B are all positive as the theorem is supposed to be true for m variables $\exists$ a n.s. matrix P over the real field $\ni$ $P'BP = I_m$

Let us now determine C $\ni$

$\begin{pmatrix} P' & 0 \\ C' & 1 \end{pmatrix} \begin{pmatrix} B & b_1 \\ b_1' & k \end{pmatrix} \begin{pmatrix} P & C \\ 0 & 1 \end{pmatrix}$ has its R.H. top corner element zero.

The product matrix equals to $\begin{pmatrix} P'B & P'b_1 \\ C'B + b_1' & C'b_1 + k \end{pmatrix} \begin{pmatrix} P & C \\ 0 & 1 \end{pmatrix}$

$$= \begin{pmatrix} P'BP & P'BC + P'b_1 \\ C'BP + b_1'P & C'BC + b_1'C + C_1'b_1 + k \end{pmatrix}$$

$\therefore$ We have $P'BC + P'\underset{\sim}{b_1} = 0$

$$\Rightarrow P'BC = -P'\underset{\sim}{b_1}$$

$$\Rightarrow BC = -\underset{\sim}{b_1}$$

$$\therefore C = (-B^{-1}\underset{\sim}{b_1})$$

Under this choice of C, we also have

$$C'BP + b_1'P = 0 \quad \text{and}$$

$$C'BC + C'\underset{\sim}{b_1} = \underset{\sim}{b_1}'(B^{-1})'BB^{-1}\underset{\sim}{b} - \underset{\sim}{b_1}'(B^{-1})'\underset{\sim}{b_1}$$

$$= \underset{\sim}{b_1}'(B^{-1})'\underset{\sim}{b_1} - \underset{\sim}{b_1}'(B^{-1})'\underset{\sim}{b_1}$$

$$= 0$$

$\therefore$ Product matrix $= \begin{pmatrix} I_m & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{b_1}'C + k \end{pmatrix}$

Taking determinant in both sides,

$$|P'||A||P| = |I_m|(\underset{\sim}{b_1}'C + k)$$

$$\text{or, } |A|\{|P|\}^2 = (\underset{\sim}{b_1}'C + k)$$

Now by assumption, $|A| > 0$ and $|P| \neq 0$.

So, $(\underset{\sim}{b_1}'C + k) > 0$

Let $\underset{\sim}{b_1}'C + k = \beta^2$

$$\therefore \begin{pmatrix} P' & 0 \\ C' & 1 \end{pmatrix} A \begin{pmatrix} P & C \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} I_m & \underset{\sim}{0} \\ \underset{\sim}{0} & \beta^2 \end{pmatrix}$$

Premultiplying and postmultiplying both sides by

$$\begin{pmatrix} I_m & \underset{\sim}{0} \\ \underset{\sim}{0} & \beta^{-1} \end{pmatrix}$$

$$\begin{pmatrix} I_m & \underset{\sim}{0} \\ \underset{\sim}{0} & \beta^{-1} \end{pmatrix} \begin{pmatrix} P' & 0 \\ c' & 1 \end{pmatrix} A \begin{pmatrix} P & c \\ 0 & 1 \end{pmatrix} \begin{pmatrix} I_m & \underset{\sim}{0} \\ \underset{\sim}{0} & \beta^{-1} \end{pmatrix} = I_{m+1}$$

or, $Q'AQ = I_{m+1}$, where $Q = \begin{pmatrix} P & c \\ 0 & 1 \end{pmatrix} \begin{pmatrix} I_m & \underset{\sim}{0} \\ \underset{\sim}{0} & \beta^{-1} \end{pmatrix}$

Since A is congruent to I_{m+1}, therefore A is p.d., i.e., the corresponding q.f. is p.d. Hence, by induction result follows.

[C.U.]

- Corollary : The necessary and sufficient condition for a q.f. $Q(\underset{\sim}{x}) = \underset{\sim}{x}'A\underset{\sim}{x}$ to be negative definite is that the leading principal minors of A starting from the first are alternatively negative and positive, i.e.

$$a_{11} < 0, \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0, \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} < 0 \; \ldots\ldots, (-1)^n|A| \gtrless 0 \text{ according as } n \text{ is even or odd.}$$

Proof :> $\underset{\sim}{x}'A\underset{\sim}{x}$ is negative definite if and only if $\underset{\sim}{x}'(-A)\underset{\sim}{x}$ is positive definite. But by virtue of the above theorem, $\underset{\sim}{x}'(-A)\underset{\sim}{x}$ is positive definite if and only if

$$-a_{11} > 0, \begin{vmatrix} -a_{11} & -a_{12} \\ -a_{21} & -a_{22} \end{vmatrix} > 0, \begin{vmatrix} -a_{11} & -a_{12} & -a_{13} \\ -a_{21} & -a_{22} & -a_{23} \\ -a_{31} & -a_{32} & -a_{33} \end{vmatrix} > 0, \ldots\ldots$$

Thus the necessary and sufficient conditions for $x'(-A)\underset{\sim}{x}$ to be p.d. or, equivalently, $\underset{\sim}{x}'A\underset{\sim}{x}$ to be n.d. is the following :

$-a_{11} > 0$, i.e. $a_{11} < 0$,

$(-1)^2 \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0$, i.e. $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0$,

$(-1)^3 \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} > 0$, i.e. $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} < 0, \ldots\ldots$

$(-1)^n|A| \gtrless 0$ according as n is even or odd.

Hence the proof.

Ex. 1. Consider the quadratic form: $4x_1^2 - 10x_1x_2 + 7x_2^2$.

Here, $A = \begin{pmatrix} 4 & -5 \\ -5 & 7 \end{pmatrix}$

As such, $a_{11} = 4 > 0$, $|A| = 3 > 0$.

Consequently, the quadratic form is positive definite (p.d.)

Ex. 2. Consider now the quadratic form: $4x_1^2 - 10x_1x_2 + 3x_2^2$.

Here, $A = \begin{pmatrix} 4 & -5 \\ -5 & 3 \end{pmatrix}$

As such, $a_{11} = 4 > 0$, $|A| = -13 < 0$,

so that this quadratic form is neither positive definite (p.d.) nor negative definite (n.d.).

1) If $\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

Show that the matrix $((\delta_{ij} + x_i x_j))$ is p.d., $i, j = 1(1)p$. (ISI)

Soln. → Let $A = ((\delta_{ij} + x_i x_j))$

$$= I_n + \underset{\sim}{x}\underset{\sim}{x}^T, \quad \text{where, } \underset{\sim}{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

Consider an associated Q.F.

$$\underset{\sim}{u}' A \underset{\sim}{u} = \underset{\sim}{u}' (I_n + xx^T) \underset{\sim}{u}$$

$$= \underset{\sim}{u}' I_n \underset{\sim}{u} + \underset{\sim}{u}' \underset{\sim}{x} \underset{\sim}{x}' \underset{\sim}{u}$$

$$= \underset{\sim}{u}'\underset{\sim}{u} + (\underset{\sim}{x}'\underset{\sim}{u})'(\underset{\sim}{x}'\underset{\sim}{u})$$

For $\underset{\sim}{u} \neq \underset{\sim}{0}$, $\underset{\sim}{u}'\underset{\sim}{u} = \sum_{i=1}^{n} u_i^2 > 0$

$\Rightarrow \underset{\sim}{u}' A \underset{\sim}{u} > 0$

Hence, $\underset{\sim}{u}' A \underset{\sim}{u}$ is positive definite,

i.e. $A = ((\delta_{ij} + x_i x_j))$ is p.d.

Problem:— 1. Reduce the equation $3x^2+5y^2+3z^2+2xy+2yz+2zx=1$ into canonical form.

Solution:- $A=\begin{pmatrix}3&1&1\\1&5&1\\1&1&3\end{pmatrix}$, $\underset{\sim}{X}=\begin{pmatrix}x\\y\\z\end{pmatrix}$

$\therefore \underset{\sim}{X}^TA\underset{\sim}{X}=1$.

The ch. equation of A is $\begin{vmatrix}3-x&1&1\\1&5-x&1\\1&1&3-x\end{vmatrix}=0$

or, $(x-2)(x-3)(x-6)=0$.

$\therefore$ The eigen values of A are 2, 3, 6.

The eigen vector corresponding to the eigen value 2 are $c\begin{pmatrix}1\\0\\-1\end{pmatrix}$; $c\neq0$.

The " " " " " 3 " $c\begin{pmatrix}1\\-1\\1\end{pmatrix}$, $c\neq0$.

The " " " " " 6 " $c\begin{pmatrix}1\\2\\1\end{pmatrix}$, $c\neq0$.

Let $\underset{\sim}{\alpha}=\begin{pmatrix}1\\0\\-1\end{pmatrix}$, $\underset{\sim}{\beta}=\begin{pmatrix}1\\-1\\1\end{pmatrix}$, $\underset{\sim}{\gamma}=\begin{pmatrix}1\\2\\1\end{pmatrix}$. Then the set $\{\underset{\sim}{\alpha},\underset{\sim}{\beta},\underset{\sim}{\gamma}\}$ is an orthogonal set. The orthonormal set of eigen vectors is

$$\left\{\frac{1}{\sqrt2}\begin{pmatrix}1\\0\\-1\end{pmatrix},\ \frac{1}{\sqrt3}\begin{pmatrix}1\\-1\\1\end{pmatrix},\ \frac{1}{\sqrt6}\begin{pmatrix}1\\2\\1\end{pmatrix}\right\}.$$

Let $P=\begin{pmatrix}\frac{1}{\sqrt2}&\frac{1}{\sqrt3}&\frac{1}{\sqrt6}\\0&-\frac{1}{\sqrt3}&\frac{2}{\sqrt6}\\-\frac{1}{\sqrt2}&\frac{1}{\sqrt3}&\frac{1}{\sqrt6}\end{pmatrix}$. Then P is an orthogonal mtx.

Let us apply the orthogonal transformation $\underset{\sim}{X}=P\underset{\sim}{X}'$, where $\underset{\sim}{X}'=\begin{pmatrix}x'\\y'\\z'\end{pmatrix}$.

Then the equation transforms to $(X')^T(P^TAP)X'=1$.

$P^TAP=(P^{-1}AP)$ is a diagonal matrix, D which has the same eigen value as those of A.

$$AP=\begin{pmatrix}3&1&1\\1&5&1\\1&1&3\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt2}&\frac{1}{\sqrt3}&\frac{1}{\sqrt6}\\0&-\frac{1}{\sqrt3}&\frac{2}{\sqrt6}\\-\frac{1}{\sqrt2}&\frac{1}{\sqrt3}&\frac{1}{\sqrt6}\end{pmatrix}=\begin{pmatrix}2\cdot\frac{1}{\sqrt2}&3\cdot\frac{1}{\sqrt3}&6\cdot\frac{1}{\sqrt6}\\2\cdot0&3\cdot-\frac{1}{\sqrt3}&6\cdot\frac{2}{\sqrt6}\\2\cdot-\frac{1}{\sqrt2}&3\cdot\frac{1}{\sqrt3}&6\cdot\frac{1}{\sqrt6}\end{pmatrix}$$

$$=\begin{pmatrix}\frac{1}{\sqrt2}&\frac{1}{\sqrt3}&\frac{1}{\sqrt6}\\0&-\frac{1}{\sqrt3}&\frac{2}{\sqrt6}\\-\frac{1}{\sqrt2}&\frac{1}{\sqrt3}&\frac{1}{\sqrt6}\end{pmatrix}\begin{pmatrix}2&0&0\\0&3&0\\0&0&6\end{pmatrix}$$

$$=PD,\text{ where } D=\begin{pmatrix}2&0&0\\0&3&0\\0&0&6\end{pmatrix}.$$

So, $P^{-1}AP=D=\begin{pmatrix}2&0&0\\0&3&0\\0&0&6\end{pmatrix}$

The equation transforms to $(X')^TDX'=1$

i.e. to $2x'^2+3y'^2+6z'^2=1$.

2. Reduce the equation $7x^2-2xy+7y^2-16x+16y-8=0$ into canonical form and determine the nature of the conic.

Solution:— Let $A=\begin{pmatrix}7 & -1\\ -1 & 7\end{pmatrix}$, $B=\begin{pmatrix}-16 & 16\end{pmatrix}$, $X=\begin{pmatrix}x\\ y\end{pmatrix}$.

Then the equation takes the form $X^TAX+BX-8I_1=0$

The eigen values of A are 8, 6.

The eigen vectors corresponding the eigen values 8 and 6 are $c\begin{pmatrix}1\\ -1\end{pmatrix}, c\neq 0;\ d\begin{pmatrix}1\\ 1\end{pmatrix}, d\neq 0$, respectively.

The orthonormal set of eigen vectors is $\left\{\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -1\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix}\right\}$.

Let $P=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}$. Then P is an orthogonal matrix.

$P^TAP=(P^{-1}AP)$ is a diagonal matrix which has the same eigenvalues as those of A.

So, $P^TAP=\begin{pmatrix}8 & 0\\ 0 & 6\end{pmatrix}$. $BP=\begin{pmatrix}-16\sqrt{2} & 0\end{pmatrix}$.

By the orthogonal transformation $X=PX'$; where $X'=\begin{pmatrix}x'\\ y'\end{pmatrix}$, the equation transforms to $8x'^2+6y'^2-16\sqrt{2}x'-8=0$

or, $8(x'-\sqrt{2})^2+6y'^2=24$.

Let us apply the translation $x''=x'-\sqrt{2}$, $y''=y'$.

$\therefore$ The equation transforms to $8x''^2+6y''^2=24$.

The canonical form is $8x^2+6y^2=24$.

The equation represents an ellipse.

Properties of Quadratic Forms

The eigenvalues of matrix A have a direct bearing on the characteristics of a quadratic form. The quadratic forms are categorized as follows by these properties:

- Positive Definite: If all eigenvalues of A are positive, Q(x) > 0 for all x not equal to 0. Such forms are frequently seen in minima-seeking optimization issues.
- Negative Definite: If all eigenvalues are negative, Q(x) < 0 for all x not equal to 0.
- Indefinite: If A has both positive and negative eigen values, Q(x) can take on positive or negative values depending on x.

The behavior of quadratic forms plays a crucial role in figuring out how the function surfaces represent the curve. For instance, positive definite forms show that the function surface is convex.

Important Applications of Quadratic Form

Quadratic forms arise in a variety of real-world situations:

- **Optimization:** Quadratic forms are used in optimization, especially in quadratic programming, to design objective functions that must be minimized or maximized. Convex problems, which are simpler to solve and ensure global minima, are linked to positive definite matrices.
- **Mechanical Systems:** In physics, potential energy in systems where the energy relies quadratically on the state variables of the system, like displacements, is frequently represented using quadratic forms.
- **Statistics:** Covariance matrices in multivariate statistics define quadratic shapes that characterize the correlation and variability of data sets. Principal component analysis and other dimensionality reduction methods require an understanding of these forms.

Signature and Classification of Quadratic Forms

The signature of a quadratic form is a pair of integers (p, q) representing the number of positive (p) and negative (q) coefficients after the form has been diagonalized, or as a triple (p, q, r) which includes the number of zero coefficients (r). The classification of a quadratic form depends on the signs of these diagonalized coefficients: a positive definite form has only positive terms, a negative definite form has only negative terms, and an indefinite form has both positive and negative terms.

Signature

Definition: The signature is a fundamental property that remains invariant under a change of variables, meaning all diagonalizations of a given quadratic form will have the same signature.

Method:

1. Diagonalize the quadratic form using an invertible linear transformation to a form like $q(x_1, \ldots, x_n) = \lambda_1 x_1^2 + \ldots + \lambda_n x_n^2$.
2. According to Sylvester's Law of Inertia, the numbers of positive, negative, and zero coefficients are constant for any diagonalization.

Representation:

- Pair (p, q): p is the count of positive squared terms and q is the count of negative squared terms. Some sources also define the signature as the single number $p - q$.
- Triple (p, q, r): p is the number of positive terms, q is the number of negative terms, and r is the number of zero terms.

The signature of a non-degenerate quadratic form

$$Q = y_1^2 + y_2^2 + \ldots + y_p^2 - y_{p+1}^2 - y_{p+2}^2 - \ldots - y_r^2$$

of rank r is most often defined to be the ordered pair $(p, q) = (p, r - p)$ of the numbers of positive, respectively negative, squared terms in its reduced form.

In the event that the quadratic form Q is allowed to be degenerate, one may write

$$Q = y_1^2 + \cdots + y_p^2 - y_{p+1}^2 - \cdots - y_{p+q}^2 + y_{p+q+1}^2 + \cdots + y_{p+q+z}^2$$

where the nonzero components $y_{p+q+1}, \ldots, y_{p+q+z}$ square to zero. In this case, the signature of Q is most often denoted by one of the triples (p, q, z) or (z, p, q).

A number of other, less common definitions are sometimes attributed to a quadratic form as its signature. In particular, the signature of Q is sometimes defined to be the number p of positive squared terms in its reduced form, as well as the quantity $2p - r$.

Classification

The classification is based on the signature and can be determined by the signs of the eigenvalues or diagonalized coefficients:

- **Positive definite**: All coefficients are positive. The signature is $(p, 0, 0)$ or just $(p, 0)$.
- **Negative definite**: All coefficients are negative.
- **Indefinite**: There are both positive and negative coefficients.
- **Non-degenerate**: This term applies when there are no zero coefficients ($r = 0$) in the diagonalization.

Sylvester's law defined

For any real symmetric matrix A, the law states that for any invertible matrix P, the congruent matrix $B = P^T A P$ has the same number of positive, negative, and zero eigenvalues as A. These three numbers are known as the **inertia** of the matrix A.

The law also applies directly to quadratic forms. Any quadratic form $Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ can be transformed by a change of basis to a diagonal form:

$$Q(\mathbf{y}) = \mathbf{y}^T D\mathbf{y} = \sum_{i=1}^{p} y_i^2 - \sum_{i=p+1}^{p+n} y_i^2$$

In this form, the number of positive coefficients (p) and negative coefficients (n) is uniquely determined by the original quadratic form, regardless of the choice of basis used for diagonalization.

Applications of Sylvester's Inertia

A primary application of Sylvester's law in real analysis is the second-derivative test for functions of several variables. To find local maxima or minima of a function $f : \mathbb{R}^n \to \mathbb{R}$, you need to analyze its Hessian matrix, H, which is the matrix of second partial derivatives. The Hessian is a real symmetric matrix.

1. **Positive definite Hessian**: If the Hessian matrix H is positive definite at a critical point, the quadratic form $\mathbf{x}^T H\mathbf{x}$ is always positive for any non-zero vector $\mathbf{x}$. By Sylvester's law, this is equivalent to all of H's eigenvalues being positive. This indicates a **local minimum**.
2. **Negative definite Hessian**: If H is negative definite, all of its eigenvalues are negative, indicating a **local maximum**.
3. **Indefinite Hessian**: If H has both positive and negative eigenvalues, it is indefinite. This corresponds to a **saddle point**, where the function increases in some directions and decreases in others.
4. **Semidefinite Hessian**: If H is semidefinite (i.e., it has a zero eigenvalue), the test is inconclusive.

Sylvester's Inequality :—

$$R(A_{n\times n} B_{n\times n}) \geq R(A) + R(B) - n.$$

Proof :- Let $R(A) = r < n$

$\exists$ n.s. matrices P and Q such that

$$PAQ = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$$

Define a square matrix C of the order $n\times n$ such that

$$PCQ = \begin{pmatrix} 0 & 0 \\ 0 & I_{n-r} \end{pmatrix}$$

Clearly, $P(A+C)Q = I_n$

Therefore, $|P(A+C)Q| = |I_n| = 1$

$\Rightarrow |P|\,|A+C|\,|Q| = 1.$

$\Rightarrow A+C$ is non singular.

Hence, $R(B) = R\left((A+C)B\right)$

$= R(AB + CB)$

$< R(AB) + R(CB)$

$\leq R(AB) + R(C)$

$= R(AB) + n - r$

$= R(AB) + n - R(A)$

$\therefore R(AB) \geq R(A) + R(B) - n.$