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Unit-I

Measure Theory

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UNIT – I

MEASURE THEORY

Introduction to Set Theory

A set is a collection of objects or groups of objects. These objects are often called elements or members of a set. For example, a group of players in a cricket team is a set.

Since the number of players in a cricket team could be only 11 at a time, thus we can say, this set is a finite set. Another example of a finite set is a set of English vowels. But there are many sets that have infinite members such as a set of natural numbers, a set of whole numbers, set of real numbers, set of imaginary numbers, etc.

Definition

Set is a well-defined collection of objects or people. Sets can be related to many real-life examples, such as the number of rivers in India, number of colours in a rainbow, etc.

Representation of Sets

Sets can be represented in two ways:

- Roster Form or Tabular form
- Set Builder Form

Roster Form

In roster form, all the elements of the set are listed, separated by commas and enclosed between curly braces { }.

Example: If set represents all the leap years between the year 1995 and 2015, then it would be described using Roster form as:

A = {1996, 2000, 2004, 2008, 2012}

Set Builder Form

In set builder form, all the elements have a common property. This property is not applicable to the objects that do not belong to the set.

Example: If set S has all the elements which are even prime numbers, it is represented as:

S = {x : x is an even prime number}

where 'x' is a symbolic representation that is used to describe the element.

':' means 'such that'

'{}' means 'the set of all'

So, S = {x : x is an even prime number } is read as 'the set of all x such that x is an even prime number'. The roster form for this set S would be S = 2. This set contains only one element. Such sets are called singleton/unit sets.

Types of sets

The sets are categorised into different types, based on elements or types of elements. These are:

- **Finite set:** The number of elements is finite. E.g. {7, 9, 11}
- **Infinite set:** The numbers of elements are infinite. E.g. Set of natural numbers.
- **Empty set:** It has no elements. It is denoted by Ø or { }.
- **Singleton set:** It has one only element. E.g. {1}, {2}, {a} etc.,
- **Equal set:** Two sets are equal if they have same elements
- **Equivalent set:** Two sets are equivalent if they have same number of elements
- **Power set:** A set of every possible subset. E.g. {A : A ⊂ B}
- **Universal set:** Any set that contains all the sets under consideration.
- **Subset:** When all the elements of set A belong to set B, then A is subset of B. It is denoted by A ⊆ B.

Symbols of Set Theory

There are several symbols that are adopted for common sets. They are given in the table below:

Table 1: Symbols denoting common sets

Symbol	**Corresponding Set**
N	Represents the set of all Natural numbers i.e. all the positive integers. This can also be represented by Z^+. Examples: 9, 13, 906, 607, etc.
Z	Represents the set of all integers The symbol is derived from the German word *Zahl*, which means number. Positive and negative integers are denoted by Z^+ and $Z^–$ respectively. Examples: -12, 0, 23045, etc.
Q	Represents the set of Rational numbers The symbol is derived from the word *Quotient*. It is defined as the quotient of two integers (with non-zero denominator). Positive and negative rational numbers are denoted by Q^+ and $Q^–$ respectively. Examples: 13/9. -6/7, 14/3, etc.
R	Represents the Real numbers i.e. all the numbers located on the number line. Positive and negative real numbers are denoted by R^+ and $R^–$ respectively. Examples: 4.3, π, $4\sqrt{3}$, etc.
C	Represents the set of Complex numbers. Examples: 4 + 3i, i, etc.

Other Notations

Symbol	Symbol Name
{ }	Set
$A \cup B$	A union B
$A \cap B$	A intersection B
$A \subseteq B$	A is subset of B
$A \not\subset B$	A is not subset B
$A \subset B$	proper subset / strict subset
$A \supset B$	proper superset / strict superset
$A \supseteq B$	Superset
$A \not\supset B$	not superset
Ø	empty set
P (C)	power set
A = B	Equal set
A^c	Complement of A
$A \in B$	a element of B
$X \notin A$	x not element of A

Set Theory Formulas

- $n(A \cup B) = n(A) + n(B) – n(A \cap B)$
- $n(A \cup B) = n(A) + n(B)$ {when A and B are disjoint sets}
- $n(U) = n(A) + n(B) – n(A \cap B) + n((A \cup B)^c)$

- $n(A \cup B) = n(A - B) + n(B - A) + n(A \cap B)$
- $n(A - B) = n(A \cap B) - n(B)$
- $n(A - B) = n(A) - n(A \cap B)$
- $n(A^c) = n(U) - n(A)$
- $n(P \cup Q \cup R) = n(P) + n(Q) + n(R) - n(P \cap Q) - n(Q \cap R) - n(R \cap P) + n(P \cap Q \cap R)$

Algebra of Sets

Commutative law

- $A \cup B = B \cup A$, $A \cap B = B \cap A$

Associative law

- $A \cup (B \cup C) = (A \cup B) \cup C$
- $A \cap (B \cap C) = (A \cap B) \cap C$

Distributive Laws

- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Complementary law

- $A \cup \overline{A} = S, A \cap \overline{A} = \phi, A \cup S = S, A \cap S = A, A \cup \phi = A, A \cap \phi = \phi$

Difference law

- $A - B = A \cap \overline{B}$
- $A - B = A - (A \cap B) = (A \cup B) - B$
- $A - (B - C) = (A - B) \cup (A - c)$, $(A \cup B) - C = (A - C) \cup (B - C)$

DeMorgan's Law

- $(A \cup B)^c = A^c \cap B^c$, $(A \cap B)^c = A^c \cup B^c$

Set Operations

Union

Union of the sets A and B, denoted A ∪ B, is the set of all objects that are a member of A, or B, or both. For example, the union of {1, 2, 3} and {2, 3, 4} is the set {1, 2, 3, 4}.

Intersection

Intersection of the sets A and B, denoted A ∩ B, is the set of all objects that are members of both A and B. For example, the intersection of {1, 2, 3} and {2, 3, 4} is the set {2, 3}.

Set difference

Set difference of U and A, denoted U \ A, is the set of all members of U that are not members of A. The set difference {1, 2, 3} \ {2, 3, 4} is {1}, while conversely, the set difference {2, 3, 4} \ {1, 2, 3} is {4}. When A is a subset of U, the set difference U \ A is also called the complement of A in U. In this case, if the choice of U is clear from the context, the notation A^c is sometimes used instead of U \ A, particularly if U is a universal set.

Symmetric difference

Symmetric difference of sets A and B, denoted A ▵ B or A ⊖ B, is the set of all objects that are a member of exactly one of A and B (elements which are in one of the sets, but not in both). For instance, for the sets {1, 2, 3} and {2, 3, 4}, the symmetric difference set is {1, 4}. It is the set difference of the union and the intersection, (A ∪ B) \ (A ∩ B) or (A \ B) ∪ (B \ A).

Cartesian product

Cartesian product of A and B, denoted A × B, is the set whose members are all possible ordered pairs (a, b), where a is a member of A and b is a member of B. For example, the Cartesian product of {1, 2} and {red, white} is {(1, red), (1, white), (2, red), (2, white)}.

Power set

Power set of a set A, denoted $P(A)$, is the set whose members are all of the possible subsets of A. For example, the power set of {1, 2} is { {}, {1}, {2}, {1, 2} }.

Examples

1. If U = {a, b, c, d, e, f}, A = {a, b, c}, B = {c, d, e, f}, C = {c, d, e}, find (A ∩ B) ∪ (A ∩ C).

Solution

A ∩ B = {a, b, c} ∩ {c, d, e, f}

A ∩ B = {c}

A ∩ C = {a, b, c} ∩ {c, d, e}

A ∩ C = {c}

∴ (A ∩ B) ∪ (A ∩ C) = {c}

2. Give examples of finite sets.

Solution

The examples of finite sets are:

- Set of months in a year
- Set of days in a week
- Set of natural numbers less than 20
- Set of integers greater than -2 and less than 3

3. If U = {2, 3, 4, 5, 6, 7, 8, 9, 10, 11}, A = {3, 5, 7, 9, 11} and B = {7, 8, 9, 10, 11}, Then find (A – B)′.

Solution

A – B is a set of member which belong to A but do not belong to B

∴ A – B = {3, 5, 7, 9, 11} – {7, 8, 9, 10, 11}

A – B = {3, 5}

According to formula,

(A − B)′ = U – (A – B)

∴ (A − B)′ = {2, 3, 4, 5, 6, 7, 8, 9, 10, 11} – {3, 5}

(A − B)′ = {2, 4, 6, 7, 8, 9, 10, 11}.

Limits of sequence of sets

SEQUENCE

Let S be a non-empty set. A function f: $\mathbb{N} \to S$ is said to be sequence in S.

If S = $\mathbb{R}$, then we call it as " Real sequence".

A sequence is denoted by $\{S_n\}$ or (S_n) or $<S_n>$

$$\{S_n\} = \{S_1, S_2, S_3, \ldots, S_n, \ldots\}$$

Where S_n denotes the n^{th} term of the sequence.

Example,

(i) $\{S_n\} = \{1, 2, 3, \ldots\}$

$S_n = n$, $S_1=1$, $S_2=2$, $S_3=3$, $S_4=4$,……

$S_n= \frac{1}{n}$, $S_1= \frac{1}{1}$, $S_2= \frac{1}{2}$, $S_3= \frac{1}{3}$,….

$S_n= (-1)$, $S_1= -1$, $S_2= 1$, $S_3= -1$, $S_4= 1$,….

RANGE OF SEQUENCE

A set of all distinct elements of a sequence is called the range set of the sequence.

Eample,

(i) S= $\{(-1)^n/n\} = \{-1, \frac{1}{2}, \frac{-1}{3}, \frac{1}{4}, \ldots\}$,

(ii) Range of S = $\{-1, \frac{1}{2}, \frac{-1}{3}, \frac{1}{4}, \ldots\}$

(iii) S= $\{1+ (-1)^n\} = \{0, 2, 0, 2, \ldots\}$

(iv) Range of S = {0,2}

CONSTANT SEQUENCE

If the range of the sequence is a singleton set, it is called constant sequence .

In other words, a sequence $\{S_n\}$ defined by $S_n = a \ \forall \ n \in \mathbb{N}$ is called a constant sequence.

Example,

If $s_n = 5$ then $\{S_n\}$ is constant sequence.

$\{S_n\} = \{5,5,5,\ldots\}$

SUBSEQUENCE

Let $\{S_n\}$ be any sequence . let $\{n_1, n_2, n_3, \ldots\}$ be a sequence of positive integers. Then the sequence $\{s_{n_1}, s_{n_2}, s_{n_3}, \ldots\}$written as $\{s_{n_k}\}$ is called a subsequence of $\{S_n\}$.

A subsequence of a sequence is a sequence whose terms are chosen from the original sequence in the same order as in the original sequence.

Example

(i) $\{S_n\} = \{1, 2, 3, \ldots.\}$

$\{s_{n_k}\} = \{2, 4, 6, \ldots.\}$ is a subsequence of $\{S_n\}$

$\{s_{n_k}\} = \{1, 4, 9, \ldots.\}$ is a subsequence of $\{S_n\}$

$\{8, 4, 6, 2, \ldots.\}$ is not a subsequence of $\{S_n\}$

{2, 4, 5, 6, 8,….} is not a subsequence of $\{S_n\}$

BOUNDED SET

A said to be bounded if it is both bounded above and bounded below.

Example,

S= {28} Lower bound = 28, Upper bound = 28

T= (-1, 1) Lower bound = -1, Upper bound = 1

R = $\{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ Lower bound = 0, Upper bound = 1

UPPER BOUND

Let S be a non-empty subset of $\mathbb{R}$.

The set S is said to be bounded above if there exists a number $u \in \mathbb{R}$ such that $s \leq u$ for all $s \in S$.

Each such number u is called an upper bound of S.

Example,

S= [2,6] Upper bound of S = {6,7,8,….}

T = {…-1, 0, 1, 2} Upper bound of T = {2,3,4,…}

LOWER BOUND

Let S non empty subset of $\mathbb{R}$.

The set S is said to be bounded above if there exists a number $w \in \mathbb{R}$ such that $w \leq s$ for all $s \in$

S.

Each such number w is called an lower bound of S.

Example,

S= [3,7] Lower bound of S= {3,2,1,…}

T= {5,10,15,…} Lower Bound of T = {5,4,3…}

UNBOUNDED

A set is said to be unbounded if it is not bounded.

Example,

A={1,4,9,16,…} →bounded below but not bounded above.

B={-∞,5 }→ bounded above but not bounded below.

C={-1, 2,-3,4,…} → neither bounded above nor bounded below.

SUPREMUM

Let S be a non empty subset of $\mathbb{R}$. If S is bounded above, then a number u is said to be a supremum (least upper bound) of S if it satisfies the conditions:

(i) u is an upper bound of S, and

(ii) if v is any upper bound of S , then $u \leq v$

Example,

S= {0, -1, -2, -3,…}

Upper bound of S = {0, 1,2,….}

Supremum (LUB) of S = 0

INFIMUM

Let S be a non empty subset of $\mathbb{R}$. If S is bounded below , then a number w is said to be an infimum (greatest lower bound) of S if it Satisfies the conditions:

(i) w is a lower bound of S, and

(ii) if t is any lower bound of S, then $t \leq w$.

Example,

S={2,4,6….}

Lower bound of S = {2,1,0,…}, Infimum(GLB) of S = 2.

Limit superior and Limit inferior

The greatest limit point of a bounded sequence $\{s_n\}$ is called the limit superior of $\{s_n\}$. Its is denoted by $\lim_{n\to\infty} \sup s_n$ or lim s_n.

The smallest limit point of a bounded sequence $\{s_n\}$ is called the limit inferior of $\{s_n\}$. its denoted by $\lim_{n\to\infty} \inf s_n$ or lim s_n.

Example:

(i) $\{s_n\} = \{2+ (-1)^n\} = \{1,3,1,3,\ldots\}$

The set of limit points of $\{s_n\} = \{1,3\}$.

(ii) $\{s_n\} = \{\frac{1}{n}\} = \{1,\frac{1}{2},\frac{1}{3},\ldots\}$ has only 0 as its limit point.

$\lim_{n\to\infty} \sup s_n = 0$ and $\lim_{n\to\infty} \inf s_n = 0$

Limit point

A real number l is said to be a limit point of a sequence, if every neighborhood of l contains an infinite number of terms of the given sequence.

In other words, a real number l is said to be a limit point of a sequence $\{s_n\}$, if for $\in>0$,

$S_n \in (l- \in, l+ \in)$ for infinitely many values of n.

Examples,

(i) $\{s_n\} = \{\frac{1}{n}\} = \{1,\frac{1}{2},\frac{1}{3},\ldots\}$ has only 0 as its limit point.

(ii) $\{s_n\} = \{n^2\} = \{1, 4, 9,\ldots\}$ has no limit point.

(iii) $\{s_n\} = \{(-1)^n\} = \{-1, 1, -1, 1\ldots\}$ has only two limit points -1 and 1.

Limits of sequence of sets

Let Ω be a non-empty set and let $A_1, A_2,\ldots$ be a sequence of subsets of Ω.

Increasing sequence of sets

If $A_1 \subseteq A_2 \subseteq \ldots..$, then we say that the sequence $\{A_n\}_n$ is non-decreasing or increasing. In this case, we write $A_n \uparrow$. If $A = \bigcup_{n=1}^{\infty} A_n$, then we say $\{A_n\}_n$ is increasing to A and denote it by $A_n \uparrow A$.

Decreasing sequence of sets

If $A_1 \supseteq A_2 \supseteq \ldots$, then we say that the sequence $\{A_n\}_n$ is non – increasing or decreasing. In this case, we write $A_n \downarrow A$. if A $\bigcap_{n=1}^{\infty} A_n$, then we say $\{A_n\}_n$ is decreasing to A and denote it by $A_n \downarrow A$.

Classes of Sets

A collection of subsets of Ω is termed as Class of subsets of Ω. It plays an important role in measure theory. They have some closure properties with respect to different set operations.

A be the class of subsets of Ω.

Complement: A is said to be closed under the complement, if for any set A ∈ **A**, $\bar{\mathbf{A}}$ is ∈ **A**

Union: **A** is said to be closed under the union if for any sets A, B∈ **A**, **A**UB is ∈ **A** Intersection: **A** is said to be closed under the intersection if for any sets A B∈**A**, **A**UB is ∈ **A**

Finite Union and Countable Union: **A** is said to be closed under the finite union if for any sets $\mathbf{A_1, A_2, \ldots A_n} \in \mathbf{A}$, $\mathrm{U}^n_{i=1} \mathbf{A_i}$, is ∈ **A** Further if n→∞ and if we have $\mathrm{U}^n_{i=1} \mathbf{A_i}$, is ∈ **A**, **A** is said to be closed under countable unions.

Finite intersection and Countable intersection: **A** is said to be closed under the finite intersection if for any sets $\mathbf{A_1, A_2, \ldots, A_n} \in \mathbf{A}$ $\cap^n_{i=1}$ is ∈ A_I Further if n→∞ and if we have $\cap^n_{i=1}\mathbf{A}_i$, is ∈ **A**, **A** is said to be closed under countable intersection. Note: Closure property for countable operation implies closed for finite operation.

classes of set

- Field (Algebra)
- Sigma field (Sigma Algebra)

Field

A class F of subsets of a non empty set Ω is called a field on Ω if

1. Ω∈F.

2. It is closed under complement.

3. It is closed under finite Union

Notationally

A class F of subsets of a non empty set Ω is called a field on Ω if

1. Ω∈ F

2. for any set A ∈ F, $\bar{A}$ ∈F.

3. for any sets $A_1, A_2, \ldots A_n \in F$, $\cup_{i=1}^{n} A_i \in F$

Following points we should keep, in mind regarding field.

Closure for complement and finite Union implies closure for intersection. So, field is closed under finite intersections.

{ φ, Ω } is a field.

Power set P(Ω), which is set of all subsets of Ω is a field.

For any A∈Ω, {Φ, Ω, **A**, $\bar{A}$ } is smallest field containing A.

For any sets A, B∈F A-B∈F, hence A **Δ** B ∈ F.

F_1 and F_2 are two fields on Ω, then $F_1 \cap F_2$, is a field.

Field is also called as an Algebra.

Sigma Field

σ- field: A class C of subsets of a non empty set Ω is called a - σ- field on Ω if

1. Ω∈ C.

2. It is closed under complement.

3. It is closed under countable Unions.

A class C of subsets of a non empty set Ω is called a σ field on 2 if

1. Ω∈ C

2. for any set A ∈ C, A^{-}∈C.

3. for any sets A, A_2,..., C, then U A∈ C

Field which is closed under countable unions is a α - field.

Like fields the intersection of arbitrary σ - field - fields is also σ – field but their union is not a σ – field.

Power set P(Ω), which is collection of all subsets of Ω is σ . - field.

Given a class of sets consisting all countable and complements of countable sets is a σ field

Minimal σ- field:

A class C of subsets of Ω is called a minimal σ- field on Ω, if it is the smallest σ field containing C

Minimal σ - field can be generated by taking intersection of all the **α** fields containing C.

If A is family of subsets of Ω, and $C_{\mathbf{A}}$ = n {C|A ⊂ C}, which is intersection of all σ - fields containing A then $C_{\mathbf{A}}$ is a minimal σ - field.

If A itself is a σ - field, then $C_{\mathbf{A}}$ = **A**

Hence onwards we term the pair (Ω,C) as a sample space

In theory of probability Ω = IR has specific features and sample space (IR, B) plays vital role.

Borel σ – field:

Let C is the class of all open intervals $(-\infty,x)$ where $x \in \mathbb{R}$, a minimal σ - field generated by C is called as Borel σ - and denoted by B.

Borel σ field has following features.

1. It is clear that $[x,\infty)$ is complement of $(-\infty,x)$ but it does not belong to C. Thus C is not closed under complement C is also not closed under countable intersections, as $\cap^{\infty}_{n=1}$ $(x-1/n, \infty) = [x, \infty)$ But B is closed under complements as well as countable unions or intersections. Hence **B** contains all intervals of the type $[x,\infty)$

2. $(-\infty,x] = \cap^{\infty}_{n=1}$ $(-\infty,x+ 1/n)$ So B contains all intervals of the type $(-\infty, x]$

3. (x,∞) is complement of $(-\infty,x]$. Thus B contains all intervals of the type (x,∞).

4. $(a,b) = (-\infty,b) \cap (a,\infty)$, where $a<b$. So B contains all intervals of the type (a, b). And contains even intervals of the type $[a, b)$, $(a, b]$ for all $a, b \in \mathbb{R}$.

Note that sets of B are called as borel sets.

Monotone class

Definition

A monotone class on Ω is a collection μ of subsets of Ω that is closed under monotone unions and monotone intersections, that is, for any sequence $A_n \in \mu$,

(i) if $A_1 \subset A_2 \subset \ldots$.. C. then $U_{n>1}$ $A_n \in \mu$, and

(ii) if $A_1 \supset A_2 \supset \ldots$ then $n \geq 1$ $U_{n>1}$ $A_n \in \mu$.

Clearly, the intersection of any collection of monotone classes is again a monotone class; consequently, by the same logic for any collection of subsets of a given space Ω there is a smallest monotone class μ containing A.

Proposition

If μ and v are probability measures on σ (A) that agree on A, where A is an algebra of subsets of Ω, then μ = v on σ(A).

Proof. Let H be the collection of all H $\in\sigma$(A) such that μ(B) = v(B). We must show that H = σ (A). Since H contains the algebra A, it will suffice, by the Monotone Class Theorem, to show that H is a monotone class.

Suppose that $H_1 \subset H_2 \subset \ldots$ is a non-decreasing sequence of sets in H, that is, such that $\mu(H_i) = v(H_i)$ for every i. Since probability measures are continuous from below,

$$\mu(\cup_{i=1}^{\infty} H_i) = \lim_{i\to\infty} \mu(H_i) \text{ and}$$

$$v(\cup_{i=1}^{\infty} H_i) = \lim_{i\to\infty} V(H_i)$$

$$\mu(\cup_{i=1}^{\infty} H_i) = v(\cup_{i=1}^{\infty} H_i).$$

Measure

A measure on an algebra of is a set function satisfying that is countably additive, i.e., if and then $A_n \in A$

$$\mu(\cup_{n\geq 1} A_n) = \Sigma_{n\geq 1}\, \mu(A_n)$$

Properties of Measures. (1) If u is a measure on an algebra A then

(a) μ is finitely additive.

(b) μ monotone, i.e., $A \subset B$ implies $\mu(A) \leq \mu(B)$.

If μ is a measure on a -algebra then

(c) μ is countably sub additive, i.e., for any sequence $A_n \in F$,

$$P(U_{i=1}^{\infty} A_n) \leq \sum_{i=1}^{\infty} P(A_n).$$

(d) If μ = P is a probability measure then P is continuous from above and below, i.e.,

$A_1 C A_2\ C A_3\ C\ldots \rightarrow\ P(U_{n\geq 1}\ A_n) = \lim_{n\to\infty}\ P(A_n)$,

$A_1 \supset A_2 \supset A_3\ldots \rightarrow P(\cap_{n\geq 1}\ A_n) = \lim_{n\to\infty}\ P(A_n)$

Measure Space

Defintion

The triple (Ω, F,) is called a measure space. Where

Ω – Sample space

F – All subset of Ω

μ – measure on a σ algebra

Measurable function

Measurable Function: An extended real valued function f defined on a measurable set E is said to be measurable function if {x| f(x) > α} is measurable for each real number α.

Theorem: A constant function with a measurable domain is measurable.

Proof: Let f be a constant function with a measurable domain E and Let f : E→ R be a constant function

i.e., f(x) = k ∀ $x \in E$ *and* k *is constant*.

To show that {x| f(x) > α} is measurable for each real number α.

$$\{x|\ f(x) > \alpha\} = \begin{cases} E, & k > \alpha \\ \varphi, & k = \alpha \\ \varphi, & k < \alpha \end{cases}$$

Since both φ *and* E are measurable, it follows that the set {x| f(x) > α} and hence f is measurable.

Theorem: Let f be an extended real valued function defined on a measurable set E, Then f is said to be measurable (Lebesgue function) if for any real $\boldsymbol{\alpha}$ any one of the following four conditions is satisfied.

(a) $\{x|\ f(x) > \alpha\}$ is measurable

(b) $\{x|\ f(x) \geq \boldsymbol{\alpha}\}$ is measurable

(c) $\{x|\ f(x) < \alpha\}$ is measurable

(d) $\{x|\ f(x) \leq \boldsymbol{\alpha}\}$ is measurable.

Proof: We show that these four conditions are equivalent. First of all we show that (a) and (b) are equivalent. Since

$$\{x|\ f(x) > \alpha\} = \{x|\ f(x) \leq \alpha\}^c$$

And also we know that complement of a measurable set is measurable, therefore (a) ⟹(d) and conversely.

Similarly since (b) and (c) are complement of each other, (c) is measurable if (b) is measurable and conversely.

Therefore, it is sufficient to prove that (a) ⟹(b) and conversely.

Firstly we show that (b) ⟹ (a).

The set $\{x|\ f(x) \geq \alpha\}$ is given to be measurable.

Now

$$\{x|\ f(x) > \alpha\} = \bigcup_{n=1}^{\infty} \{x|\ f(x) \geq \alpha + \frac{1}{n}\}$$

But by (b),

$$\{x|\ f(x) \geq \alpha + \frac{1}{n}\}$$

is measurable. Also we know that countable union of measurable sets is measurable. Hence $\{x|\ f(x) > \alpha\}$ is measurable which implies that (b) ⟹(a).

Conversely, let (a) holds. We have

$$\{x|\ f(x) \geq \alpha\} = \bigcap_{n=1}^{\infty} \{x|\ f(x) \geq \alpha - \frac{1}{n}\}$$

The set

$$\{x|\ f(x) > \alpha - \frac{1}{n}\}$$

is measurable by (a). Moreover, intersection of measurable sets is also measurable . Hence $\{x|\ f(x) \geq \alpha\}$ is also measurable . Thus (a) $\Longrightarrow$ (b).

Hence the four conditions are equivalent.

Properties of a Measurable Function

Following are some important and useful properties of a measurable function:

- If f is a measurable function defined on measurable sets E_n for all n being natural numbers, and if $E = \bigcup E_n$, then f is measurable on E as well.
- If f is a measurable function on a measurable set A and $B \subset A$ is a measurable set then f is measurable on B.
- If f is a continuous function defined on set E which is a measurable set, then f is a measurable function.
- A continuous function on a closed interval is measurable.
- A function f will be a measurable function on measurable set A, if and only if, for any open set G in R, $f^{-1}(G)$ is a measurable set.
- If f and g are measurable functions, then f + g and fg are also measurable functions.
- If f is a measurable function on a measurable set E, and g is a continuous function defined on the range of f, then g is a measurable function on E.