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Unit-II

Functions of Random Variables and their Distributions

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UNIT II

Function of random variables and their distribution

Let x be a random variable with known distribution. Let another random variable y be a function of x.

$$y = g(x)$$

Where g: R→R.

Strictly increasing function

When the function g is strictly increasing on the support of x

[I.e, $\forall\ x_{1,}x_2 \in Rx_{p,}x_1 > x_2 \Rightarrow g(x_1) > x_2$]

Then g admits an inverse defined on the support of y(i.e.)a function $g^{'}(y)$such that $x = g^{'}(y)$. And function more $g^{'}(y)$is itself increasing.

Strictly increasing functions of a discrete random variable

Let x be a discrete random variable with support of R_x and probability mass function P_x(x).Let g:R→R be strictly increasing on the support of X. Then the support of Y=g(x)is

$$R_y = \{Y = g(x): x\ \in R_x\}\ and\ its\ P.M.F\ is$$

$$P_y(y)=\begin{cases} P_y\left(g^{-1}(y)\right) if\ y \in f\ R_y \\ 0 \qquad\qquad if\ y \in f\ R_y \end{cases}$$

Strictly increasing functions of a continuous random variable

Let x be continuous random variable with support R_x and probability density function bef_x(x). Let g:R→R be strictly increasing and differentiable on the support of X. Then the support of Y=g(x)is

$$R_y = \{Y = g(x): x\ \in R_x\}\ and\ its\ P.D.F\ is$$

$$F_Y(Y) = \begin{cases} f_X\left(g^{-1}(y)\right)\frac{dy^{-1}(y)}{dy}, if\ y \in R_y \\ 0 \qquad\qquad\qquad\quad , if\ y \in R_y \end{cases}$$

Strictly decreasing function

Let x be a random variables with support R_x and distribution $F_x(x)$. Let g:R→R be strictly increasing on the support of X. Then the support of Y=g(x)is

$$R_y = \{Y = g(x): x \in R_x\}\ and\ the\ distribution\ of\ y\ is$$

$$F_Y(Y)=\begin{cases}0 & ,if y < x, \forall x \in R_x \\ 1 - F_x\left(g^{-1}(y)\right) + p(x = g^{-1}(y), if\ y \in R_x \\ 1 & ,if\ y \in R_x\end{cases}$$

- **Strictly decreasing functions of a discrete random variable**

Let x be a discrete random variable with support of R_x and probability mass function P_x(x).Let g:R→R be strictly decreasing on the support of X. Then the support of Y=g(x)is

R_y={Y=g(x): x ∈ R_x } and its P.M.F is

$$P_y(y)=\begin{cases}P_y\left(g^{-1}(y)\right) if\ y \in f\ R_y \\ 0 \qquad\quad if\ y \notin f\ R_y\end{cases}$$

- **Strictly decreasing functions of a continuous random variable**

Let x be continuous random variable with support R_x and probability density function bef_x(x). Let g:R→R be strictly DEcreasing and differentiable on the support of X. Then the support of Y=g(x)is

R_y={Y=g(x): x ∈ R_x } and its P.D.F is

$$F_Y(Y) = \begin{cases}-f_X\left(g^{-1}(y)\right)\frac{dy^{-1}(y)}{dy}, if\ y \in R_y \\ 0 \qquad\qquad\quad ,if\ y \in R_y\end{cases}$$

Invertible functions

In this case, in which the function g(x) is neither strictly increasing nor strictly decreasing.

Cumulative distribution function

The cumulative distribution function (C.D.F) of a random variable may be defined as the probability that the random variable x takes a value less than are equal to x.

Mathematically C.D.F [$F_X(x)$] may be defined as C.D.F [$F_X(x)$]= p(X≤ x)

Ex; Let S take an example of [experiment of tossing 3 coins at the same time]

S={HHH,HHT,HTH,HTT,THT,TTH,TTT,THH}
Here let S suppose the no. of tails is the random variable x
X={$x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8$}
={0,1,1,2,2,2,3,1}
C.D.F mar be defined is for

- Continuous random variable
- Discrete random variable

Other name of C.D.F

- Distribution function of the random variable
- Cumulative Probability distribution function.

Properties of Cumulative distribution function (C.D.F).

Property 1:

As C.D.F is defined as $F_X(x) = p(X \leq x)$ so its values is always between 0 and 1.

$0 \leq F_X(x) \leq 1$.

Property 2:

$F_X(-\infty) = 0$.

$F_X(\infty) = 1$.

Property 3:

$F_X(x_1) \leq F_X(x_2)$ if $x_1 \leq x_2$

So, according to this property $F_X(x)$ (i.e,) C.D.F is a monotone non decreasing function of x.

Note.

$p(X = x_1) = p(x_1 < X < \infty)$

$= F(\infty) - F(x_1)$

$= 1 - F(x_1)$

C.D.F for discrete random variable

If x is a discrete random variable then it has taken on value of discrete period.

Cumulative distribution function can be defined for this case as for w.r.to

C.D.F= $[F_X(x)] = p(X = x)$

Suppose X=$\{x_1, x_2, \ldots \ldots x_n\}$

So, C.D.F for a discrete random variable for the complete range of x can be defined

$$F_X(x)=\begin{cases} 0 & \text{for } -\infty \leq x \leq x_1 \\ \sum_{i=1}^{n} p(X = x_i) & \text{for } x_1 \leq x \leq n \\ 1 & \text{for } x_n \leq x < \infty \end{cases}$$

C.D.F of a discrete random variable may contain event is equal t6o the summation of the probability of random variables up to the contain event.

As a various from $-\infty$ to ∞ to graph of C.D.F (I.e,) $F_X(x)$ reasonable a staincase with upward steps having heights $p(X = x_i)$ at each $(X = x_i)$. But note one thing that the graph of $F_X(x)$ remains constants between the two steps on event.

C.D.F and P.D.F problems

Problem 1

Determine whether the problem following function is cumulative distribution function

$$F_X(x)=\begin{cases} 0 & ,for\ x<-a \\ \frac{1}{2}\left(\frac{x}{a}+1\right) & for-a\le x\le a \\ 1 & ,if\ x>0 \end{cases}$$

Solution

$$F_X(dx)=\frac{d}{dx}\ F_X(x)$$

$$F_X(x)=\begin{cases}\frac{1}{2a} & for-a\le x\le a \\ 0 & for\ elsewhere\end{cases}$$

The required condition for a P.D.F is $\int_{-\infty}^{\infty}\ F_X(x)\ dx=1$

$$\int_{-\infty}^{\infty} F_X(x)\ dx=\int_{-\infty}^{-a} F_X(x)\ dx+\int_{-a}^{a} F_X(x)\ dx+\int_{a}^{\infty} F_X(x)\ dx$$

$$=\mathbf{0}+\int_{-a}^{a}\frac{1}{2a}\ dx+0$$

$$=\frac{1}{2a}[x]_{-a}^{a}$$

$$=\frac{1}{2a}[a+a]$$

$$\int_{-\infty}^{\infty} F_X(x)\ dx=1$$

$F_X(x)$ satisfies the condition of P.D.F. Hence $F_X(x)$ is C.D.F

Problem 2

Verify the condition $F_X(x)=\begin{bmatrix} 2x & for\ 0<x<1 \\ 4-2x & for\ 1<x<2 \\ 0 & for\ x>2 \end{bmatrix}$

Whether probability density function or not?

Solution

$$F_X(x)=\begin{bmatrix} 2x & for\ 0<x<1 \\ 4-2x & for\ 1<x<2 \\ 0 & for\ x>2 \end{bmatrix}$$

$$\int_{-\infty}^{\infty} F_X(x)\,dx = \int_{-\infty}^{0} F_X(x)\,dx + \int_{0}^{1} F_X(x)\,dx + \int_{1}^{2} F_X(x)\,dx + \int_{2}^{\infty} F_X(x)\,dx$$

$=0+\int_0^1 F_X(x)\,dx + \int_1^2 F_X(x)\,dx + 0$

$$= 2\left[\frac{x^2}{2}\right]_0^1 + \left[\frac{4x - 2x^2}{2}\right]_1^2$$

$=1+4-3$

$$\int_{-\infty}^{\infty} F_X(x)\,dx = 2$$

$$\int_{-\infty}^{\infty} F_X(x)\,dx \neq 1$$

Hence it is not a probability density function.

Jacobian of a transformation

The concept of a derivation of a co-ordinate transformation which is known as the jacobian of the transformation

If we let $u = (uv), p = (pq)\ and\ x = (x, y)\ then\ (x, y) = T(u, v)$ is given in vector notation by

$$X = T(u)$$

This notation allows us to extend the concept of a total derivation to the total derivation of a co-ordinate transformation.

Matrix of Jacobian

A co-ordinate transformation T(u) is differential at a point P if there exists a matrix J(p) for which,

$$\lim_{u \to p} \frac{\|T(u) - T(P) - J(P)(u - p)\|}{\|(u - p)\|} = 0 \rightarrow (1)$$

When it exists J(P) is the trail derivative of J(u) at p

In non vector notation says that the total derivative at a point (p,q)of a co ordinate transformation T(u,v) is a matrix J(u,v) evaluate at (p,q). in a manner analogues to that it can be shown that this matrix is given by

$$J(u,v) = \begin{pmatrix} xu & xv \\ yu & yu \end{pmatrix}$$

The total derivative is also known as the jocobian matrix of the transformation T(u,v).

EX.1

What is the jocobian matrix for the polar co ordinates transformation?

Soln.

Since $x = rcos\theta\ y = rsin\theta$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$\frac{\partial x}{\partial r} = cos\theta, \frac{\partial x}{\partial \theta} = -rsin\theta$$

$$\frac{\partial y}{\partial r} = sin\theta, \frac{\partial y}{\partial \theta} = rcos\theta$$

$$= \begin{vmatrix} cos\theta & -rsin\theta \\ sin\theta & rcos\theta \end{vmatrix}$$

$$= rcos^2\theta + rsin^2\theta$$

$$= r[cos^2\theta + sin^2\theta] = r$$

Jacobian determinant

The determinant of the Jacobian matrix of a transformation is given by

$$\det J = \begin{vmatrix} xu & xv \\ yu & yv \end{vmatrix}$$

$$= \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

However, we often use a notation for det(J) that is more suggeation of the determinant is calculated

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

The remainder of this section explores the Jacobian determinant and some of its more important properties.

EX:2

Calculate the Jacobian determinant of $T(u,v) = u^2 - yu^2 + v$. If we identify

$$x = \boldsymbol{u^2 + v}, y = \boldsymbol{u^2 + v}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

$$= 2u - 2u(-1) = 4u$$

To begin with if $x(u,v)$and $y(u,v)$ are differentiable function, then

$$\frac{\partial(y,x)}{\partial(u,v)} = -\frac{\partial x}{\partial v}\frac{\partial y}{\partial u} - \frac{\partial x}{\partial u}\frac{\partial y}{\partial v}$$

$$= -\left(\frac{\partial x}{\partial v}\frac{\partial y}{\partial u} - \frac{\partial x}{\partial u}\frac{\partial y}{\partial v}\right) = -\frac{\partial(x,y)}{\partial(u,v)} = 0$$

Note

If $f(u,v), g(u,v)\ and\ h(u,v)$ are differential then,

$$\frac{\partial(f,g,h)}{\partial(u,v)} = \frac{\partial(f+g)}{\partial u}\frac{\partial h}{\partial v} - \frac{\partial(f+g)}{\partial v}\frac{\partial h}{\partial u}$$

$$= \frac{\partial f}{\partial u}\frac{\partial h}{\partial v} + \frac{\partial g}{\partial u}\frac{\partial h}{\partial v} - \frac{\partial f}{\partial v}\frac{\partial h}{\partial u} - \frac{\partial g}{\partial v}\frac{\partial h}{\partial u}$$

$$= \left(\frac{\partial f}{\partial u}\frac{\partial h}{\partial v} - \frac{\partial g}{\partial u}\frac{\partial h}{\partial v}\right) + \left(\frac{\partial f}{\partial v}\frac{\partial h}{\partial u} - \frac{\partial g}{\partial v}\frac{\partial h}{\partial u}\right)$$

$$\frac{\partial(f,g,h)}{\partial(u,v)} = \frac{\partial(f,h)}{\partial(u,v)} + \frac{\partial(f,g)}{\partial(u,v)}$$

Inverse function theorem

Let T(u,v) be a co ordinate transformation or an open region s in the uv plane and let (pq) be a point in s if

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right|_{(u,v)=(p,q)} \neq 0$$

Then there is an open region 'u' containing (p,q) and an open region 'v' containing $(x,y) = T(p,q)$ such that T^{-1} exists and maps v and u.

Proof:

Inverse function theorem follows from the fact of that the Jacobian matrix of $T^{-1} = (x,y)$. when it exists by the inverse of the Jacobian of T.

$$J^{-1}(x,y) = \left(\frac{\partial(x,y)}{\partial(u,v)}\right)^{-1} = \begin{pmatrix} yv & -xv \\ -yu & xu \end{pmatrix}$$

Which features a Jacobian determinate with a negative power. Thus J^{-1} exists only if the determinate of J(u,v) is non- zero.

Moment generating function

The M.G.F of random variable x (about origin) having the probability function f(x)is given by

$$M_X(t) = \mathrm{E}(e^{tx}) = \begin{Bmatrix} \int (e^{tx}) f(x), & for\ continuous\ p.d \\ \sum (e^{tx})\, p(x), & for\ discrete\ \ p.d \end{Bmatrix} \rightarrow (1)$$

The integration or summation being extended to the endire range of x,t being the real parameter and it is being assumed that the right hand side of (1) is absolutely convergent for some positive number 4 such that –h<t<h. thus

$$M_X(t) = \mathrm{E}(e^{tx}) = E\left(1 + tx + \frac{t^2x^2}{2!} + \cdots + \frac{t^r x^r}{r!} + \cdots\right)$$

$$= 1 + tE(x) + \frac{t^2E(x^2)}{2!} + \cdots + \frac{t^r E(x^r)}{r!} + \cdots$$

$$= 1 + t\mu_1^{'} + \frac{t^2\mu_2^{'}}{2!} + \cdots + \frac{t^r\mu_r^{'}}{r!} + \cdots \quad \rightarrow (2)$$

$$= \sum_{\mathrm{r}=0}^{\infty} \frac{\mathrm{t}^{\mathrm{r}}}{\mathrm{r!}} \mu_r^{'}$$

where $\mu_r^{'} = E(X^r) = \begin{Bmatrix} \int_{-\infty}^{\infty} x^r f(x)dx & for\ continuous\ p.d \\ \sum_x^n x^r\, f(x), & for\ discrete\ \ p.d \end{Bmatrix}$

Is the r [th] moment of x about origin thus we see that the coefficient of $\frac{\mathrm{t}^{\mathrm{r}}}{\mathrm{r!}}$ in $\mathrm{M_X(t)}$ gives $\mu_r^{'}$ (about origin .since $\mathrm{M_X(t)}$ generates moments, it is known as m.g.f

Diff, (2) w.r.t r and put t=0,

$$\left|\frac{d^r}{dt^r}\mathrm{M_x(t)}\right|_{t=0} = \left|\mu_r^{'} + \mu_{r+1}^{'}.t + \mu_{r+2}^{'}\frac{\mathrm{t}^2}{2} + \right|_{t=0} = \mu_r^{'} \rightarrow (3)$$

In general, the moment generating function x about the point x=a is defined as

$$M_{x-a}(t) = E(e^{t(x-a)})$$

$$= E\left(1 + t(x-a) + \frac{t^2(x-a)^2}{2!} + \cdots + \frac{t^r(x-a)^r}{r!} + \cdots\right)$$

$$= 1 + tE(x-a) + \frac{t^2E(x-a)^2}{2!} + \cdots + \frac{t^rE(x-a)^r}{r!} + \cdots$$

$$= 1 + t\mu_1^{'} + \frac{t^2\mu_2^{'}}{2!} + \cdots + \frac{t^r\mu_r^{'}}{r!} + \cdots \quad \rightarrow (4)$$

Where $\mu_r^{'} = E[(x-a)^r]$ is the rth moment about the point x=a.

Example 1

Let as consider a discrete random variable with probability function to find m.g.f

$$f(x) = \frac{1}{x(x+1)}\ , x = 1,2, \ldots.$$

Soln.

$$f(x) = \frac{1}{x(x+1)}\ , x = 1,2, \ldots$$

$$E(x) = \sum_{x=1}^{\infty} x\, f(x)$$

$$= \sum_{x=1}^{\infty} x \frac{1}{x(x+1)}$$

$$= \sum_{x=1}^{\infty} \frac{1}{(x+1)}$$

$$= \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots..$$

$$= \left(\sum_{x=1}^{\infty} \frac{1}{x}\right) - 1$$

Since $\sum_{x=1}^{\infty} \frac{1}{x}$ is a diverged series, then E(x) does not exist and consequently no moments of x exists However the m.g.f of x is given by

$$M_x(t) = \sum_{x=1}^{\infty} e^{tx} f(x)$$

$$= \sum_{x=1}^{\infty} e^{tx} \frac{1}{x(x+1)}$$

$$= \sum_{x=1}^{\infty} \frac{z^x}{x(x+1)} \; z = e^t$$

$$= \frac{z}{1.2} + \frac{z^2}{2.3} + \frac{z^3}{3.4} \ldots$$

$$= \left(\frac{z}{1} + \frac{z^2}{2} + \frac{z^3}{3} \ldots \right) - \frac{z}{2} - \frac{z^2}{3} - \frac{z^3}{4} \ldots$$

$$= \left(\frac{z}{1} + \frac{z^2}{2} + \frac{z^3}{3} + \cdots \right) - \left(\frac{z}{2} + \frac{z^2}{3} + \frac{z^3}{4} + \cdots \right)$$

$$= -\log(1-z) - \frac{1}{z}\left(\frac{z^2}{2} + \frac{z^3}{3} + \frac{z^4}{4} + \cdots \right)$$

$$= -\log(1-z) + \frac{1}{z}(\log(1-z) + z)$$

$$= -\log(1-z) + \frac{1}{z}(\log(1-z) + 1)$$

$$= 1 + \left(\frac{1}{z} - 1\right)\log(1-z), \lceil z \rceil < 1$$

$= 1 + (e^t - 1)\log(1 - e^t)$,t<0

Also $M_x(t) = 1$ for t=0

Hence $M_x(t)$ exist for t≤0.

Properties of moment generating function

Property 1

If $M_x(t)$ is the m.g.f of a random variable x then $M_{xc}(t)$is the m.g.f of the random variable [x, where c is any constant]

Proof

By defn of m.g.f

$\mathrm{M_x(t)} = E(e^{tx})$, $t \epsilon R$

$$\mathrm{M_{cx}(t)} = E(e^{t(cx)})$$

$= E(e^{(tc)x})$

$= \mathrm{M_x(ct)}.$

Property 2:

If $X_1, X_2, \ldots \ldots . X_n$ are independent random variable then

$$\mathrm{M}_{X_1, X_2, \ldots\ldots X_n}(\mathrm{t}) = \mathrm{M}_{X_1}(\mathrm{t}), \mathrm{M}_{X_2}(\mathrm{t}), \ldots \ldots \mathrm{M}_{X_n}(\mathrm{t})$$

Proof

$$\mathrm{M_x(t)} = E(e^{tX})$$

$$\mathrm{M}_{X_1, X_2, \ldots\ldots X_n}(\mathrm{t}) = E[e^{t(X_1, X_2, \ldots\ldots X_n)}]$$

$= E[e^{tX_1} e^{tX_2} \ldots \ldots e^{tX_n}]$

$= E[e^{tX_1}]. E[e^{tX_2}] \ldots \ldots E[e^{tX_n}]$

$\mathrm{M}_{X_1, X_2, \ldots\ldots X_n}(\mathrm{t}) = \mathrm{M}_{X_1}(\mathrm{t}), \mathrm{M}_{X_2}(\mathrm{t}), \ldots \ldots \mathrm{M}_{X_n}(\mathrm{t})$

Property 3:

Effect of chance of origin and scale on M.G.F. let us transform x to the new variable u by changing both the origin and scale in x as follows $u = \frac{x-a}{h}$. Where a and h are constants than

$$\mathrm{M_u(t)} = (e^{-at/h})\ \mathrm{M_x(t/h)}$$

Proof

$$\mathrm{M_x(t)} = E(e^{tX})$$

$$\mathrm{M_u(t)} = E(e^{tu})$$

$= E(e^{t\frac{x-a}{h}})$

$= E(e^{\frac{(tx-at)}{h}})$

$= E(e^{\frac{(tx)}{h}} e^{\frac{-at)}{h}})$

$= e^{\frac{-at}{h}} E(e^{\frac{(tx)}{h}})$

$\mathrm{M_u(t)} = e^{\frac{-at}{h}} \mathrm{M_x(t/h)}$

Uniqueness theorem of moment generating function

The m.g.f of a distribution if it exist, uniquely determine the distribution their means that corresponding to a given probability distribution, there is only one m.g.f (if it exist)and corresponding to a given m.g.f there is only one probability distribution.

$$\mathrm{M_{x+y}(t)} = E(e^{t(X+y)})$$

$$= E(e^{tX}.e^{ty)})$$

$$= E(e^{tX})\, E(e^{ty})$$

$$= X \text{ and } Y \text{ are identical distribution}$$

$$= \mathrm{M_x(t)M_y(t)}.$$

Characteristic function

The Characteristic function of random variable x is denoted by $\varphi_X(t)$ and is defined as

$\varphi_X(t) = \mathrm{E}(e^{itx})\ t \epsilon R$

$$\varphi_X(t) = \begin{cases} \int (e^{itx}) f(x)\, dx, & for\ continuous\ p.d \\ \sum (e^{itx})\, p(x), & for\ discrete\ \ p.d \end{cases}$$

Properties of Characteristic function

Property 1:

The Characteristic function $\varphi_X(t)$ is bounded by 1. (i.e.,)$\lceil \varphi_X(t) \rceil \leq 1$ for all t.

Proof

Let x is a continuous random variable by definition of characteristic function, we have

$$\lceil \varphi_X(t) \rceil = \lceil \mathrm{E}(\mathrm{e^{itx}}) \rceil$$

$$= \left\lceil \int_{-\infty}^{\infty} \mathrm{e^{itx}\, f(x) dx} \right\rceil$$

$$\leq \int_{-\infty}^{\infty} \lceil \mathrm{e^{itx}\, f(x)} \rceil\, dx$$

$$\leq \int_{-\infty}^{\infty} \lceil e^{itx} \rceil f(x)\, dx$$

$$\leq \int_{-\infty}^{\infty} f(x)\, f(x) dx$$

$$\lceil \varphi_X(t) \rceil \leq 1$$

Because f(x) is p.d.f

Hence $\lceil \varphi_X(t) \rceil \leq 1$ for all t.

Property 2:

For a Characteristic function $\varphi_X(t)$ we have

$\overline{\varphi_X(t)} = \varphi_X(-t)$

Proof

By defn of $\varphi_X(t)$

$\varphi_X(t) = E(e^{itx})$

$\overline{\varphi_X(t)} = \overline{E(e^{itx})}$

$$= E(e^{\overline{itx}})$$

$$= E(e^{-itx})$$

$$\overline{\varphi_X(t)} = \varphi_X(-t)$$

Property 3:

If a random variable x has a symmetrical distribution about the origin f(-x)=f(x) then $\varphi_X(t)$ is real valued solution in even function of t.

Proof

$$\varphi_X(t) = \int_{-\infty}^{\infty} e^{itx}\ f(x)dx\ [x = -y, dx = -dy]$$

$$= \int_{-\infty}^{\infty} e^{-itx}\ f(-x)dx\ [x = -x]$$

$$= \int_{-\infty}^{\infty} e^{-itx}\ f(x)dx\ [f(-x) = f(x)]$$

$$\varphi_X(t) = \varphi_X(-t)$$

$\varphi_X(t)$ is an even function of t

From property 2 and 3, we get

$$\varphi_X(t) = \varphi_X(-t) = \overline{\varphi_X(t)}$$

Hence, $\varphi_X(t)$ is a real valued and even function of t.

Property 4:

Additive property (or) if x and y are independent random variable then,

$$\varphi_{X+y}(t) = \varphi_X(t)\,\varphi_y(t)$$

Proof

$$\varphi_{X+y}(t) = E(\mathrm{e}^{\mathrm{itx+y}})$$

$$= E(\mathrm{e}^{\mathrm{itx}}.\,\mathrm{e}^{\mathrm{ity}})$$

$$= E(\mathrm{e}^{\mathrm{itx}})E(\mathrm{e}^{\mathrm{ity}})$$

$$\varphi_{X+y}(t) = \varphi_X(t)\,\varphi_y(t)$$

Property 5:

If we transform the variable x to a new variable y by changing both the origin and the scale on x

i.e, y= $\frac{x-a}{h}$ then $\varphi_y(t) = e^{\frac{-ait}{h}}\,\varphi_X(t/h)$.

Proof

By defn of characteristics function

$$\varphi_y(t) = E(\mathrm{e}^{\mathrm{ity}})$$

$$= E[\mathrm{e}^{\mathrm{it}\left(\frac{x-a}{h}\right)}]$$

$= E(e^{\frac{(itx)}{h}} e^{\frac{-ait)}{h}})$

$= e^{\frac{-ait}{h}} E(e^{\frac{(itx)}{h}})$

$$\varphi_y(t) = e^{\frac{-ait}{h}}\,\varphi_X(t/h)$$

Property 6:

If x is a random variable with characteristics function $\varphi_x(t)$ and if $\mu_r^{'} = E(X^r)$ then

$\mu_r^{'} = \frac{1}{i^r}\frac{dr}{dt^r}\ \varphi_X(t)$ at t=0

Proof

$$\varphi_X(t) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{itx}}\ \mathrm{f(x)dx}$$

Diff w.r.t t is r times

$$\frac{dr}{dt^r}\ \varphi_X(t) = \int_{-\infty}^{\infty} (ix)^r \mathrm{e}^{\mathrm{itx}}\ \mathrm{f(x)dx}$$

$$= i^r \int_{-\infty}^{\infty} x^r \mathrm{e}^{\mathrm{itx}}\ \mathrm{f(x)dx}$$

Put t=0

$$\frac{dr}{dt^r}\ \varphi_X(t) = i^r \int_{-\infty}^{\infty} x^r \mathrm{e}^{\mathrm{i0x}}\ \mathrm{f(x)dx}$$

$$= i^r \int_{-\infty}^{\infty} x^r\ \mathrm{f(x)dx}$$

$$= i^r E(X^r)$$

Hence

$$\mu_r^{'} = E(X^r)$$

$$\mu_r^{'} = \frac{1}{i^r}\frac{dr}{dt^r}\ \varphi_X(t)\ at\ t = 0$$

Uniqueness theorem of characteristics function

The characteristics function uniquely determine the distribution (i.e,) a necessary and sufficient condition for then distribution with p.d.f $f_1(.)$and $f_2(.)$ to be identified that the c.f $\varphi_1(t)$ and $\varphi_2(t)$ are identical.

Proof

If $f_1(.) = f_2(.)$ then from the definition of c.f

$$\varphi_1(t) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{itx}} f_1(x)dx$$

$$\varphi_2(t) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{itx}} f_2(x)dx$$

Conversely if,

$$\varphi_1(t) = \varphi_2(t)$$

$$f_1(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{itx}}\ \varphi_1(t)dt$$

$$f_2(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{itx}}\ \varphi_2(t)dt$$

This are to one corresponding between c.f and p.d.f's enables us to identify the form of the p.d.f that of c.f.