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Unit-IV

Continuous distribution

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Exponential distribution

Definition

A continuous random variable X assuming non- negative values is said to have an exponential distribution with parameter $\theta > 0$ and its p.d.f is given by

$$f(x) = \theta e^{-\theta x}; x \geq 0$$

Moment generating function of Exponential distribution

$$M_x(t) = E(e^{tx})$$

$$= \int_0^\infty e^{tx} f(x)dx = \int_0^\infty e^{tx}\, \theta e^{-\theta x} dx = \theta \int_0^\infty e^{tx}\, e^{-\theta x} dx$$

$$= \theta \int_0^\infty e^{-(\theta - t)x}\, dx = \theta \left[\frac{e^{-(\theta-t)x}}{-(\theta - t)}\right]_0^\infty = \frac{\theta}{-(\theta - t)} \left[e^{-(\theta-t)x}\right]_0^\infty$$

$$= \frac{\theta}{-(\theta-t)} [0 - 1] = \frac{\theta}{-(\theta-t)} (-1)$$

$$M_x(t) = \frac{\theta}{(\theta - t)}$$

$$\mu_r' = \left[\frac{\theta - t}{\theta}\right]^{-1}$$

$$= \left[1 - \frac{t}{\theta}\right]^{-1} = \sum_{r=0} \left(\frac{t}{\theta}\right)^r = 1 + \frac{t}{\theta} + \frac{t^2}{\theta^2} + \cdots.$$

$$\mu_r' = E(X)^r = co\ efficient\ of\ \frac{t^r}{r!}\ is\ M_x(t)$$

$$\mu_r' = \frac{r!}{\theta^r}$$

Put r=1

$$\mu_1' = \frac{1!}{\theta^1} = \frac{1}{\theta}$$

r=2

$$\mu_2' = \frac{2!}{\theta^2} = \frac{2}{\theta^2}$$

$$variance\ \mu_2 = \mu_2' - (\mu_1')^2$$

$$= \frac{2}{\theta^2} - \left(\frac{1}{\theta}\right)^2 = \frac{2}{\theta^2} - \frac{1}{\theta^2}$$

$$\mu_2 = \frac{1}{\theta^2}$$

Lack of memory

If X has exponential distribution then for every constant a<0, 1 has probability of X $p(Y \leq x\ /X \geq a) = p(X = x)\ \forall x,$ where Y=x-a

Proof

The p.d.f of exponential distribution with parameter θ is

$$f(x) = \theta e^{-\theta x}; x \geq 0$$

We have

$$p(Y \leq x\ /X \geq a) = \frac{p(Y \leq x\ \cap X \geq a)}{p(X \geq a)} = \frac{p(X - a \leq x\ \cap X \geq a)}{p(X \geq a)}$$

$$= \frac{p(X \leq a + x\ \cap X \geq a)}{p(X \geq a)}$$

$$p(Y \leq x\ /X \geq a) = \frac{p(\ a \leq X \leq a + x)}{p(X \geq a)}$$

$$p(\ a \leq X \leq a + x) = \int_a^{a+x} f(x)\ dx$$

$$= \int_a^{a+x} \theta e^{-\theta x}\, dx = \theta \int_a^{a+x} e^{-\theta x}\, dx = \theta \left\{\frac{e^{-\theta x}}{-\theta}\right\}_a^{a+x}$$

$$= -\left\{e^{-\theta x}\right\}_a^{a+x} = -\left[e^{-\theta(a+x)} - e^{-\theta a}\right] = e^{-\theta a} - e^{-\theta(a+x)}$$

$$= e^{-\theta a} - e^{-\theta a} e^{-\theta x}$$

$$p(\ a \leq X \leq a + x) = e^{-\theta a}\left[1 - e^{-\theta x}\right]$$

And

$p(X \geq a) = \int_a^{\infty} f(x)\, dx$

$$= \int_a^{\infty} \theta e^{-\theta x}\, dx = \theta \int_a^{\infty} e^{-\theta x}\, dx$$

$$= \theta \left\{\frac{e^{-\theta x}}{-\theta}\right\}_a^{\infty} = -\left\{e^{-\theta x}\right\}_a^{\infty} = -[0 - e^{-\theta a}]$$

$$p(X \geq a) = e^{-\theta a}$$

Now,

$$p(Y \leq x\,/X \geq a) = \frac{p(\,a \leq X \leq a + x)}{p(X \geq a)}$$

$$= \frac{e^{-\theta a}\left[1 - e^{-\theta x}\right]}{e^{-\theta a}}$$

$$p(Y \leq x\,/X \geq a) = \left[1 - e^{-\theta x}\right] \to L.H.S$$

$$p(X \leq x) = \int_0^x f(x)\,dx = \int_0^x \theta e^{-\theta x}\,dx = \theta \int_0^x e^{-\theta x}\,dx$$

$$= \theta \left\{\frac{e^{-\theta x}}{-\theta}\right\}_0^x = -\left\{e^{-\theta x}\right\}_0^x = -[e^{-\theta x} - 1]$$

$$p(X \leq x) = \left[1 - e^{-\theta x}\right] \to R.H.S$$

$$\therefore L.H.S = R.H.S$$

Hence proved

Standard Laplace (double exponential distribution)

A continuous random variable X is said to follow Standard Laplace distribution if its p.d.f is

$$f(x) = \frac{1}{2} e^{-|x|};\ -\infty < x < \infty$$

Characteristic function of double exponential distribution

$$\varphi_x(t) = E(e^{itx})$$

$$= \int_{-\infty}^{\infty} e^{itx} f(x) dx = \int_{-\infty}^{\infty} e^{itx} \frac{1}{2} e^{-|x|} dx$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} e^{itx}\, e^{-|x|} dx = \frac{1}{2}\int_{-\infty}^{\infty} [costx + isintx]\, e^{-|x|} dx$$

$$= \frac{1}{2}\left[\int_{-\infty}^{\infty} costx\ e^{-|x|} dx + \int_{-\infty}^{\infty} isintx\, e^{-|x|} dx\right] = \frac{1}{2} 2 \int_0^{\infty} costx\ \ e^{-|x|} dx + \frac{1}{2}(0)$$

$$= \int_0^{\infty} costx\ e^{-x} dx = \int_0^{\infty} e^{-x}\, costx\ dx = \left[e^{-x} \frac{sintx}{t}\right]_0^{\infty} + \int_0^{\infty} e^{-x} \frac{sintx}{t} dx$$

$$= 0 + \frac{1}{t}\int_0^{\infty} e^{-x}\, sintx\ dx = \frac{1}{t}\left[e^{-x} - \frac{costx}{t}\right]_0^{\infty} - \int_0^{\infty} e^{-x} - \frac{costx}{t}\ dx$$

$$= \frac{1}{t}\left(\frac{1}{t} - \left[\frac{1}{t}\int_0^{\infty} e^{-x}\, costx\, dx\right]\right)$$

$$\varphi_x(t) = \frac{1}{t^2} - \frac{1}{t^2}\varphi_x(t)$$

$$\varphi_x(t) + \frac{1}{t^2}\varphi_x(t) = \frac{1}{t^2}$$

$$\varphi_x(t)\left[1 + \frac{1}{t^2}\right] = \frac{1}{t^2}$$

$$= \frac{1}{t^2} \times \frac{t^2}{1 + t^2}$$

$$\varphi_x(t) = \frac{1}{1 + t^2}$$

The mean is zero standard deviation$\sqrt{2}$ and variance 2, mean deviation 1.

Two Parameter Laplace (Double Exponential Distribution)

A continuous random variate X is said to have a double exponential distribution with two parameter λ and μ if it p.d.f is given by

$$f(x, \mu, \lambda) = \frac{1}{2\lambda} exp[-|x - \mu|/\lambda]\,;\; -\infty < x < \infty, \lambda > 0$$

We write

$X \sim Lap(\lambda, \mu)$

$$Y = \frac{X - \mu}{\lambda}$$

$$X = \mu + \lambda Y$$

The p.d.f g(.) of Y is given by

$$g(y) = f(x)\frac{dx}{dy}$$

$$= \frac{1}{2\lambda} exp(-|y|)\lambda$$

$$= \frac{1}{2} exp(-|y|);\; -\infty < y < \infty$$

Which is the p.d.f of Standard Laplace Distribution. y is a Standard Laplace Distribution with p.d.f g(y).

Characteristic function of Two Parameter Laplace (Double Exponential Distribution)

$$\varphi_x(t) = E(e^{itx}) = E\left[e^{it(\mu+\lambda Y)}\right] = e^{it\mu}\, E\left[e^{it\lambda Y)}\right]$$

$$\varphi_x(t) = e^{it\mu}\varphi_x(\lambda t)$$

w.k.t

$$E\left[e^{itx)}\right] = \varphi_x(t)$$
$$E\left[e^{it\lambda Y)}\right] = \varphi_y(\lambda t)$$

And

$$\varphi_x(t) = \frac{1}{1+t^2}$$
$$\varphi_y(t) = \frac{1}{1+\lambda t^2}$$

Therefore,

$$\varphi_x(t) = \frac{e^{it\mu}}{1+\lambda t^2}$$

Moments of Two Parameter Laplace (Double Exponential Distribution

$$\mu_r' = E(X^r) = \frac{1}{2\lambda}\int_{-\infty}^{\infty} x^r\, f(x) dx$$

$$= \frac{1}{2\lambda}\int_{-\infty}^{\infty} x^r\, exp[-|x-\mu|/\lambda] dx = \frac{1}{2\lambda}\int_{-\infty}^{\infty} (z\lambda+\mu)^r\, e^{-]z[}\lambda\, dz$$

$$= \frac{1}{2}\int_{-\infty}^{\infty}\left[\sum_{k=0}^{\infty}\binom{r}{k}(z\lambda)^k(\mu)^{r-k}\right]e^{-]z[}\, dz$$

$$\mu_r' = \frac{1}{2}\sum_{k=0}^{\infty}\int_{-\infty}^{\infty}\left[\binom{r}{k}(z\lambda)^k(\mu)^{r-k}\right]e^{-]z[}\, dz$$

$$= \frac{1}{2}\sum_{k=0}^{\infty}\binom{r}{k}(\lambda)^k(\mu)^{r-k}\left[\int_{-\infty}^{0}[z^k]e^{-]z[}\, dz + \int_{0}^{\infty}[z^k]e^{-]z[}\, dz\right]$$

$$\mu_r' = \frac{1}{2}\sum_{k=0}^{\infty}\binom{r}{k}(\lambda)^k(\mu)^{r-k}\left[(-1)^k\int_{0}^{\infty}[z^k]e^{-]z[}\, dz + \int_{0}^{\infty}[z^k]e^{-]z[}\, dz\right]$$

$$= \frac{1}{2}\sum_{k=0}^{\infty}\binom{r}{k}(\lambda)^k(\mu)^{r-k}\left[\sqrt{k+1}(-1)^k + \sqrt{k+1}\right]$$

$$= \frac{1}{2}\sum_{k=0}^{\infty}\binom{r}{k}(\lambda)^k(\mu)^{r-k}k!\,[1+(-1)^k]$$

Mean

$$\mu_1' = \frac{1}{2}\sum_{k=0}^{1}\binom{1}{k}(\lambda)^k(\mu)^{1-k}k!\,[1+(-1)^k]$$

$$= \frac{1}{2}\sum_{k=0}^{1}\binom{1}{0}(\lambda)^0(\mu)^{1-0}0!\,[1+(-1)^0] + \frac{1}{2}(1)\,(\lambda)^1(\mu)^{1-1}1!\,[1+(-1)^1]$$

Mean

$$\mu_1' = \mu$$

$$\mu_2' = \frac{1}{2}\sum_{k=0}^{2}\binom{2}{k}(\lambda)^k(\mu)^{2-k}k!\,[1+(-1)^k]$$

$$= \frac{1}{2}\binom{2}{0}(\lambda)^0(\mu)^{2-0}0!\,[1+(-1)^0] + \frac{1}{2}\binom{2}{1}(\lambda)^1(\mu)^2 1!\,[1+(-1)^1]$$

$$+\frac{1}{2}\binom{1}{2}(\lambda)^2(\mu)^{2-2}2!\,[1+(-1)^2]$$

$$\mu_2' = \mu^2 + 2\lambda^2$$

Variance

$$\mu_2 = \mu_2' - (\mu_1')^2 = \mu^2 + 2\lambda^2 - \mu^2$$

$$\mu_2 = 2\lambda^2$$

Log normal distribution:

The positive random variable x is said to have log normal distribution if $log\,{}_e^X$ is normally distributed. Let we defined by

$$f_x(u) = \begin{cases} \frac{1}{u\sigma\sqrt{2\pi}} e^{\left\{-\frac{(logu-\mu)^2}{2\sigma^2}\right\}} & : u > 0 \\ 0 & : otherwise \end{cases}$$

Moments of Log normal distribution

$$f(u) = \frac{1}{u\sigma\sqrt{2\pi}} e^{\left\{-\frac{(logu-\mu)^2}{2\sigma^2}\right\}}$$

$$\mu_r' = E(u^r) = \int u^r\, f(u)du$$

$$= \int_{-\infty}^{\infty} u^r \frac{1}{\sigma\sqrt{2\pi}} \frac{e^{-1/2[(logu-\mu)/\sigma]^2}}{u} du = \int_{-\infty}^{\infty} e^{ry} \frac{1}{\sigma\sqrt{2\pi}} \frac{e^{-1/2[(y-\mu)/\sigma]^2}}{e^y} e^y dy$$

$$= \int_{-\infty}^{\infty} e^{ry}\, e^{-y} e^y \frac{1}{\sigma\sqrt{2\pi}} e^{-1/2[(y-\mu)/\sigma]^2} dy = \int_{-\infty}^{\infty} e^{ry}\ f(y{:}\,\mu\sigma)dy$$

$$= E(e^{ry})$$

M.G,F of normal distribution

$$\mu_r' = e^{\mu r} + \frac{1}{2}\ \sigma^2 r^2$$

Mean

$$\mu_1' = e^{\mu(1)+\frac{1}{2}\sigma^2(1)^2} = e^{\mu+\frac{1}{2}\sigma^2}$$

Variance

$$\mu_2 = \mu_2' - (\mu_1')^2$$

$$= e^{2\mu+2\,\sigma^2} - \left(e^{\mu+\frac{1}{2}\sigma^2}\right)^2$$

$$= e^{2\mu+2\,\sigma^2} . e^{\,\sigma^2-1}.$$

Beta distribution

Beta distribution of first kind

A random variable X is said to have a beta distribution of first kind with parameters u $and\ v\ \ (u > 0, v > 0)$ if its p.d.f is given by

$$f(x) = \frac{1}{\beta(u,v)} x^{u-1}(1-x)^{v-1}; (u,v) > 0, 0 < x < 1$$

Constant of Beta distribution of first kind

$$\mu_r' = \int_0^1 x^r\, f(x)dx$$

$$= \int_0^1 x^r \frac{1}{\beta(u,v)} x^{u-1}(1-x)^{v-1}dx = \frac{1}{\beta(u,v)} \int_0^1 x^r\, x^{u-1}(1-x)^{v-1}dx$$

$$= \frac{1}{\beta(u,v)} \int_0^1 x^{r+u-1}\,(1-x)^{v-1}dx \left\{\therefore \int_0^1 x^{m-1}\,(1-x)^{n-1}dx = \beta(m,n)\right\}$$

$$= \frac{1}{\beta(u,v)} \beta(u+r,v)\ \ \left\{\therefore \beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}\right\}$$

$$= \frac{\Gamma(u+v)}{\Gamma(u)\Gamma(v)}\ \frac{\Gamma(u+r)\Gamma(v)}{\Gamma(u+r+v)}$$

$$\mu_r' = \frac{\Gamma(u+v)\Gamma(u+r)}{\Gamma(u+r+v)\Gamma(u)}$$

Mean

Put r=1

$$\mu_r' = \frac{\Gamma(u+v)\Gamma(u+1)}{\Gamma(u+v+1)\Gamma(u)}\ \ [\therefore \Gamma(m+1) = m\Gamma(m)]$$

$$= \frac{\Gamma(u+v)\, u\Gamma(u)}{(u+v)\Gamma(u+v)\Gamma(u)} = \frac{u\Gamma(u)}{(u+v)\Gamma(u)}$$

$$\mu_r' = \frac{u}{(u+v)}$$

Put r=2

$$\mu_2' = \frac{\Gamma(u+v)\,\Gamma(u+2)}{\Gamma(u+v+2)\,\Gamma(u)}$$

$$= \frac{\Gamma(u+v)\,\Gamma(u+1+1)}{\Gamma(u+v+1+1)\,\Gamma(u)} = \frac{\Gamma(u+v)(u+1)\,\Gamma(u+1)}{(u+v+1)\,\Gamma(u+v+1)\,\Gamma(u)}$$

$$= \frac{\Gamma(u+v)(u+1)u\Gamma(u)}{(u+v+1)(u+v)\Gamma(u+v)\Gamma(u)}$$

$$\mu_2' = \frac{(u+1)u}{(u+v+1)(u+v)}$$

Variance $(\mu_2) = \mu_2' - (\mu_1')^2$

$$\mu_2 = \frac{(u+1)u}{(u+v+1)(u+v)} - \left(\frac{u}{(u+v)}\right)^2$$

$$= \frac{(u+1)u}{(u+v+1)(u+v)} - \frac{u^2}{(u+v)^2} = \frac{u}{u+v}\left[\frac{(u+1)}{(u+v+1)} - \frac{u}{u+v}\right]$$

$$= \frac{u}{u+v}\left[\frac{u+v(u+1)}{(u+v+1)} - \frac{u(u+v+1)}{u+v}\right] = \frac{u}{u+v}\left[\frac{u^2+u+uv+v-(u^2+uv+u)}{(u+v+1)u+v}\right]$$

$$= \frac{u}{u+v}\left[\frac{u^2+u+uv+v-u^2-uv-u}{(u+v+1)u+v}\right] = \frac{u}{u+v}\left[\frac{v}{(u+v+1)u+v}\right]$$

$$\mu_2 = \frac{uv}{(u+v)^2(u+v+1)}$$

Beta distribution of second kind

A random variable X is said to have a beta distribution of second kind with parameters u and v $(u > 0, v > 0)$ if its p.d.f is given by

$$f(x) = \frac{1}{\beta(u,v)}\frac{x^{u-1}}{(1+x)^{u+v}}; (u,v) > 0, 0 < x < \infty$$

Constant of Beta distribution of first kind

$$\mu_r' = \int_0^\infty x^r f(x)dx = \int_0^\infty x^r \frac{1}{\beta(u,v)} \frac{x^{u-1}}{(1+x)^{u+v}}\ dx$$

Add and substruct to r

$$= \frac{1}{\beta(u,v)} \int_0^\infty \frac{x^{r+u-1}}{(1+x)^{u+v-r+r}}\ dx = \frac{1}{\beta(u,v)} \beta(r+u)(v-r)$$

$$\mu_r' = \frac{\text{г}(u+v)\text{г}(v-r)}{\text{г}(u+v+r-r)} . \frac{\text{г}(u+v)}{\text{г}u\ \ \text{г}v}\ [\therefore \text{г}(m+1) = m\text{г}(m)]$$

$$\mu_r' = \frac{\text{г}(u+r)\text{г}(v-r)}{\text{г}u\ \ \text{г}v}$$

$$\mu_1' = \frac{\text{г}(u+1)\text{г}(v-1)}{\text{г}u\ \ \text{г}v} = \frac{u\ \text{г}u\ \text{г}v-1}{\text{г}u\ (v-1)\text{г}v-1} = \frac{u}{v-1}$$

$$\mu_2' = \frac{\text{г}(u+2)\text{г}(v-2)}{\text{г}u\ \ \text{г}v} = \frac{(u+1)\ \text{г}(u+1)\ \ \text{г}(v-2)}{\text{г}u\ (v-1)\text{г}(v-1)} = \frac{(u+1)u\ \text{г}(u)\ \ \text{г}(v-2)}{\text{г}u\ (v-1)(v-2)\text{г}(v-2)}$$

$$\mu_2' = \frac{u(u+1)}{(v-1)(v-2)}$$

Variance $(\mu_2) = \mu_2' - (\mu_1')^2$

$$= \frac{u(u+1)}{(v-1)(v-2)} - \left(\frac{u}{v-1}\right)^2 = \frac{u^2+u}{(v-1)(v-2)} - \frac{u^2}{(v-1)^2}$$

Gamma distribution

A random variable X is said to have a gamma distribution with parameter $\lambda > 0$, if its p.d.f is given by

$$f(x) = \frac{e^{-x} x^{\lambda-1}}{ꙅ(\lambda)}; \lambda > 0, 0 < x < \infty$$

Moment generating function of Gamma distribution

$$M_x(t) = E(e^{tx})$$

$$= \int_0^\infty e^{tx} f(x)dx = \frac{1}{ꙅ(\lambda)} \int_0^\infty e^{tx}\, e^{-x} x^{\lambda-1} dx$$

$$= \frac{1}{ꙅ(\lambda)} \int_0^\infty e^{-(1-t)x}\, x^{\lambda-1} dx = \frac{1}{ꙅ(\lambda)} \frac{ꙅ(\lambda)}{(1-t)^\lambda} = \frac{1}{(1-t)^\lambda}$$

$$M_x(t) = (1-t)^{-\lambda}$$

Cumulant generating function of Gamma distribution

$$k_x(t) = \log M_x(t)$$
$$= \log(1-t)^{-\lambda} = -\lambda \log(1-t)$$
$$= -\lambda\left[-\frac{t}{1}-\frac{t^2}{2}-\frac{t^3}{3}-\cdots.\right]$$

$$k_x(t) = \lambda\left[\frac{t}{1}+\frac{t^2}{2}+\frac{t^3}{3}+\cdots.\right]$$

$$mean(\mu) = k_1 = co\ efficient\ of\ t\ in\ k_x(t) = \lambda$$

$$variance(\mu_2) = k_2 = co\ efficient\ of\ \frac{t^2}{2!}\ in\ k_x(t) = \lambda$$

$$\mu_3 = k_3 = co\ efficient\ of\ \frac{t^3}{3!}\ in\ k_x(t) = 2\lambda$$

$$k_4 = co\ efficient\ of\ \frac{t^4}{4!}\ in\ k_x(t) = 6\lambda$$

$$\mu_4 = k_4 + 3(k_2)^2$$
$$\mu_4 = 6\lambda + 3\lambda^2$$
$$k_r = co\ efficient\ of\ \frac{t^r}{r!}\ in\ k_x(t) = \lambda(r-1)!$$
$$\beta_1 = \frac{{\mu_3}^2}{{\mu_2}^3} = \frac{(2\lambda)^2}{\lambda^3}$$
$$\beta_1 = \frac{4}{\lambda}$$
$$\beta_2 = \frac{\mu_4}{{\mu_2}^2} = \frac{6\lambda + 3\lambda^2}{\lambda^2}$$
$$\beta_2 = \frac{6}{\lambda} + 3.$$

Additive property of Gamma distribution

Statement

The sum of independent gamma variate is also a Gamma variate .more precisely if $X_1, X_{2,} \ldots\ldots X_k$ are independent gamma variate with parameter $\lambda_1, \lambda_2, \ldots.\lambda_k$ respectively then $X_1 + X_2 + \cdots. + X_k$ is also gamma variate with parameter $\lambda_1 + \lambda_2 + \cdots + \lambda_k$.

Proof

Since X_i is a $\gamma(\lambda_1)$ variate,

$$M_x(t) = (1-t)^{-\lambda}$$

The m.g.f of the sum $X_1 + X_2 + \cdots . + X_k$ is given by

$$M_{X_1+X_2+\cdots.+X_k}(t) = M_{X_1}(t). M_{X_2}(t). M_{X_3}(t) \ldots \ldots M_{X_k}(t)$$

$$= (1-t)^{-\lambda_1}. (1-t)^{-\lambda_2}. \ldots \ldots (1-t)^{-\lambda_k}$$

$$= (1-t)^{-(\lambda_1+\lambda_2+\cdots+\lambda_k)}$$

Which is the m.g.f of a gamma variate with parameter $\lambda_1 + \lambda_2 + \cdots + \lambda_k$ Hence the result follows by the uniqueness theorem of m.g.f's.

Cauchy distribution

A random variable X is said to have a standard Cauchy distribution if its p.d.f is given by

$$f(x) = \frac{1}{\pi(1+x^2)} : \; -\infty < x < \infty$$

Where x is a standard Cauchy variable more generally Cauchy distribution with parameter λ and μ has the p.d.f is given by

$$g_Y(y) = \frac{\lambda}{\pi[\lambda^2 + (y-\mu)^2]} : \; -\infty < y < \infty : \lambda > 0 \; and$$

We write :$X \sim c(\lambda, \mu)$

Characteristic function of Cauchy distribution

$$\varphi_x(t) = E\left(e^{itx}\right) = \int_{-\infty}^{\infty} e^{itx} f(x) dx$$

$$= \int_{-\infty}^{\infty} e^{itx} \frac{1}{\pi(1+x^2)} dx = \int_{-\infty}^{\infty} \frac{cos\; tx + i \sin tx}{\pi(1+x^2)} dx$$

$$= \int_{-\infty}^{\infty} \frac{cos\; tx}{\pi(1+x^2)} dx + \int_{-\infty}^{\infty} \frac{i \sin tx}{\pi(1+x^2)} dx = \int_{-\infty}^{\infty} \frac{cos\; tx}{\pi(1+x^2)} dx = \frac{2}{\pi} \int_{0}^{\infty} \frac{cos\; tx}{(1+x^2)} dx$$

$$\varphi_x(t) = \frac{2}{\pi} \times \frac{\pi}{2} . e^{-|t|} = e^{-|t|}.$$

Additive property of Cauchy distribution

Statement:

If $X_1 \; and \; X_2$ are independent Cauchy variate with parameter $(\lambda_1, \mu_1) \; and \; (\lambda_2, \mu_2)$ respectively, then $X_1 + X_2$ is a Cauchy variate with parameters $(\lambda_1 + \lambda_2, \mu_1 + \mu_2)$.

Proof:

$$\varphi_{x_i}(t) = e^{\{it - \lambda_j |t|\}} \; where \; j = 1,2,3, \ldots\ldots$$

$$\varphi_{X_1+X_2}(t) = \varphi_{X_1}(t).\varphi_{X_2}(t) = e^{\{i\mu_1 t-\lambda_1|t|\}}\, e^{\{i\mu_2 t-\lambda_2|t|\}}$$
$$= e^{it(\mu_1+\mu_2)-(\lambda_1+\lambda_2)|t|}$$

Thus

$$X_1 + X_2 \sim (\lambda_1 + \lambda_2, \mu_1 + \mu_2).$$

Moments of Cauchy's distribution

$$E(y) = \int_{-\infty}^{\infty} y\, f(y) dy = \int_{-\infty}^{\infty} y \frac{\lambda}{\pi[\lambda^2 + (y-\mu)^2]} dy$$

$$= \frac{\lambda}{\pi}\int_{-\infty}^{\infty} y \frac{1}{[\lambda^2 + (y-\mu)^2]} dy = \frac{\lambda}{\pi}\int_{-\infty}^{\infty} \frac{y+\mu-\mu}{[\lambda^2 + (y-\mu)^2]} dy$$
$$= \frac{\lambda}{\pi}\int_{-\infty}^{\infty} \frac{y-\mu}{[\lambda^2 + (y-\mu)^2]} dy + \mu\frac{\lambda}{\pi}\int_{-\infty}^{\infty} \frac{1}{[\lambda^2 + (y-\mu)^2]} dy$$

$$E(y) = \frac{\lambda}{\pi}\int_{-\infty}^{\infty} \frac{y-\mu}{[\lambda^2 + (y-\mu)^2]} dy + \mu$$

$$put\ y-\mu = x: dy = dx$$
$$E(y) = \frac{\lambda}{\pi}\int_{-\infty}^{\infty} \frac{x}{[\lambda^2 + (x)^2]} dx + \mu$$
$$put\ \lambda^2 + x^2 = t: 2xdx = dt: xdx = \frac{1}{2}dt$$
$$E(y) = \frac{\lambda}{2\pi}\int_{-\infty}^{\infty} \frac{1}{t} dt + \mu = \frac{\lambda}{2\pi}[\log|t|]_{-\infty}^{\infty} + \mu = \frac{\lambda}{2\pi}\{\infty - \infty\} + \mu$$

$\lim_{\substack{n\to\infty \\ n'\to\infty}} \int_{-n}^{n'} \frac{z}{\lambda^2+z^2} dz,$ *does not* exist, Its principal value viz, $\lim_{n\to\infty} \int_{-n}^{n'} \frac{z}{\lambda^2+z^2} dz,$ exist and is equal to zero.

Thus in the general sense the mean of cauchy distribution does not exist. But, If we conventionally agree to assume that the mean of cauchy distribution exists by taking the principal value), then it is located at x>μ also, obviously, then probability curve in symmetrical about the point x=μ. Hence, for this distribution the mean, median & mode coincide at the point x=μ.

Let $\lim_{n\to\infty}$

$$E(y) = \frac{\lambda}{2\pi}\lim_{n\to\infty}\int_{-n}^{n} \frac{1}{t} dt + \mu \ = \frac{\lambda}{2\pi}\lim_{n\to\infty}[\log|t|]_{-n}^{n} + \mu$$

$$= \frac{\lambda}{2\pi}\lim_{n\to\infty}[\log|\lambda^2 + x^2|]_{-n}^{n} + \mu = \frac{\lambda}{2\pi}\lim_{n\to\infty}[\log|\lambda^2 + x^2 + y^2|]_{-n}^{n} + \mu$$

$$= \frac{\lambda}{2\pi} \lim_{n\to\infty} \log|\lambda^2 + (n-\mu)^2| - \log|\lambda^2 + (-n-\mu)^2| + \mu$$

$$= \frac{\lambda}{2\pi} \lim_{n\to\infty} \left[\frac{\log|\lambda^2 + (n-\mu)^2|}{\lambda^2 + (-n-\mu)^2}\right] + \mu = \frac{\lambda}{2\pi} \lim_{n\to\infty} \left[log \ \frac{\frac{n^2}{n^2}(\lambda^2 + (n-\mu)^2)}{\frac{n^2}{n^2}(\lambda^2 + (-n-\mu)^2)}\right]$$

$$= \frac{\lambda}{2\pi} \lim_{n\to\infty} \left[log \ \frac{n^2\left(\frac{\lambda^2}{n^2} + \left(\frac{n-\mu}{n}\right)^2\right)}{n^2\left(\frac{\lambda^2}{n^2} + \left(\frac{-n-\mu}{n}\right)^2\right)}\right] + \mu = 0 + \mu$$

$$mean = E(y) = \mu$$

Variance $(\mu_2) = E(X-\mu)^2$

$$\mu_2 = \int_{-\infty}^{\infty} (X-\mu)^2 f(y)dy = \frac{\lambda}{2\pi}\int_{-\infty}^{\infty} \frac{(X-\mu)^2}{\lambda^2 + (x-\mu)^2}dx$$

$$= \frac{\lambda}{2\pi}\int_{-\infty}^{\infty} \frac{(X-\mu)^2 + \lambda^2 - \lambda^2}{\lambda^2 + (x-\mu)^2}dx = 1 - \frac{\lambda}{\pi}\int_{-\infty}^{\infty} \frac{\lambda^2}{\lambda^2 + (x-\mu)^2}dx\text{: } does\ not\ exist.$$

Compound poisson distribution

Let x be a p(y) so that

$$p(y) = \frac{e^{-\lambda}\lambda^r}{r!} : r = 0,1,2 ....$$

Where λ itself is a continuous random variable with generalized gamma density,

$$g(y) = \begin{cases} \frac{a^u}{\Gamma v} e^{-a\lambda}\lambda^{v-1} & : \lambda > 0, a > 0, v > 0. \\ 0 & : \lambda \leq 0 \end{cases}$$

Let us consider the two dimensional random, vector (X, λ) in which one variable is discrete and the other is continuous. For a constant h>o and $\lambda_1 > 0$. The joint density of X and λ is given by

$$p(X = rn, \lambda_1 \leq \lambda < \lambda_1 + h) = p(\lambda_1 \leq \lambda \leq \lambda_1 + h)p(X = r/(\lambda_1 \leq \lambda))$$

Dividing both side by h and proceeding to the limits as $h \to 0$, we get,

$$\lim_{h\to u} \frac{p(X = r \cap (\lambda_1 \leq \lambda \leq \lambda_1 + h))}{h} = \lim_{h\to 0} p(X = r)\lambda_1 \leq \lambda$$
$$\leq \lambda_1 + h \times \lim_{h\to 0} \frac{p(\lambda_1 \leq \lambda \leq \lambda_1 + h)}{h}$$

$$\lim_{h\to 0}\frac{p(\lambda_1 \le \lambda \le \lambda_1 + h)}{h} = \lim_{h\to 0}\frac{G(\lambda_1 + h) - G(\lambda_1)}{h} = G(\lambda_1) = g(\lambda_1).$$

Where G (.) is the distribution function and g(.) p.d.f of λ

$$\therefore \lim_{h\to u}\frac{p(X = r \cap \lambda_1 \le \lambda \le \lambda_1 + h)}{h} = \frac{e^{-\lambda}{\lambda_1}^r}{r!}\frac{a^u}{\Gamma v}e^{-a\lambda}{\lambda_1}^{v-1}$$

Integrating w.r.t λ_1 over 0 to ∞ and using gamma integral, the marginal probability function and of X is given by

$$p(X = r) = \frac{a^u}{\Gamma v\, r!}\int_0^\infty e^{-(1+a)^h}\lambda^{r+v-1}\,dy = \frac{a^u}{\Gamma v\, r!}\frac{\Gamma r + v}{(1+a)^{r+v}}$$

$$= \left(\frac{a}{1+a}\right)^u \frac{r(v+1)(v+2)\ldots\ldots(v+r-1)}{(1+a)^u r!} = \left(\frac{a}{1+a}\right)^u (-1)^r \binom{-v}{r}\left(\frac{1}{1+a}\right)^r$$

$$= \binom{-v}{r} p^v\,(-q)^r\ where\ p = \left(\frac{a}{1+a}\right) : q = 1 - p = \frac{1}{1+a}$$

Thus the marginal distribution of x is negative binomial with parameter (v, p).

Sampling Distribution:-

The sampling distribution of a statistic is the distribution of all possible values taken by the statistic when all possible samples of a fixed size n are taken from the population

If is a theoretical idea, we do not actually build it. The sampling distribution of a statistic is the Probability distribution of that statistic Sampling distribution of the sample mean, we take may random samples of a given size 'n' from population with mean " μ " and standard deviation.

Some sample mean will be above the population mean μ and some will be below, making up the sampling distribution. For any population with mean μ and standard deviation σ.

- The mean (or) center of the sampling distribution of $\bar{X}$ is equal to the population mean

$$\mu\colon \mu_{\bar{X}} - \mu$$

- The standard deviation of the sampling distribution is $\frac{\sigma}{\sqrt{n}}$. where n is the sample size,

$$\sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}}$$

- Mean of a sampling distribution of $\bar{x}$ no tendency for a sample mean to fall systematically above (or) below μ even if the distribution of the raw data is skewed

Thus, the mean of the sampling distribution is an unbiased estimate of the population mean μ It will be "correct on average" in many samples

- SD of a sampling distribution measures how much the sample statistic varies from sample to sample It is smaller than the standard deviation of the population by a factor of √n
- Averages are less variables than individuals observations. For normally distributed populations. When a variable in populations is normally distributed the sampling distribution of $\bar{x}$ for all possible samples of size n is also normally. If the population is $N(\mu, \sigma)$ then the sample means distribution is $N(\mu, \sigma/\sqrt{n})$

Student's 't' distribution:

Let $X_i(i = 1,2,3, \ldots n)$ be a random sample of size n from a normal population with mean μ variance σ^2.Then Student's't' is defined by the statistic

$$t = \frac{\bar{x} - \mu}{s/\sqrt{n}}$$

Where, $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_{i,}$is the sample mean and $s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$ is an unbiased estimate of the population variance σ^2, and if follows Student's 't' distribution with v=(n-1) d.f with p.d.f

$$f(t) = \frac{1}{\sqrt{v}\beta\left(\frac{1}{2}, \frac{v}{2}\right)} \cdot \frac{1}{\left(1 + \frac{t^2}{v}\right)\frac{(v+1)}{2}} : -\infty < t < \infty.$$

Derivation of Student's 't' distribution:

$$t = \frac{\bar{x} - \mu}{s/\sqrt{n}}$$

$$t^2 = \frac{(x^2 - \mu)^2}{s^2/n} = \frac{(x - \mu)^2/\sigma^2}{\frac{ns^2}{(n-1)n}/\sigma^2} \qquad \therefore ns^2 = (n-1)s^2 \to s^2 = \frac{ns^2}{n-1}$$

$$\frac{t^2}{n-1} = \frac{(x - \mu)^2/\sigma n}{ns^2/\sigma^2} = \frac{\left[(x - \mu)/\sigma\sqrt{n}\right]^2}{ns^2/\sigma^2}$$

$$f(x) = \frac{1}{\beta(m,n)} \cdot \frac{x^{m-1}}{(1+x)^{m+n}} : 0 \le x \le \infty = \frac{t^2}{n-1} \sim \beta\left(\frac{1}{2}, \frac{n-1}{2}\right)$$

$$f\left(\frac{t^2}{n-1}\right)=\frac{1}{\beta\left(\frac{1}{2},\frac{n-1}{2}\right)}.\frac{\left(\frac{t^2}{n-1}\right)^{\frac{1}{2}-1}}{\left(1+\frac{t^2}{n-1}\right)^{\frac{1}{2}+\frac{n+1}{2}}}:0\le\frac{t^2}{n-1}<\infty$$

Fisher’s t distribution:

It is the ratio of a standard normal variate to the square root of an Independent χ^2-variate divided by its degrees of freedom. If $\sum_j$ is a N$(0,1)$ and χ^2 is an Independent χ^2 variate with n d.f, then Fishers t's given by

$$t=\frac{\varepsilon}{\sqrt{\chi^2/n}}$$

And it follows students t distribution with n degrees of freedom.

Moments:

$$t=\frac{\varepsilon}{\sqrt{\chi^2/n}}\sim t_n$$

$$\varepsilon\sim(0,1)$$

$$f(x)=\frac{1}{2\pi}e^{-\varepsilon^2/2}:-\infty<\varepsilon<\infty$$

$$p(\chi^2)=\frac{1}{2^{\frac{n}{2}}\sqrt{\frac{n}{2}}}e^{-\frac{\chi^2}{2}}x^{\frac{n}{2}-1}:0<\chi^2<\infty$$

By Jacobian transformation,

$$f(x)=\begin{vmatrix}\frac{\partial\varepsilon}{\theta t} & \frac{\partial\varepsilon}{\partial u}\\ \frac{\partial\chi^2}{\partial t} & \frac{\partial\chi^2}{\partial u}\end{vmatrix}=\begin{vmatrix}\frac{\sqrt{u}}{n} & \frac{t}{\sqrt{n}}.\frac{1}{2\sqrt{u}}\\ 0 & 1\end{vmatrix}=\sqrt{\frac{u}{n}}\qquad\therefore t=\frac{\varepsilon}{\sqrt{\frac{\chi^2}{n}}}:u=\chi^2$$

Joint p.d.f t & u g(t,u)

$$f_u(t,u)=f_{\varepsilon\chi^2}(\varepsilon\chi^2)|J|=f_\varepsilon(\varepsilon)f_{\chi^2}(\chi^2)|J|=f_\varepsilon\left(t.\sqrt{\frac{u}{n}}\right)f_{\chi^2}(u)\left(\sqrt{\frac{u}{n}}\right)$$

$$=\frac{1}{\sqrt{2\pi}}e^{-t^2u/2n}\frac{1}{2^{\frac{n}{2}}\sqrt{\frac{n}{2}}}e^{-u/2}u^{\frac{n}{2}-1}\sqrt{\frac{n}{n}}$$

$$f(t,u)=\frac{1}{\sqrt{2\pi}\,2^{\frac{n}{2}}\sqrt{\frac{n}{2}}}e^{-\frac{u}{2}(1+\frac{t^2}{n})}u^{\frac{n+1}{2}-1}$$

$$f(t) = \int f(t,u)\,du = \int_0^{\infty} \frac{1}{\sqrt{2\pi}\, 2^{\frac{n}{2}} \sqrt{\frac{n}{2}}\sqrt{n}} e^{-\frac{u}{2}(1+\frac{t^2}{n})} u^{\frac{n+1}{2}-1} du$$

$$= \left[\int_0^{\infty} e^{-\frac{1}{2}(1+\frac{t^2}{n})u} u^{\frac{n+1}{2}-1} du\right] \frac{1}{\sqrt{2\pi}\, 2^{\frac{n}{2}} \sqrt{\frac{n}{2}}\sqrt{n}}$$

w.k.t $\int_0^{\infty} e^{-ax} x^{\lambda-1} dx = \sqrt{\frac{\lambda}{a^{\lambda}}}$

$$f(t) = \frac{1}{\sqrt{2\pi}\, 2^{\frac{n}{2}} \sqrt{\frac{n}{2}}\sqrt{n}} \left[\frac{\sqrt{\frac{n+1}{2}}}{\left(\frac{1}{2}(1+\frac{t^2}{n})\right)^{\frac{n+1}{2}}}\right] = \frac{1}{\sqrt{1/2}\, \sqrt{\frac{n}{2}}\sqrt{n}} \left[\frac{\sqrt{\frac{n+1}{2}}}{\left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}}}\right]$$

$$= \frac{\frac{1}{\sqrt{\frac{1}{2}}\sqrt{\frac{n}{2}}}}{\sqrt{\frac{n}{2}+\frac{1}{2}}\sqrt{n}} \frac{1}{\left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}}}$$

$$f(t) = \frac{1}{\sqrt{n}\,\beta\left(\frac{1}{2},\frac{n}{2}\right)} \frac{1}{\left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}}} : -\infty < t < \infty$$

Applications t-distribution

- To test if the sample mean ($\bar{x}$) differs significantly from the hypothetical value μ of the population mean.
- To test the significance of the difference between two sample means.
- To test the significance of an observed sample correlation coefficient and sample regression coefficient.
- To test the significance of an observed partial and multiple correlation coefficients.

Central F distribution

If X and Y are two independent chi square variate with $v_1 and\ v_2$ d.f respectively then F statistic is defined by

$$F = \frac{(X/v_1)}{(Y/v_2)}$$

In other words, F is defined as the ratio of two independent chi - square variate divided by their respective degrees of freedom and it follows Sneducer's F distribution with (v_1, v_2) d.f with p.f given by

$$f(F)=\frac{\left(\frac{v_1}{v_2}\right)^{\frac{v_1}{2}}}{\beta\left(\frac{v_1}{2},\frac{v_2}{2}\right)}.\frac{(F)^{\frac{v_1}{2}-1}}{\left(1+\frac{v_1}{v_2}F\right)^{\frac{v_1+v_2}{2}}}\ ;0\le F<\infty$$

Derivation

Since X,Y are independent chi-square variate with $v_1 and\ v_2\ d.f$, their joint p.d.f is given by

$$f(x,y)=\left[\frac{1}{2^{\frac{v_1}{2}}\Gamma\left(\frac{v_1}{2}\right)}e^{(-x/2)}.x^{\frac{v_1}{2}-1}\right]\left[\frac{1}{2^{\frac{v_2}{2}}\Gamma\left(\frac{v_2}{2}\right)}e^{(-y/2)}.y^{\frac{v_2}{2}-1}\right]$$

$$=\left[\frac{1}{2^{\frac{v_1+v_2}{2}}\Gamma\left(\frac{v_1}{2}\right)\left(\frac{v_2}{2}\right)}e^{[-\frac{x+y}{2}]}.x^{\frac{v_1}{2}-1}y^{\frac{v_2}{2}-1}\right]$$

$F=\frac{(x/v_1)}{(\mathrm{y}/v_2)}\ ,u=y$

$$=\frac{(x/v_1)}{(\mathrm{u}/v_2)}$$

$$Fu=\frac{(x/v_1)}{(1/v_2)}$$

$$\frac{Fu}{v_2}=\frac{x}{v_1}$$

$$x=\frac{v_1}{v_2}Fu$$

Joccoban transformation.

$$J=\frac{\partial(x,y)}{\partial(F,u)}=\begin{vmatrix}\frac{\partial x}{\partial F} & \frac{\partial x}{\partial u}\\ \frac{\partial y}{\partial F} & \frac{\partial y}{\partial u}\end{vmatrix}$$

$$=\begin{vmatrix}\frac{v_1}{v_2}u & \frac{v_1}{v_2}F\\ 0 & 1\end{vmatrix}$$

$$J=\frac{v_1}{v_2}u$$

Joint p.d.f

$$f(F,u)=\left[\frac{1}{2^{\frac{v_1+v_2}{2}}\Gamma\left(\frac{v_1}{2}\right)\left(\frac{v_2}{2}\right)}e^{-\frac{\left(\frac{v_1}{v_2}Fu+u\right)}{2}}.\left(\frac{v_1}{v_2}Fu\right)^{\frac{v_1}{2}-1}u^{\frac{v_2}{2}-1}\right]$$

$$=\left[\frac{1}{2^{\frac{v_1+v_2}{2}}\Gamma\left(\frac{v_1}{2}\right)\left(\frac{v_2}{2}\right)}e^{\left\{-\frac{u}{2}\left(1+\frac{v_1}{v_2}F\right)\right\}}.\left(\frac{v_1}{v_2}Fu\right)^{\frac{v_1}{2}-1}u^{\frac{v_2}{2}-1}\right]$$

$$= \left[\frac{1}{2^{\frac{v_1+v_2}{2}} \Gamma\left(\frac{v_1}{2}\right)\left(\frac{v_2}{2}\right)} e^{\left\{-\frac{u}{2}\left(1+\frac{v_1}{v_2}F\right)\right\}} . \frac{v_1}{v_2} u \, (F)^{\frac{v_1}{2}-1} u^{\frac{v_1}{2}+\frac{v_2}{2}-1} \right]$$

$$= \left[\frac{\left(\frac{v_1}{v_2}\right)^{\frac{v_1}{2}} (F)^{\frac{v_1}{2}-1}}{2^{\frac{v_1+v_2}{2}} \Gamma\left(\frac{v_1}{2}\right)\left(\frac{v_2}{2}\right)} e^{\left\{-\frac{u}{2}\left(1+\frac{v_1}{v_2}F\right)\right\}} . u^{\frac{v_1}{2}+\frac{v_2}{2}-1} \right]$$

Integration of w.r.t (u)

$$f(F) = \left[\frac{\left(\frac{v_1}{v_2}\right)^{\frac{v_1}{2}} (F)^{\frac{v_1}{2}-1}}{2^{\frac{v_1+v_2}{2}} \Gamma\left(\frac{v_1}{2}\right) \Gamma\left(\frac{v_2}{2}\right)} \int_0^\infty e^{\left\{-\frac{u}{2}\left(1+\frac{v_1}{v_2}F\right)\right\}} . u^{\frac{v_1}{2}+\frac{v_2}{2}-1} du \right]$$

$$= \frac{\left(\frac{v_1}{v_2}\right)^{\frac{v_1}{2}} (F)^{\frac{v_1}{2}-1}}{2^{\frac{v_1+v_2}{2}} \Gamma\left(\frac{v_1}{2}\right) \Gamma\left(\frac{v_2}{2}\right)} \frac{\Gamma\left(\frac{v_{1+v_2}}{2}\right)}{\left[\frac{1}{2}\left(1+\frac{v_1}{v_2}F\right)\right]^{\frac{v_{1+v_2}}{2}}} = \frac{\left(\frac{v_1}{v_2}\right)^{\frac{v_1}{2}} (F)^{\frac{v_1}{2}-1}}{2^{\frac{v_1+v_2}{2}} \Gamma\left(\frac{v_1}{2}\right) \Gamma\left(\frac{v_2}{2}\right)} \frac{\Gamma\left(\frac{v_{1+v_2}}{2}\right) 2^{\frac{v_{1+v_2}}{2}}}{\left(1+\frac{v_1}{v_2}F\right)^{\frac{v_{1+v_2}}{2}}}$$

$$= \frac{\left(\frac{v_1}{v_2}\right)^{\frac{v_1}{2}} (F)^{\frac{v_1}{2}-1}}{\Gamma\left(\frac{v_1}{2}\right) \Gamma\left(\frac{v_2}{2}\right)} \frac{\Gamma\left(\frac{v_{1+v_2}}{2}\right)}{\left(1+\frac{v_1}{v_2}F\right)^{\frac{v_{1+v_2}}{2}}}$$

$$f(F) = \frac{\left(\frac{v_1}{v_2}\right)^{\frac{v_1}{2}} (F)^{\frac{v_1}{2}-1}}{\beta\left(\frac{v_1}{2}, \frac{v_2}{2}\right)} \frac{1}{\left(1+\frac{v_1}{v_2}F\right)^{\frac{v_{1+v_2}}{2}}} ; 0 \le F < \infty$$

Constant's of F distribution

$$\mu_r' = E(F^r) = \int_0^\infty F^r f(F) dF = \int_0^\infty F^r \frac{\left(\frac{v_1}{v_2}\right)^{\frac{v_1}{2}}}{\beta\left(\frac{v_1}{2}, \frac{v_2}{2}\right)} \frac{(F)^{\frac{v_1}{2}-1}}{\left(1+\frac{v_1}{v_2}F\right)^{\frac{v_{1+v_2}}{2}}} dF$$

$$\mu_r' = \frac{\left(\frac{v_1}{v_2}\right)^{\frac{v_1}{2}}}{\beta\left(\frac{v_1}{2}, \frac{v_2}{2}\right)} \int_0^\infty \frac{(F)^{\frac{v_1}{2}+r-1}}{\left(1+\frac{v_1}{v_2}F\right)^{\frac{v_{1+v_2}}{2}}} dF$$

Let $\frac{v_1}{v_2} F = y \rightarrow F = \frac{v_2}{v_1} y : dF = \frac{v_2}{v_1} dy$

$$\mu_r' = \frac{\left(\frac{v_1}{v_2}\right)^{\frac{v_1}{2}}}{\beta\left(\frac{v_1}{2}, \frac{v_2}{2}\right)} \int_0^\infty \frac{(\frac{v_2}{v_1} y)^{\frac{v_1}{2}+r-1}}{(1+y)^{\frac{v_{1+v_2}}{2}}} \frac{v_2}{v_1} dy = \frac{\left(\frac{v_1}{v_2}\right)^{\frac{v_1}{2}} (\frac{v_2}{v_1})^{\frac{v_1}{2}+r-1}}{\beta\left(\frac{v_1}{2}, \frac{v_2}{2}\right)} \int_0^\infty \frac{(y)^{v_1+r-1}}{(1+y)^{\frac{v_{1+v_2+r-r}}{2}}} dy$$

$$= \frac{\left(\frac{v_1}{v_2}\right)^{\frac{v_1}{2}} (\frac{v_2}{v_1})^{\frac{v_1}{2}+r-1+1}}{\beta\left(\frac{v_1}{2},\frac{v_2}{2}\right)} \int_0^\infty \frac{(y)^{v_1/v_2+r-1}}{(1+y)^{\frac{v_1}{2}+r+\frac{v_2}{2}-r}} dy = \frac{(\frac{v_2}{v_1})^r}{\beta\left(\frac{v_1}{2},\frac{v_2}{2}\right)} \frac{\Gamma\frac{v_1}{2}+r\Gamma\frac{v_2}{2}-r}{\Gamma\frac{v_1}{2}+r+\frac{v_2}{2}-r}$$

$$\mu_r' = (\frac{v_2}{v_1})^r \frac{\Gamma\frac{v_1}{2}+r\Gamma\frac{v_2}{2}-r\Gamma\frac{v_1}{2}+\frac{v_2}{2}}{\Gamma\left(\frac{v_1}{2}+\frac{v_2}{2}\right)+\Gamma\frac{v_1}{2}\Gamma\frac{v_2}{2}} = (\frac{v_2}{v_1})^r \frac{\Gamma\frac{v_1}{2}+r\ \Gamma\frac{v_2}{2}-r}{\Gamma\frac{v_1}{2}\Gamma\frac{v_2}{2}}$$

Mean and variance of F distribution

$$\mu_r' = (\frac{v_2}{v_1})^r \frac{\Gamma\frac{v_1}{2}+r\ \Gamma\frac{v_2}{2}-r}{\Gamma\frac{v_1}{2}\Gamma\frac{v_2}{2}}$$

If r=1

$$\mu_1' = (\frac{v_2}{v_1})^1 \frac{\Gamma\frac{v_1}{2}+1\ \Gamma\frac{v_2}{2}-1}{\Gamma\frac{v_1}{2}\Gamma\frac{v_2}{2}} = \frac{v_2}{v_1} \frac{\frac{v_1}{2}\Gamma\frac{v_1}{2}\ \Gamma\frac{v_2}{2}-1}{\Gamma\frac{v_1}{2}\Gamma\frac{v_2}{2}+1-1} = \frac{\frac{v_2}{v_1}\frac{v_1}{2}\Gamma\left(\frac{v_1}{2}\ \left[\frac{v_2}{2}-1\right]\right)}{\Gamma\frac{v_1}{2}\Gamma\frac{v_2}{2}+1\Gamma\frac{v_2}{2}-1}$$

$$= \frac{v_2}{2}\left(\frac{1}{\frac{v_2-2}{2}}\right) = \frac{v_2}{v_2-2} : v_2 > 2$$

$$mean = \mu_1' = \frac{v_2}{v_2-1}$$

If r=2

$$\mu_2' = (\frac{v_2}{v_1})^2 \frac{\Gamma\frac{v_1}{2}+2\ \Gamma\frac{v_2}{2}-2}{\Gamma\frac{v_1}{2}\Gamma\frac{v_2}{2}} = (\frac{v_2}{v_1})^2 \frac{\left(\frac{v_1}{v_2}+1\right)\frac{v_1}{2}\Gamma\frac{v_1}{2}\ \Gamma\frac{v_2}{2}-2}{\Gamma\frac{v_1}{2}\left(\frac{v_2}{2}-1\right)\left(\frac{v_2}{2}-2\right)\Gamma\frac{v_2}{2}-2}$$

$$= \frac{(\frac{v_2}{v_1})^2\left(\frac{v_1+2}{2}\right)\left(\frac{v_1}{2}\right)}{\left(\frac{v_2-2}{2}\right)\left(\frac{v_2-4}{2}\right)} = \frac{(v_2)^2(v_1+2)}{v_1(v_2-2)(v_2-4)}$$

$$\mu_2 = \mu_2' - (\mu_1')^2$$

$$\mu_2 = \frac{(v_2)^2(v_1+2)}{v_1(v_2-2)(v_2-4)} - \left(\frac{v_2}{v_2-1}\right)^2 = \frac{(v_2)^2}{(v_2-2)}\left[\frac{(v_1+2)}{v_1(v_2-4)} - \frac{1}{(v_2-2)}\right]$$

$$= \frac{(v_2)^2}{(v_2-2)}\left[\frac{(v_1+2)(v_2-2)-v_1(v_2-4)}{v_1(v_2-4)(v_2-2)}\right]$$

$$= \frac{(v_2)^2}{(v_2-2)}\left[\frac{v_1(v_2-2)+2(v_2-2)-v_1(v_2-2)+2v_1}{v_1(v_2-4)(v_2-2)}\right]$$

$$\mu_2 = \frac{(2v_2)^2(v_1+v_2-2)}{v_1(v_2-2)^2(v_2-4)}: (v_2>4)$$

Mode of F distribution:

$$f(F) = \frac{\left(\frac{v_1}{v_2}\right)^{\frac{v_1}{2}}(F)^{\frac{v_1}{2}-1}}{\beta\left(\frac{v_1}{2},\frac{v_2}{2}\right)}\frac{1}{\left(1+\frac{v_1}{v_2}F\right)^{\frac{v_1+v_2}{2}}}; 0 \le F < \infty$$

$$logf(F) = log\left[\frac{\left(\frac{v_1}{v_2}\right)^{\frac{v_1}{2}}}{\beta\left(\frac{v_1}{2},\frac{v_2}{2}\right)}\right] + log\left[(F)^{\frac{v_1}{2}-1}\right] - log\left[\left(1+\frac{v_1}{v_2}F\right)^{\frac{v_1+v_2}{2}}\right]$$

$$logf(F) = C + \left(\frac{v_1}{v_2}-1\right)logF - \frac{v_1+v_2}{2}log\left(1+\frac{v_1}{v_2}F\right)$$

Diff. both sides w.r.t F

$$\frac{1}{f(F)}f'(F) = C + \left(\frac{v_1}{2}-1\right)(1/F) - \frac{v_1+v_2}{2}\left(\frac{1}{1+\frac{v_1}{v_2}F}\right)\frac{v_1}{v_2}$$

$$\frac{f'(F)}{f(F)} = \left(\frac{v_1-2}{2}\right)(1/F) - \frac{v_1+v_2}{2}\left(\frac{1}{1+\frac{v_1}{v_2}F}\right)\frac{v_1}{v_2}$$

$$f'(F) = f(F)\left[\left(\frac{v_1-2}{2}\right)(1/F) - \frac{{v_1}^2+v_2v_1}{2(v_2+v_1F)}\right]$$

$$f'(F) = 0$$

$$\left(\frac{v_1-2}{2}\right)(1/F) - \frac{{v_1}^2+v_2v_1}{2(v_2+v_1F)} = 0$$

$$\left(\frac{v_1-2}{2}\right)(1/F) = \frac{{v_1}^2+v_2v_1}{2(v_2+v_1F)}$$

$$\left(\frac{v_2+v_1F}{F}\right) = \frac{{v_1}^2+v_2v_1}{(v_1-2)} \rightarrow \left(\frac{v_2}{F}\right) = \frac{{v_1}^2+v_2v_1}{(v_1-2)} - v_1$$

$$\left(\frac{v_2}{F}\right) = \frac{{v_1}^2 + v_2 v_1 - {v_1}^2 + 2v_1}{(v_1 - 2)}$$

$$\frac{v_2(v_1 - 2)}{v_1(v_2 + 2)} = F = mode\ now f'(F) < 0$$

Chi square distribution:

The square of a standard normal variate is known as a chi square variate (pronounced as ki-sky without s) with 1 d.f.

Thus, if $X \sim N(\mu, \sigma^2)$ then $z = \frac{X-\mu}{\sigma} \sim N(0,1)$,and $z^2 = \left(\frac{X-\mu}{\sigma}\right)^2$ is a chi square variate with 1 d.f

In general if $X_i (i = 1,2, \dots n)$ are n independent normal variate with means μ and variance $\sigma_i^2, (i = 1,2, \dots n)$then,

$\chi^2 = \sum_{i=1}^{n} \left(\frac{X_i - \mu_i}{\sigma_i}\right)^2$ is a chi square variate of n d.f

Derivation of chi square distribution:

$$\chi^2 = \sum_{i=1}^{n} \left(\frac{X_i - \mu_i}{\sigma_i}\right)^2 = \sum_{i=1}^{n} (z_i)^2$$

$$\mu_z^2(t) = E\left(e^{tz^2}\right) = \int_{-\infty}^{\infty} e^{tz^2} \varphi(z) dz = \int_{-\infty}^{\infty} e^{tz^2} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\, dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tz^2} e^{-z^2/2}\, dz = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}(1-2t)}\, dz$$

$$= \frac{1}{\sqrt{2\pi}} 2 \int_{0}^{\infty} e^{-\frac{z^2}{2}(1-2t)}\, dz = \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} e^{-u(1-2t)} \frac{du}{\sqrt{2u}}$$

$$= \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} e^{-u(1-2t)}\, u^{-\frac{1}{2}}\, du = \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} e^{-u(1-2t)}\, u^{\frac{1}{2}-1}\, du = \frac{1}{\sqrt{\pi}} \frac{\sqrt{\frac{1}{2}}}{(1-2t)^{\frac{1}{2}}}$$

$$M_{\chi^2}(t) = (1-2t)^{-\frac{1}{2}}$$

$$M_{\chi^2}(t) = M_{u_1^2}(t) M_{u_2^2}(t) \dots \dots M_{u_n^2}(t) = (1-2t)^{=\frac{1}{2}} (1-2t)^{=\frac{1}{2}} \dots \dots (1-2t)^{=\frac{1}{2}}$$

$$M_{\chi^2}(t) = (1-2t)^{-\frac{n}{2}} = \left(1 - \frac{t}{\frac{1}{2}}\right)^{-\frac{n}{2}}$$

$$a = \frac{1}{2} : \lambda = \frac{n}{2}$$

$$f(\chi^2) = f(X) = \frac{\left(\frac{1}{2}\right)^{n/2}}{\sqrt{n/2}} e^{-\chi^2/2} (\chi^2)^{\frac{n}{2}-1} : 0 < \chi^2 < \infty$$

$$f(\chi^2) = \frac{1}{2^{n/2}\sqrt{n/2}} e^{-\chi^2/2} (\chi^2)^{\frac{n}{2}-1} : 0 \le \chi^2 < \infty$$

M.G.F of χ^2-distribution:

$$f(x) = \frac{1}{2^{n/2}\sqrt{n/2}} e^{-x^2/2} (x)^{\frac{n}{2}-1}$$

$$M_X(t) = E(e^{tx}) = \int e^{tx} f(x) dx = \int_0^\infty e^{tx} \frac{1}{2^{\frac{n}{2}}\sqrt{\frac{n}{2}}} e^{-\frac{x^2}{2}} (x)^{\frac{n}{2}-1} dx$$

$$= \frac{1}{2^{\frac{n}{2}}\sqrt{\frac{n}{2}}} \int_0^\infty e^{tx} e^{-\frac{x^2}{2}} (x)^{\frac{n}{2}-1} dx = \frac{1}{2^{\frac{n}{2}}\sqrt{\frac{n}{2}}} \int_0^\infty e^{-x(\frac{1}{2}-t)} (x)^{\frac{n}{2}-1} dx = \frac{1}{2^{\frac{n}{2}}\sqrt{\frac{n}{2}}} \frac{\sqrt{n/2}}{\left(\frac{1}{2}-t\right)^{\frac{n}{2}}}$$

$$= \frac{1}{2^{\frac{n}{2}}\sqrt{\frac{n}{2}}} \frac{\sqrt{n/2}}{\left(\frac{1-2t}{2}\right)^{\frac{n}{2}}} = \frac{1}{2^{\frac{n}{2}}} \frac{2^{\frac{n}{2}}}{(1-2t)^{\frac{n}{2}}} = \frac{1}{(1-2t)^{\frac{n}{2}}} = (1-2t)^{-\frac{n}{2}}$$

Characteristic function of χ^2-distribution

$$f(x) = \frac{1}{2^{n/2}\sqrt{n/2}} e^{-x^2/2} (x)^{\frac{n}{2}-1}$$

$$\varphi_X(t) = E\left(e^{itx}\right) = \int e^{itx} f(x) dx = \int_0^\infty e^{itx} \frac{1}{2^{\frac{n}{2}}\sqrt{\frac{n}{2}}} e^{-\frac{x^2}{2}} (x)^{\frac{n}{2}-1} dx$$

$$= \frac{1}{2^{\frac{n}{2}}\sqrt{\frac{n}{2}}} \int_0^\infty e^{itx} e^{-\frac{x^2}{2}} (x)^{\frac{n}{2}-1} dx = \frac{1}{2^{\frac{n}{2}}\sqrt{\frac{n}{2}}} \int_0^\infty e^{-x(\frac{1}{2}-t)i} (x)^{\frac{n}{2}-1} dx = \frac{1}{2^{\frac{n}{2}}\sqrt{\frac{n}{2}}} \frac{\sqrt{n/2}}{\left(\frac{1}{2}-it\right)^{\frac{n}{2}}}$$

$$\varphi_X(t) = \frac{1}{2^{\frac{n}{2}}\sqrt{\frac{n}{2}}} \frac{\sqrt{n/2}}{\left(\frac{1-2it}{2}\right)^{\frac{n}{2}}} = \frac{1}{2^{\frac{n}{2}}} \frac{2^{\frac{n}{2}}}{(1-2it)^{\frac{n}{2}}} = \frac{1}{(1-2it)^{\frac{n}{2}}} = (1-2it)^{-\frac{n}{2}}$$

C.G.F of χ^2-distribution:

$$k_X(t) = logM_X(t) = log\left[(1-2t)^{-\frac{n}{2}}\right] = -\frac{n}{2}\log(1-2t)$$
$$= \frac{n}{2}\left[2t + \frac{2t^2}{2!} + \frac{2t^3}{3!} \dots..\right]$$
$$k_1 = co\ effeicient\ of\ t\ in\ k(t) = \frac{n}{2} \times 2 = n$$
$$k_2 = co\ effeicient\ of\ \frac{t^2}{2!}\ in\ k(t) = \frac{n}{2!} \times 4 = 2n$$
$$k_3 = co\ effeicient\ of\ \frac{t^3}{3!}\ in\ k(t) = 8n$$
$$k_4 = co\ effeicient\ of\ \frac{t^4}{4!}\ in\ k(t) = 48n$$

In general

$$k_r = co\ effeicient\ of\ \frac{t^n}{n!}\ in\ k(t) = n2^{r-1}(r-1)!$$
$$\text{Mean} = \mu_1 = k_r = n$$
$$variance = \mu_2 = k_2 = 2n$$
$$\mu_3 = k_3 = 8n$$
$$\mu_4 = k_4 + 3{k_2}^2 = 48n + 12n^2$$

Limiting form of χ^2-distribution:

$If\ X \sim {\chi^2}_n$

$$M_X(t) = (1-2t)^{-\frac{n}{2}}$$

M.G.F of standard χ^2 variate z,

$$M_z(t) = M_{\frac{X-\mu}{\sigma}}(t) = M_{X-\mu}(t/\sigma) = M_X(t/\sigma)M(\mu t/\sigma)$$
$$= M_X(t/\sigma)E\left(e^{-\mu t/\sigma}\right) = M_X(t/\sigma)e^{-\mu t/\sigma} = e^{-\mu t}(1-2t/\sigma)^{-n/2}$$
$$= e^{-nt/\sqrt{2n}}(1-2t/\sqrt{2n})^{-n/2}$$

Also

$$k_2(t) = \log M_2(t) = log\left[e^{-nt/\sqrt{2n}}(1-2t/\sqrt{2n})^{-n/2}\right]$$
$$= log\left[e^{-nt/\sqrt{2n}}\right] + \log(1-2t/\sqrt{2n})^{-n/2} = -t\ \left[\frac{n}{2} - \frac{n}{2}\ log(1-2t/\sqrt{2/n})\right]$$
$$= -t\sqrt{\frac{n}{2}} + t\sqrt{\frac{n}{2}} + \frac{t^2}{2} + 0\left(\frac{n-1}{2}\right) = \frac{t^2}{2} + 0 = \frac{t^2}{2}$$
$$\lim_{n\to\infty} k_2(t) = \frac{t^2}{2}$$

$$M_2(t) = e^{k_2/2} \; as \; n \to \infty$$

Mode of χ^2-distribution:

$$f(\chi^2) = \frac{1}{2^{n/2}\sqrt{n/2}} e^{-\chi^2/2} (\chi^2)^{\frac{n}{2}-1} : 0 \le \chi^2 < \infty$$

$$F'^{(\chi^2)} = \frac{1}{2^{n/2}\sqrt{n/2}} \left[e^{-\frac{\chi^2}{2}} \left(-\frac{1}{2}\right) (\chi^2)^{\frac{n}{2}-1} + e^{-\frac{\chi^2}{2}} (\frac{n}{2} - 1)(\chi^2)^{\frac{n}{2}-2} \right]$$

$$= \frac{1}{2^{n/2}\sqrt{n/2}} e^{-\frac{\chi^2}{2}} (\chi^2)^{\frac{n}{2}-1} \left[-\frac{1}{2} + (\frac{n}{2} - 1)(\chi^2)^{-1} \right]$$

$$F'^{(\chi^2)} = 0 \to \frac{1}{2^{n/2}\sqrt{n/2}} e^{-\frac{\chi^2}{2}} (\chi^2)^{\frac{n}{2}-1} \left[-\frac{1}{2} + (\frac{n}{2} - 1)(\chi^2)^{-1} \right] = 0$$

$$\left[-\frac{1}{2} + (\frac{n}{2} - 1)(\chi^2)^{-1} \right] = 0 \to \frac{1}{2} - \left(\frac{n}{2} - 1\right)\left(\frac{1}{\chi^2}\right) \to \frac{1}{2} = \left(\frac{n-2}{2}\right)\left(\frac{1}{\chi^2}\right) \to \chi^2 = n - 2$$

$$F''^{(\chi^2)} = 0 \to \chi^2 = n - 2 \; is \; the \; mode.$$

Additive property of χ^2 variate

The sum of independent χ^2 variate is also a χ^2 variate more precisely, if $X_i(i = 1,2, \dots k)$ are independent χ^2 variates with n d.f respectively. The sum of $\sum_{i=1}^{k} X_i$ is also a χ^2 variate with $\sum_{i=1}^{k} n_i \; d.f$

Proof

We have $M_{X_i}(t) = (1-2t)^{-\frac{n_i}{2}} : (i = 1,2, \dots . k)$. the m.g.f of the sum $\sum_{i=1}^{k} X_i$ is given by,

$$M_{\sum X_i}(t) = M_{X_1}(t) . M_{X_2}(t) \dots . . M_{X_k}(t) \; \{\therefore \; X_i \; is \; are \; independent\}$$

$$= (1-2t)^{-\frac{n_1}{2}} (1-2t)^{-\frac{n_2}{2}} \dots \dots (1-2t)^{-\frac{n_k}{2}}$$

$$= (1-2t)^{-\frac{n_1+n_2 \dots n_k.}{2}} X_i/2$$

Which is the m.g.f of a χ^2variate with $n_1 + n_2 \dots n_k \; d.f$. Hence by uniqueness theprem of m.g.f ‘s $\sum_{i=1}^{k} X_i$ is a χ^2variate with $\sum_{i=1}^{k} n_i \; d.f$.