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Unit-V

Non-Central Distribution

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Non central t-Distribution

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-\mu)^2}$$

$$g(y) = \frac{e^{-y/2} y^{(n/2)-1}}{2^{n/2}\Gamma(n/2)}$$

$$t = \frac{x}{\sqrt{y/n}},\ x = \sqrt{u}\ (Y = u)$$

$$J = \begin{vmatrix} \sqrt{u/n} & \frac{1}{2\sqrt{u/n}} \\ 0 & 1 \end{vmatrix} = \sqrt{u/n}$$

$$f_{t,u}(t,u) = f_{x,y}(x,y)|J| = f_x(x) f_y(y)|J|$$

$$= \frac{1}{\sqrt{2\pi}} e^{-ut^2/2n} \frac{u^{n/2-1} e^{-u/2}}{2^{n/2}\Gamma(n/2)} \frac{\sqrt{u}}{\sqrt{n}}$$

$$= \frac{1}{\sqrt{n\pi}} \frac{1}{\Gamma(n/2) 2^{(n+1)/2}} u^{n/2-1/2} e^{-u/2}[1 + t^2/n]$$

$$f_T(t) = \int_0^\infty f_{T,U}(t,u) du$$

$$\frac{2\Gamma(n+1)}{\Gamma(\frac{n+1}{2})\sqrt{n\pi}\,\Gamma(\frac{n}{2})} \int_0^\infty u^{\frac{n+1}{2}-1} e^{-\frac{u}{2}(1+\frac{t^2}{n})} du$$

$$= \frac{1}{\sqrt{n\pi}} \frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2})} [\frac{\Gamma(\frac{n+1}{2})}{(\frac{1}{2}(1+\frac{t^2}{n}))^{\frac{n+1}{2}}}]^{-1}$$

$$= \frac{1}{\sqrt{n\pi}} \frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2})} \sqrt{\frac{n+1}{2}} [2(1+\frac{t^2}{n})^{-1}]$$

Joint P.D.F of K order Statistics

$1 \le r_1 \le r_2 \le r_3 ... \le r_K \le n; 1 \le K \le n$

$$x_1 \le x_2 \le x_3 \le ... \le x_n$$

comparing 2^{nd} order statistic and K-order

$F_{r_1,r_2} ... r_K(x_1, x_2, ...x_n)$:

$$F_{r,s}(x, y)$$

$$F_{r_1,r_2,r_3}(x_1, x_2, x_3)$$

$$= n! \frac{f(x_{r_1} F(x_{r_2}) - F(x_{r_1}))^{r_2-r_1-1}}{(r_1 - 1)!(r_2 - r_1 - 1)!(r_3 - r_2 - r_1)!} ....(n - r)!$$

$$x f(x_{r_1})[F(x_{r_2}) - F(x_{r_1})]^{r_2-r_1-1}$$

$$x f(x_K)[1 - F(x_K)]^{n-K+r}$$

Joint Density function of n order statistics

Joint p.d.f

$$X_{(1)}, X_{(2)}, ...X_{(n)}$$

$$f(x_1, x_2, \dots, x_n) = n! f(x_1) f(x_2) \dots f(x_n)$$

we know that

$$\lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x} = F'(x) = f(x)$$

$$\frac{d}{dx}(F(x)) = f(x)$$

$$f'(y) = f(y)$$

$$F_{r,s}(x, y) = n! f^{r-1}(x) f(x) f(y) - f(x + \delta x)^{s-r-1} \dots$$

$$f(y)[1 + f(y) + \delta\delta y]^{n-s}$$

P.d.f of two order statistics:
Joint P.d.f of $X_{(r)}$ & $X_{(s)}$ is given by $F_{r,s}(x, y)$
where $1 \leq r \leq s \leq n$

definition of joint p.d.f

$$F_{r,s}(x, y) = \lim_{\delta x \to 0, \delta y \to 0} \frac{P(x \leq X_{(r)} \leq x + \delta x, y \leq X_{(s)} \leq y + \delta y)}{\delta x \delta y}$$

$$E = (x \leq X_{(r)} \leq x + \delta x, y \leq X_{(s)} \leq y + \delta y)$$

$P_1 = P(X_i \leq x) = F(x) \ldots$ definition $F(x)$

$$P_2 = P(x \leq X_i \leq x + \delta x) = F(x + \delta x) - F(x)$$

$$P(a \leq X \leq b) = F(b) - F(a)$$

$$P_3 = P[x + \delta x \leq X_i \leq y] = F(y) - F(x + \delta x)$$

$$P_4 = P(y \leq X_i \leq y + \delta y) = F(y + \delta y) - F(y)$$

$$P_5 = P[X_i > y + \delta y] = 1 - P(X_i \leq y + \delta y)$$

$$= 1 - F(y + \delta y)$$

using multinomial distribution

$P(E) = \dfrac{n!}{(r-1)!(S-r-1)!(n-S)!}$ P1 P2 P4 P5...

$$F_{rs}(x, y) = \lim_{\delta x \to 0} \frac{P(E)}{\delta x \delta y}$$

$$F_{rs}(x, y) = \lim_{\delta y \to 0} \frac{P(E)}{\delta x \delta y}$$

$$F_{rs}(x, y) = n! f(x)^{r-1} \frac{[f(y) - f(x + \delta x)]^{S-r-1}}{(r-1)!(S-r-1)!(n-S)!}$$

$$\lim_{\delta y \to 0} \frac{f(y + \delta y) - f(y)}{\delta y} [1 + f(y + \delta y)]^{n-S}$$

P.d.f of single order statistics

$$f_r(x) = \frac{d}{dx}\left(F_r(x)\right)$$

$$f_r(x) = \frac{d}{dx}\left[B(r, n-r+1)F^{r-1}(1-F)^{n-r+1}\right]$$

$$g(t) = \int t^{r-1}(1-t)^{n-r}dt$$

$$g'(t) = t^{r-1}(1-t)^{n-r}$$

$$I_P[a,b] = \frac{1}{B(a,b)}\int_a^b t^{a-1}(1-t)^{b-1}dt$$

$$\int_0^{F(x)} t^{r-1}(1-t)^{n-r}dt = [g(t)]_0^{F(x)}$$

$$= g[F(x)] - g$$

$\frac{d}{dx}\int_0^{F(x)} t^{r-1}(1-t)^{n-r}dt$ {using Leibniz theorem}

$$= g'(F(x)) \cdot f(x)$$

$$f_r(x) = [F(x)]^{r-1}[1-F(x)]^{n-r}f(x)$$

$$g'(b) = t^{7-1}(1-t)^{b-7}$$

$f(x) = B(7, b-7)[t^{7-1}(1-t)^{b-7}f'(x)]$ is the p.d.f of single order statistics

Cumulative Distribution of single order statistics

$$F_n(x) = P(X_{(n)} \le x) = P(x_i \le x; i = 1, 2, 3\ldots)$$

$X_1, X_2, \ldots X_n$ are Independent.

$$F_n(x) = P(X_1 \le x \cap X_2 \le x \cap \ldots \cap X_n \le x)$$

$$F_n(x) = P(X_1 \le x)P(X_2 \le x)\ldots P(X_n \le x)$$

p.d.f of $X_1 = f(x)$
p.d.f of $X_2 = f(x)$
p.d.f of $X_n = f(x)$
so that,

$$F_n(x) = f(x) \cdot f(x) \cdot f(x) \dots f(x)$$

$$F_n(x) = [f(x)]^n$$

Smallest order statistics is given by:

$$F_1(x) = P(X_{(1)} \leq x)$$

$$[\because P(X \leq x) = 1 - P(X > x)]$$

$$F_1(x) = 1 - P(X_{(1)} > x)$$

$$F_1(x) = 1 - P[X_i > x; i = 1, 2, 3 \dots]$$

$$F_1(x) = 1 - [P(X_1 > x \cap X_2 > x \cap \dots \cap X_n > x)]$$

$$(x_i > x)$$

$$F_1(x) = 1 - [P(X_1 > x)P(X_2 > x) \dots P(X_n > x)]$$

$$F_1(x) = 1 - \prod_{i=1}^{n} P(X_i > x)$$

$$[\because P(X \leq x) = 1 - P(X > x)]$$

$$= 1 - [1 - P(X \leq x)]$$

$$= 1 - [(1 - f(x))(1 - f(x)) \dots]$$

$$F_1(x) = 1 - P[X_i > x; i = 1, 2, 3 \dots]$$

$$F_1(x) = 1 - P(X > x)^n$$

$$[\because P(X \leq x) = 1 - P(X > x)]$$

$$b(x) = 1 - [1 - f(x)]$$

Constant of t-distribution:

$$\mu'_{0\gamma} = E(t^{2\gamma}) = \int_{-\infty}^{\infty} t^{2\gamma} f(t) dt = 2\int_{0}^{\infty} t^{2\gamma} f(t) dt$$

$$= \frac{1}{\sqrt{n}\, B(\frac{1}{2}, \frac{n}{2})} \int_{0}^{\infty} \frac{t^{2\gamma}}{(1+t^2)^{\frac{n+1}{2}}}\, dt$$

Put $t^2/n = y \rightarrow t^2 = ny \rightarrow t^2/n + 1 = 1/y - 1 + 1 = 1/y$

$t^2 = n[\frac{1-y}{y}];\, t = \sqrt{n(\frac{1-y}{y})}$

$$\frac{2t\, dt}{n} = \frac{-1}{y^2}\, dy$$

$$dt = \frac{-n}{2y^2 t}\, dy$$

$$dt = \frac{-n}{2y^2\sqrt{\frac{n(1-y)}{y}}}\, dy$$

$$= \frac{1}{B(\frac{1}{2}, \frac{n}{2})} \int_{0}^{1} \frac{[n(\frac{1-y}{y})]^{\gamma}}{(\frac{1}{y})^{\frac{n+1}{2}}} \times \frac{-n}{2y^2\sqrt{\frac{n(1-y)}{y}}}\, dy$$

$$= \frac{n^{\gamma}}{B(\frac{1}{2}, \frac{n}{2})} \int_{0}^{1} y^{\frac{n}{2}-\gamma-1}(1-y)^{\gamma+\frac{1}{2}-1} dy$$

$$\mu'_{2\gamma} = \frac{n^{\gamma}}{B(\frac{1}{2}, \frac{n}{2})}\, B(\frac{n}{2} - \gamma, \gamma + \frac{1}{2})$$

$$= \frac{n^r}{\sqrt{\frac{1}{2}}\sqrt{\frac{n}{2}}}$$

$$\frac{\sqrt{\frac{n}{2} - r}}{\sqrt{\frac{n}{2}}}$$

$$H_0 : \pi = \frac{\sqrt{n/2 - r)}}{\sqrt{n/2}}$$

Non-central chi-square m.g.f.

$$M_x(t) = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{tx^2 - 1/2(x-\mu)^2} dx$$

Let $I = e^{[tx^2 - 1/2(x-\mu)^2]}$

$$= e^{[tx^2 - 1/2(x^2 + \mu^2 - 2\mu x)]}$$

$$= e^{[(t-1/2)x^2 + (1/2\mu^2 - \mu x)]}$$

$$= e^{[-(\frac{1-2t}{2})x^2 - \mu x + \mu^2/2]}$$

$$I = e^{[-(\frac{1-2t}{2})x^2 - \frac{2\mu x(1-2t)}{(1-2t)} + \frac{1-2t(\mu^2)}{2(1-2t)}]}$$

$$= e^{[-(\frac{1-2t}{2})} \{x^2 - \frac{2\mu x}{(1-2t)} + \frac{\mu^2}{1-2t}\}]$$

$$= e^{[-(\frac{1-2t}{2})(\frac{x-\mu}{1-2t})^2]}$$

$$= e^{[-(\frac{1-2t}{2})(\frac{x-\mu}{1-2t})^2]}$$

$$= e^{[-(\frac{1-2t}{2})(\frac{x-\mu}{1-2t})^2]}$$

$$= e^{(\frac{t\mu^2}{1-2t})} e^{(-(\frac{1-2t}{2})(\frac{x-\mu}{1-2t})^2)}$$

$$M_{X^2}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{(\frac{t\mu^2}{1-2t})} e^{[-(\frac{1-2t}{2})(\frac{x-\mu}{1-2t})^2]} dx$$

$$= e^{(\frac{t\mu^2}{1-2t})} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{(-\frac{1}{2}u^2)} \frac{du}{(1-2t)^{1/2}}$$

$$= (1-2t)^{-1/2} e^{(t\mu^2/1-2t)} \rightarrow$$

$$1 - 2t = 0 \Leftrightarrow t < 1/2$$

If X_i $(i = 1, 2, ..., n)$ are Independent $N(\mu_i, 1)$ then the m.g.f. of the non-central χ^2-variate $\chi'^2 = \sum_{i=1}^{n} X_i^2$ is given by

$$M_{\chi'^2}(t) = M_{\sum_{i=1}^{n} X_i^2}(t) = \prod_{i=1}^{n} M_{X_i^2}(t)$$

By equ ①

$$M_{\chi'^2}(t) = \prod_{i=1}^{n} [(1-2t)^{-1/2} e^{(\frac{t\mu_i^2}{1-2t})}] = (1-2t)^{-n/2} e^{[\sum_{i=1}^{n} \frac{t\mu_i^2}{1-2t}]}; t < 1/2 \rightarrow$$

Where $\lambda = \frac{1}{2} \sum_{i=1}^{n} \mu_i^2$ is the non-centrality parameter.

Equ ② can be re-written as,

$$M_{\chi'^2}(t) = (1-2t)^{-n/2} e^{[\lambda(-1+\frac{1}{1-2t})]}$$

$$= (1-2t)^{-n/2} e^{-\lambda} e^{\lambda/1-2t}$$

$$= (1-2t)^{-n/2} e^{-\lambda} \sum_{r=0}^{\infty} (\frac{\lambda}{1-2t})^r \times \frac{1}{r!}$$

$$= \sum_{r=0}^{\infty} \frac{e^{-\lambda} \lambda^r}{r!} (1-2t)^{-(r+n/2)}; t < 1/2 \rightarrow$$

Thus the m.g.f of a non-central χ^2 distribution is seen to be a convex-combination m.g.f.s (with d.f. $n_1, n_2, n_3, \ldots$). The co-efficients appearing in the convex combination are merely the poisson probabilities.

Hence by the uniqueness theorem of m.g.f the p.d.f of non-central χ^2 distribution with n.d.f n & with non-centrality parameter λ is given by

$$f(\chi^2; n, \lambda) = \sum_{r=0}^{\infty} \frac{e^{-\lambda}\lambda^r}{r!} p(\chi^2; n+2r)$$

where,

$$p(\chi^2; n+2r) = \frac{1}{2^{(n+2r)/2}\Gamma(\frac{n+2r}{2})} e^{-\chi^2/2}(\chi^2)^{\frac{n+2r}{2}-1}$$

$$0 \leq \chi^2 < \infty$$

i.e. the p.d.f of central χ^2 distribution with $(n+2r)$ d.f.

Additive (or) Reproductive property of non-central χ^2-square Distribution.

If Y_i $(i = 1, 2, \ldots, k)$ are independent non-central χ^2-variates with n_i d.f. and non-centrality element λ_i, then $Y = \sum_{i=1}^{k} Y_i$ is also a non-central χ^2-variate with $n = \sum n_i$ d.f. and non-centrality element $\lambda = \sum \lambda_i$.

PROOF:

We have from [equ (2)] non-central χ^2-square m.g.f.

$$M_{Y_i}(t) = (1-2t)^{-n_i/2} \exp\left[\frac{\lambda_i t}{1-2t}\right]$$

for $i = 1, 2, \ldots, k$.

$$\therefore M_Y(t) = \prod_{i=1}^{k} M_{Y_i}(t) = \prod_{i=1}^{k}(1-2t)^{-n_i/2} \exp\left[\frac{\lambda_i t}{1-2t}\right]$$

$$= (1-2t)^{-\sum n_i/2} \exp\left[\frac{t\sum\lambda_i}{1-2t}\right]$$

which is the m.g.f of a non-central χ^2 variate with n f.d.f and non-centrality parameter $\lambda = \sum \lambda_i^2$. Hence by uniqueness theorem of m.g.f's

$$\sum_{i=1}^{n} Y_i \sim \chi^2 \sum \lambda_i^2 (\sum \lambda_i^2)$$

cumulants of Non-central chi-square distribution.

cumulant generating function is given by

$$K_x(t) = \log M_x(t) = -\frac{n}{2}\log(1-2t) + \frac{2\lambda^2 t}{(1-2t)}$$

$$= \frac{n}{2}[t + 2t^2 + \ldots + \frac{(2t)^r}{r!} + \ldots] + 2\lambda t[1 + 2t + (2t)^2 + \ldots + (2t)^{r-1} + \ldots]$$

the expansion being valid for $t < \frac{1}{2}$.

$$K_x(t) = (n+2\lambda)t + (n+4\lambda)\frac{t^2}{2!} + \ldots + \frac{(2\gamma-1)!}{2^{\gamma-1}}\frac{2\lambda(2\gamma-1)!}{2^{\gamma-1}}t^{\gamma}\ldots$$

$K_\gamma =$ co-efficient of $\frac{t^\gamma}{\gamma!}$ in $K_x(t)$

$$= \gamma!\left[\frac{n+2\lambda(\gamma-1)}{2^{\gamma-1}(\gamma-1)!}\right]$$

$$=2^{\gamma-1} (\gamma-1)! (n+2\lambda) \to \circledast$$

$$K_{\gamma-1} = 2^{\gamma-2}(\gamma-2)![n+2\lambda(\gamma-2)]$$

$$\frac{d}{d\lambda}(K_{\gamma-1}) = \frac{n}{2^{\gamma-2}(\gamma-2)!}(\gamma-2)!2(\gamma-1) = \frac{2\lambda}{2^{\gamma-2}}(\gamma-1)!$$

$$\frac{n+2\lambda}{2^{\gamma-2}(\gamma-2)!} \to \text{from equ} \circledast$$

$$K_\gamma = (n+2\lambda\gamma)\frac{d}{d\lambda}(K_{\gamma-1})$$

Non-central F distribution.

$$F = x/n_1$$

$$y/n_2$$

$$P(x,y) = \beta_3(x) = P_2(4)$$

$$P_1(x) = \sum_{i=0}^{\infty} \frac{e^{-\lambda}\lambda^i}{i!} \frac{e^{-(n_1+2i)x/2} x^{(n_1+2i)/2-1}}{2^{(n_1+2i)/2}\Gamma[(n_1+2i)/2]}$$

$$P_2(y) = \frac{e^{-n_2 y/2} y^{n_2/2-1}}{2^{n_2/2}\Gamma(n_2/2)}$$

$$P(x,y) = \sum_{i=0}^{\infty} \frac{e^{-\lambda}\lambda^i}{i!} \frac{e^{-(n_1+2i)x/2} x^{(n_1+2i)/2-1}}{2^{(n_1+2i)/2}\Gamma[(n_1+2i)/2]} \frac{e^{-n_2 y/2} y^{n_2/2-1}}{2^{n_2/2}\Gamma(n_2/2)}$$

$F_1 = n_2 x/n_1 y$ let $u = y/x \Longrightarrow y = ux;\ x = n_1 u F_1/n_2$

$$I = \int\int P(x,y)dxdy \quad P(F,u) = P_1(x)\cdot P_2(y)|J|$$

$$J = (x,y) \to (F,u)$$

$$g(F,u) = \sum_{i=0}^{\infty} \frac{e^{-\lambda}\lambda^i}{i!} \frac{e^{-(n_1+2i)x/2} x^{(n_1+2i)/2-1}}{2^{(n_1+2i)/2}\Gamma[(n_1+2i)/2]} \frac{e^{-n_2 y/2} y^{n_2/2-1}}{2^{n_2/2}\Gamma(n_2/2)} |J|$$

$$= \sum_{i=0}^{\infty} \frac{e^{-\lambda}\lambda^i}{i!} \exp[-\lambda(1 + n_1 F/n_2)] u^{i+(n_1+n_2)/2-1} \frac{(n_1/n_2)^{(n_1+2i)/2}}{2^{(n_1+n_2)/2}\Gamma[(n_1+2i)/2]\Gamma(n_2/2)} \cdot (\dots)$$

w.r.t u b/w limits 0 to ∞ and gamma Integral.

$$g(F) = \sum_{i=0}^{\infty} \frac{e^{-\lambda}\lambda^i}{i!} \frac{(n_1/n_2)^{(n_1+2i)/2}\Gamma[i + (n_1+n_2)/2]}{2^{(n_1+n_2)/2}\Gamma[(n_1+2i)/2]\Gamma(n_2/2)}$$

$$\frac{(\eta_1+\eta_2)/2}{\left(\frac{1}{2}(\eta_1 F') + (\eta_1+\eta_2)/2\right)^2}$$

$$\phi f' = \sum_{r=0}^{\infty} \frac{e^{-\lambda/2}(\lambda/2)^r}{r!} \frac{\Gamma((\eta_1+\eta_2)/2 + r)}{\Gamma(\eta_1/2 + r)\Gamma(\eta_2/2)} \frac{1}{(1 + \eta_1 F'/\eta_2)^{(\eta_1+\eta_2)/2+r}}$$

where $0 \leq F' \leq \infty$

Non-central F-Distribution

The ratio of two independent variate each divided by the corresponding d.f. has a non-central F-distribution if the numerator has non-central χ^2-distribution and the denominator has a central χ^2-distribution. Thus, if X has a non-central χ^2-distribution with η_1 d.f. and non-centrality parameter λ i.e., $X \sim \chi^2(\eta_1, \lambda)$, and Y is an independent (central) χ^2-variate with η_2 d.f., $Y \sim \chi^2(\eta_2)$, then the non-central F-statistic is defined as:

$$F = \frac{X/\eta_1}{Y/\eta_2} = \frac{X}{Y}\frac{\eta_2}{\eta_1} = \frac{\eta_2 X}{\eta_1 Y}$$

Non-central t-distribution

The non-central t-distribution is the distribution of the ratio of a normal variate, with possible non-zero mean and variance unity, to the square root of an independent χ^2-variate divided by its degrees of freedom. If $X \sim N(\mu, 1)$ and Y is an

Independent t^2-variate with ind. $N(0, 1)$ then

$$t' = \frac{x}{\sqrt{y/n}}$$

is said to have a non-central t-distribution with ndf and non centrality parameter λ. non-central t-distribution is required for the power functions of certain tests concerning normal population.

Hyper Geometric Distribution:

A Discrete random variable x is said to follow the hypergeometric distribution with parameter N, M and n if it assumes only non-negative values and its probability mass-function is given by:

$$P(x = k) = h(K; N, M, n) = \begin{cases} \frac{\binom{M}{k}\binom{N-M}{n-k}}{\binom{N}{n}} & k = 0, 1, 2, \ldots \min(n, m) \\ 0 & \text{otherwise} \end{cases}$$

non-central χ^2-distribution:
The χ^2-distribution defined as the sum of the square of independent standard normal variates is often referred to as the central χ^2-distribution. The distribution of the sum of the squares of independent normal variates, each having unit variance but with possibly non-zero means is known as non-central chi-square distribution.
Thus if x_i $(i = 1, 2, \ldots, n)$ are independent $N(\mu_i, 1)$ variates then,

$$\chi'^2 = \sum x_i^2$$

Has the non-central χ^2 distribution with n d.f.
Intuitively, this distribution sould seem to depend upon the n parameters $\mu_1, \mu_2, \ldots, \mu_n$ but it will be seen that it depends on these parameters only through the non-centrality parameter,

$$\lambda = \frac{1}{2}(\mu_1^2 + \mu_2^2 + \ldots + \mu_n^2)$$

and we write $\chi^2 \sim \chi_c^2(n, \lambda)$.