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Unit-V

Reliability Theory

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Unit v
Reliability

RELIABILITY

Define

Reliability is defined as the probability that a product, system, or service will perform its intended function adequately for a specified period of time, or will operate in a defined environment without failure.

The most important components of this definition must be clearly understood to fully know how reliability in a product or service is established:

- **Probability:** the likelihood of mission success
- **Intended function:** for example, to light, cut, rotate, or heat
- **Satisfactory:** perform according to a specification, with an acceptable degree of compliance
- **Specific period of time:** minutes, days, months, or number of cycles
- **Specified conditions:** for example, temperature, speed, or pressure

Stated another way, reliability can be seen as:

- Probability of success
- Durability
- Dependability
- Quality over time
- Availability to perform a function

Types

There are several general classes of reliability estimates:

- Inter-rater reliability assesses the degree of agreement between two or more raters in their appraisals. For example, a person gets a stomach ache and different doctors all give the same diagnosis.
- Test-re test reliability assesses the degree to which test scores are consistent from one test administration to the next. Measurements are gathered from a single rater who uses the same methods or instruments and the same testing conditions. This includes intra-rater reliability.
- Inter-method reliability assesses the degree to which test scores are consistent when there is a variation in the methods or instruments used. This allows inter-rater reliability to be ruled out. When dealing with forms, it may be termed parallel-forms reliability.
- Internal consistency reliability assesses the consistency of results across items within a test.

Meaning of Reliability

The idea behind reliability is that any significant results must be repeatable. Other researchers must be able to perform exactly the same experiment, under same conditions and generate the same results. This will vindicate the findings and ensure that all researchers will accept the hypothesis. Without this replication of statistically significant results, experiment and research have not fulfilled all of the requirements of testability. This prerequisite is essential to a hypothesis establishing itself as an accepted scientific truth. For example, if you are performing a time critical experiment, you will be using some type of stopwatch.

Generally, it is reasonable to assume that the instruments are reliable and will keep true and accurate time. However, scientists take measurements many times, to minimize the chances of malfunction and maintain validity and reliability. At the other extreme, any experiment that uses human judgment is always going to come under question. Human judgment can vary as individual observer may rate things differently depending upon time of day and current mood. This means that such experiments are more difficult to repeat and are inherently less reliable.

Reliability is a necessary ingredient for determining the overall validity of a scientific experiment and enhancing the strength of the results. Reliability is the consistency of your measurement, or the degree to which an instrument measures the same way each time it is used under the same condition with the same subjects. In short, it is the repeatability of measurement. A measure is considered reliable if a person's score on the same test given twice is similar.

It is important to remember that reliability is not measured, it is estimated. For instance, if a test is constructed to measure a particular trait; say, neuroticism, then each time it is administered, it should yield same results. A test is considered reliable if we get same result repeatedly.

- According to Anastasi (1957), the reliability of test refers to the consistency of scores obtained by the individual on different occasions or with different sets of equivalent items.
- According to Stodola and Stordahl (1972), the reliability of a test can be defined as the correlation between two or more sets of scores of equivalent tests from the same group of individuals.
- According to Guilford (1954), reliability is the proportion of the true variance in obtained test scores.

The reliability of test is also defined from another angle. Whenever we measure something, measurement involves some kind of measure. Error of measurement is generally between true scores and the observed score. However, in psychological term, word error does not imply the mistake has been made. In other words, error in psychological testing implies that there is always some inaccuracy in measurement. Hence, goal of psychological measurement remains to find out the magnitude of such error and develop ways to minimize them.

Methods of estimating reliability

There are number of ways of estimating reliability of an instrument. Various procedures can be classified into two groups:

1. External consistency procedures
2. Internal consistency procedures

1. External Consistency Procedures

External consistency procedures compare findings from two independent process of data collection with each other as a means of verifying the reliability of the measure. Two methods are as beneath.

(I)Test Re-test Reliability

The most frequently used method to find the reliability of a test is by repeating the same test on same sample, on two different time periods. The reliability coefficient in this case would be the correlation between the score obtained by the same person on two administrations of the test. Test-Retest reliability is estimated, when same test is administered on same sample.

Therefore, if refers to the consistency of a test among on two different time periods different administrations. The assumption behind this approach is that there will be no substantial changes in the measurement of the construct in question, upon administration on separate occasions.

The time gap that is given between measures is of critical value, the shorter the time gap, higher the correlation value and vice versa. If the test is reliable, the scores that are attained on first administration should be more or less equal to those obtained on second time also. The relationship between the two administrations should be highly positive.

Limitations

There are a few limitations which include the following:

I. Memory Effect/carry over Effect
II. Practice effect,
III. Absence.

I) **Memory effect /carry over effect:**

One of the common problems with tester test reliability is that of memory effect. This argument particularly holds true when, the two administrations takes place within short span of time,

for example, when a memory related experiment including nonsense syllables is conducted whereby, the subjects are asked to remember a list in a serial wise order, and the next experiment is conducted within 15 minutes, most of the times, subject is bound to remember his/her responses, as a result of which there can be prevalence of artificial reliability coefficient since subjects give response from memory instead of the test. Same is the condition when pre-test and post-test for a particular experiment is being conducted.

ii) **Practice effect:**

This happens when repeated tests are being taken for the improvement of test scores, as is typically seen in the case of classical IQ where there is improvement in the scores as we repeat these tests.

iii) **Absence:** People remaining absent for re-tests.

(II) Parallel Forms Reliability

Parallel-Forms Reliability is known by the various names such as Alternate forms reliability, equivalent form reliability and comparable form reliability. Parallel forms reliability compares two equivalent forms of a test that measure the same attribute. The two forms use different items. However, the rules used to select items of a particular difficulty level are the same. When two forms of the test are available, one can compare performance on one form versus the other. Sometimes the two forms are administered to the same group of people on the same day.

The Pearson product moment correlation coefficient is used as an estimate of the reliability. When both forms of the test are given on the same day, the only sources of variation are random error and the difference between the forms of the test. Sometimes the two forms of the test are given at different times. In these cases, error associated with time sampling is also included in the estimate of reliability.

The method of parallel forms provides one of the most rigorous assessments of reliability commonly in use. Unfortunately the use of parallel forms occurs in practice less often than is desirable. Often test developers find it burdensome to develop two forms of the same test, and practical constraints make it difficult to retest the same group of individuals. Instead many test developers prefer to base their estimate or reliability on a single form of a test.

In practice, psychologists do not always have two forms of a test. More often they have only one test form and must estimate the reliability for this single group of items.

Assess the different sources of variation within a single test in many ways. One method is to evaluate the internal consistency of the test by dividing it into subcomponents.

2) Internal Consistency Procedures

The idea behind internal consistency procedures is that items measuring same phenomena should produce similar results. Following internal consistency procedures are commonly used for estimating reliability.

Split Half Reliability

In this method, as the name implies, we randomly divide all items that intends to measure same construct into two sets .The complete instrument is administered on sample of people and total scores are calculated for each randomly divided half; the split half reliability is then, the simply the correlation between these two scores.

System Reliability Concepts

The analysis of the reliability of a system must be based on precisely defined concepts. Since it is readily accepted that a population of supposedly identical systems, operating under similar conditions, fall at different points in time, then a failure phenomenon can only be described in probabilistic terms. Thus, the fundamental definitions of reliability must depend on concepts from probability theory. This chapter describes the concepts of system reliability engineering.

These concepts provide the basis for quantifying the reliability of a system. They allow precise comparisons between systems or provide a logical basis for improvement in a failure rate. Several distribution models are discussed and the resulting hazard functions are derived describes a new concept of system ability.

Several system ability functions of various system configurations such as series, parallel, and k-out-of-n, are various reliability aspects of systems with multiple failure modes. Stochastic processes including Markov process, Poisson process, renewal process, quasi-renewal process, and no homogeneous.

In general, a system may be required to perform various functions, each of which may have a different reliability. In addition, at different times, the system may have a different probability of successfully performing the required function under stated conditions. The term failure means that the system is not capable of performing a function when required. The term capable used here is to define if the system is capable of performing the required function.

Reliability Measures

Reliability is the probability of success or the probability that the system will perform its intended function under specified design limits. More specific, reliability is the probability that a product or part will operate properly for a specified period of time (design life) under the design operating conditions (such as temperature, volt, *etc.*) without failure.

In other words, reliability may be used as a measure of the system's success in providing its function properly. Reliability is one of the quality characteristics that consumers require from the manufacturer of products. Mathematically, reliability $R(t)$ *is* the probability that a system will be successful in the interval from time 0 to time t:

$$R(t) = P(T > t); t \geq 0$$

Where T is a random variable denoting the time-to-failure or failure time
Unreliability F(t), a measure of failure, is defined as the probability that the system will fail by time t:

$$F(t) = P(T \leq t) \; for \; t \geq 0$$

In other words, F (t) is the failure distribution function. If the time-to-failure random variable T has a density function f (t), then

$$R(t) = \int_{r}^{s} f(s)\,ds$$

or, equivalently,

$$f(t) = -\frac{d}{dt}[R(t)]$$

The density function can be mathematically described in terms of T:

$$\lim_{\Delta t \to 0} P(t < T \leq t + \Delta t)$$

This can be interpreted as the probability that the failure time T will occur between the operating time t and the next interval of operation, $t + \Delta t$.Consider a new and successfully tested system that operates well when put into service at time $t = 0$. The system becomes less likely to remain successful as the time interval increases. The probability of success for an infinite time interval, of course, is zero. Thus, the system functions at a probability of one and eventually decreases to a probability of zero. Clearly, reliability is a function of mission time.

For example, one can say that the reliability of the system is 0.995 for a mission time of 24 hours. However, a statement such as the reliability of the system is 0.995 is meaningless because the time interval is unknown.

The Reliability Function

The reliability function can be derived using the previous definition of the cumulative dens ity function.

Note that the probability of an eventhappening by time t(based on a continuous distributio n given by f(x), or f(t).

Since our random variable of interest in life data analysis is time, or t) is given by:

$$F(t) = \int_{0,\gamma}^{t} f(s)\, ds$$

One could also equate this event to the probability of a unit failing by time t, since the even t of interest in life data analysis is the failure of an item.

From this fact, the most commonly used function in reliability engineering can then be obtained, the Reliability function, this enables the determination of the probability of success of a unit, in undertaking a mission of a prescribed duration. To mathematically show this, we first define the unreliability Function Q(t), which is the probability of failure, or the probability that our time to failure is in the region of 0 (or γ) and t.

So, from the previous equation, we have:

$$F(t) = Q(t) = \int_{0,\gamma}^{t} f(s)\, ds$$

In this situation, there are only two situations that can occur: success or failure. The se two states are also mutually exclusive. Since reliabilityand unreliability is the probabiliti es of these two mutually Exclusive states, the sum of these probabilities are always equal to unity. So then:

$$\mathrm{Q(t) + F(t) = 1}$$

$$\mathrm{R(t) = 1 - Q(t)}$$

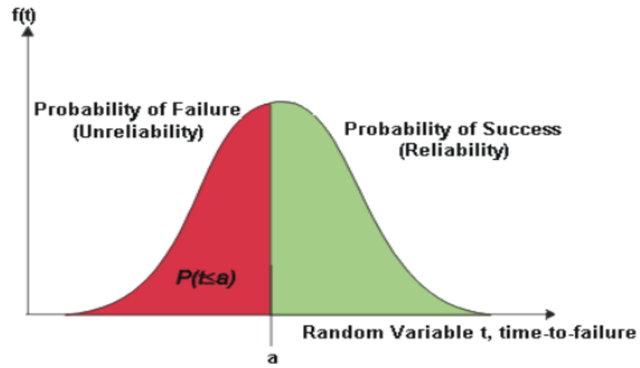
$$\mathrm{R(t) = 1 - \int_{0,\gamma}^{t} f(s)\, ds}$$

$$\mathrm{R(t)} = \int_{\mathrm{t}}^{\infty} \mathrm{f(s)}\, ds$$

Where R(t) is the reliability function. Conversely, the pdf can be defined in terms o f the reliability Function as:

$$f(t) = \frac{d}{dt}[R(t)]$$

The following figure illustrates the relationship between the reliability function and the cdf , or the Unreliability function.



The reliability function derivation process with the exponential distribution. The pd f of the exponential distribution is given by:

$$f(t) = \lambda e^{-\lambda t}$$

Where λ(lambda) is the sole parameter of the distribution. This form of the expone ntial is a oneparameter distribution. Based on the previous definition of the Reliability function, It is a relatively easy matter to derive the reliability function for the exponential distribution:

$$\begin{aligned} \mathrm{R(t)} &= 1 - \int_0^{\mathrm{t}} \lambda e^{-\lambda s}\,\mathrm{ds} \\ &= 1 - \left[1 - e^{-\lambda t}\right] \\ &= e^{-\lambda t} \end{aligned}$$

Failure Rate Function

The probability of a system failure in a given time interval [t_1, t_2] can be expressed in terms of the reliability function as

$$\int_{\mathrm{t}_1}^{t_2} f(t)\,dt = \int_{\mathrm{t}_1}^{\infty} f(t)\,dt - \int_{t_2}^{\infty} f(t)\,dt$$

$$= R(\mathrm{t}_1) - \mathrm{R}(\mathrm{t}_2)$$

or in terms of the failure distribution function (or the unreliability function) as

$$\int_{\mathrm{t}_1}^{t_2} f(t)\,dt = \int_{-\infty}^{t_2} f(t)\,dt - \int_{-\infty}^{\mathrm{t}_1} f(t)\,dt$$

$$= F(\mathrm{t}_2) - \mathrm{F}(\mathrm{t}_1)$$

The rate at which failures occur in a certain time interval [[t_1, t_2] is called the failure rate. It is defined as the probability that a failure per unit time occurs in the interval, given that a failure has not occurred prior to t1, the beginning of the interval. Thus, the failure rate is

$$\frac{R(\mathrm{t}_1) - \mathrm{R}(\mathrm{t}_2)}{(\mathrm{t}_2 - \mathrm{t}_1)\mathrm{R}(\mathrm{t}_1)}$$

Note that the failure rate is a function of time. If we redefine the interval as [t,t+Δt], the above expression becomes

$$\frac{R(t) - R(t + \Delta t)}{\Delta t R(t)}$$

The rate in the above definitions is expressed as failures per unit time, when in reality, the time units might be in terms of miles, hours, etc. The hazard function is defined as the limit of the failure rate as the interval approaches zero. Thus, the hazard function h(t) is the instantaneous failure rate, and is defined by

$$h(t) = \lim_{\Delta t \to 0} \frac{R(t) - R(t + \Delta t)}{\Delta t R(t)}$$

$$= \frac{1}{R(t)}\left(-\frac{d}{dt}[R(t)]\right) \rightarrow (1)$$

$$= \frac{f(t)}{R(t)}$$

The quantity $h\ (t)\ dt$ represents the probability that a device of age t will fail in the small interval of time t to (t + dt). The importance of the hazard function is that it indicates the change in the failure rate over the life of a population of components by plotting their hazard functions on a single axis.

For example, two designs may provide the same reliability at a specific point in time, but the failure rates up to this point in time can differ. The death rate, in statistical theory, is analogous to the failure rate as the force of mortality is analogous to the hazard function. Therefore, the hazard function or hazard rate or failure rate function is the ratio of the probability density function (pdf) to the reliability function.

Lifetime Distributions

A statistical distribution is fully described by its pdf (or probability density function). The functions most commonly used in reliability engineering and life data analysis, namely the reliability function, failure rate function, mean time function and median life function, can be determineddirectly from the pdf definition, or f(t). Different di stributions exist, such as the normal, exponential etc., and each one of them has a predefined f(t). These distributions were formulated by statisticians, mathematicians and /or engineers to mathematically model or represent certain behavior.

For example, the Weibull distribution was formulated by Walloddi Weibull and thu s it bears his name. Some distributions tend to betterrepresent life data and are most comm only referred to as lifetime Distributions.The pdf of the well known normal, or Gaussian, distribution is given by:

$$f(t) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{(t-\mu)}{\sigma}\right)^2}$$

In this definition, note that t is our random variable which represents time and the Greek let ters μ and σ represent what are commonly referred to as the parameters of the distribution. Depending on the values of μand σ, f(t) will take on different shapes. The normal distribution is a two parameter distribution, with two parameters μ and σ. For any distribution, the parameter or parameters of the distribution are estimated from the data.

For example, in the case of the normal distribution, μ, the mean, and σ, the standard deviation, are its parameters. Both of these parameters are estimated from the data, i.e. the meanand standard deviation of the data. Once these parameters are estimated, the p df function f(t) is fully defined and we can obtain any value for f(t) given any value of t. Given the mathematical representation of a distribution, we can also derive all of the functi ons needed for life data analysis, such as the reliabilityfunction. Once again, this will only depend on the value of tafter the value of the distribution parameter or parameters are esti mated from data.

Exponential Distribution

Exponential distribution plays an essential role in reliability engineering because it has a constant failure rate. This distribution has been used to model the lifetime of electronic and electrical components and systems. This distribution is appropriate when a used component that has not failed is as good as a new component – a rather restrictive assumption.

Therefore, it must be used diplomatically since numerous applications exist where the restriction of the memory less property may not apply. For this distribution,

$$f(t) = \frac{1}{\theta} e^{-\frac{t}{\theta}} = \lambda e^{-\lambda t}, t \geq 0$$

$$R(t) = e^{-\frac{t}{\theta}} = e^{-\lambda t}, t \geq 0$$

Where θ=1/λ>0 is an MTTF's parameter and λ≥0 is a constant failure rate. The hazard function or failure rate for the exponential density function is constant, i.e.,

$$h(t) = \frac{f(t)}{R(t)} = \frac{\frac{1}{\theta} e^{-\frac{t}{\theta}}}{e^{-\frac{t}{\theta}}} = \frac{1}{\theta} = \lambda$$

The failure rate for this distribution is λ, a constant, which is the main reason for this widely used distribution. Because of its constant failure rate property, the exponential is an excellent model for the long flat "intrinsic failure" portion of the bathtub curve. Since most parts and systems spend most of their lifetimes in this portion of the bathtub curve, this justifies frequent use of the exponential (when early failures or wear out is not a concern).

The exponential model works well for inter-arrival times. When these events trigger failures, the exponential lifetime model can be used. We will now discuss some properties of the exponential distribution that are useful in understanding its characteristics, when and where it can be applied.

Property 1: (*Memoryless property*) The exponential distribution is the only continuous distribution satisfying

$$P[T \geq t] = P\{\langle T \geq t + \delta | T \geq \delta \rangle\} \quad for \;\; t > 0, s > 0 \;\rightarrow (1)$$

This result indicates that the conditional reliability function for the lifetime of a component that has survived to time *s* is identical to that of a new component. This term is the so-called "used-as-good-as-new" assumption. The lifetime of a fuse in an electrical distribution system may be assumed to have an exponential distribution. It will fail when there is a power surge causing the fuse to burn out. Assuming that the fuse does not undergo any degradation over time and that power surges that cause failure are likely to occur equally over time, then use of the exponential lifetime distribution is appropriate, and a used fuse that has not failed is as good as new.

Property 2: If $T_1, T_2, \ldots\ldots T_n$, are independently and identically distributed exponential random variables (RVs) with a constant failure rate λ, then

$$2\lambda \sum_{i=1}^{n} T_1 \sim \chi^2\,(2n)$$

Where χ^2 $(2n)$ is a chi-squared distribution with degrees of freedom $2n$. This result is useful for establishing a confidence interval for λ

Problem: 1

A manufacturer performs an operational life test on ceramic capacitors and finds they exhibit constant failure rate with a value of 3×10^{-8} failure per hour. What is the reliability of a capacitor at 10^4hours?

Solution:

The reliability of a capacitor at 10^4 hour is

$$R(t) = e^{-\lambda t} = e^{-3\times10^{-8}\mathrm{t}} = e^{-3\times10^{-4}} = 0.9997$$

The resulting reliability plot is shown in figure 2.1

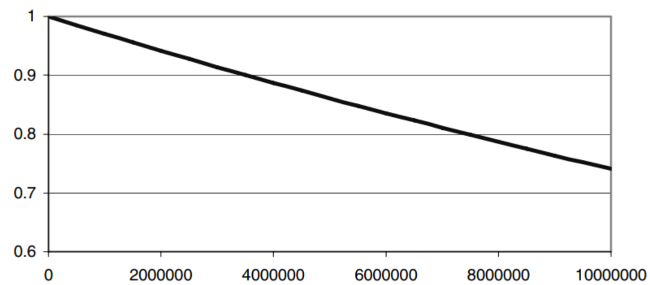


Figure 2.1. Reliability function *vs* time

Gamma Distribution

Gamma distribution can be used as a failure probability function for components whose distribution is skewed. The failure density function for a gamma distribution is

$$f(t) = \frac{t^{\alpha-1}}{\beta^{\alpha}\,\Gamma(\alpha)} e^{-\frac{t}{\beta}} \quad for\ t \geq 0, \alpha, \beta > 0 \ \rightarrow (1)$$

Where α is the shape parameter and β is the scale parameter. Hence,

$$R(t) = \int_t^{\infty} \frac{1}{\beta^{\alpha}\Gamma(\alpha)} s^{\alpha-1} e^{-\frac{s}{\beta}} ds$$

If α is an integer, it can be shown by successive integration by parts that

$$R(t) = e^{-\frac{t}{\beta}} \sum_{i=0}^{\alpha-1} \frac{\left(\frac{t}{\beta}\right)^i}{i!} \quad \rightarrow (2)$$

And

$$h(t) = \frac{f(t)}{R(t)} = \frac{\frac{t^{\alpha-1}}{\beta^{\alpha}\,\Gamma(\alpha)} e^{-\frac{t}{\beta}}}{e^{-\frac{t}{\beta}} \sum_{i=0}^{\alpha-1} \frac{\left(\frac{t}{\beta}\right)^i}{i!}}$$

The gamma density function has shapes that very similar to the Weibull distribution. At α = 1, the gamma distribution becomes the exponential distribution with the constant failure rate 1/β. The gamma distribution can also be used to model the time to the *n*th failure of a system if the underlying failure distribution is exponential. Thus, if *Xi* is exponentially distributed with parameter θ = 1/β, then $T = X_1, X_2, \ldots\ldots X_n$ is gamma distributed with parameters β and *n*.

Problem:

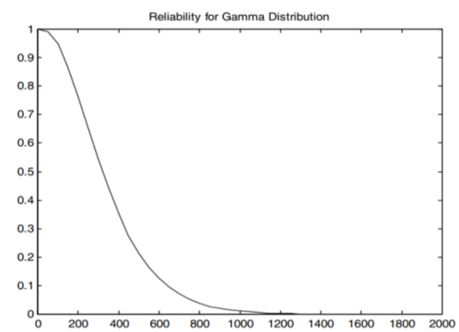
A mechanical system time to failure is gamma distribution with α=3 and 1/β=120. Find the system reliability at 280 hours.

Solution:

The system reliability at 280 hours is given by

$$R(280) = e^{-\frac{280}{120}} \sum_{i=0}^{2} \frac{\left(\frac{280}{120}\right)^2}{k!} = 0.85119$$

The gamma model is a flexible lifetime model that may offer a good fit to some sets of failure data. It is not, however, widely used as a lifetime distribution model for common failure mechanisms. A common use of the gamma lifetime model occurs in Bayesian reliability applications.



Weibull Distribution

The exponential distribution is often limited in applicability owing to the memory less property. The Weibull distribution (Weibull 1951) is a generalization of the exponential distribution and is commonly used to represent fatigue life, ball bearing life, and vacuum tube life. The Weibull distribution is extremely flexible and appropriate for modeling component lifetimes with fluctuating hazard rate functions and for representing various types of engineering applications. The three-parameter probability density function is

$$f(t) = \frac{\beta(t-\gamma)^{\beta-1}}{\theta^\beta} e^{-(\frac{t-\gamma}{\theta})^\beta}: \ t \geq \gamma \geq 0$$

Where θ and β are known as the scale and shape parameters, respectively, and γ is known as the location parameter. These parameters are always positive. By using different parameters, this distribution can follow the exponential distribution, the normal distribution, *etc*. It is clear that, for $t \geq \gamma$ the reliability function *R*(*t*) *is*

$$R(t) = e^{-(\frac{t-\gamma}{\theta})^\beta} \ for\ t > \gamma > 0, \beta > 0, \theta > 0 \ \rightarrow (2.13)$$

Hence,

$$h(t) = \frac{\beta(t-\gamma)^{\beta-1}}{\theta^{\beta}};\ t > \gamma > 0, \beta > 0, \theta > 0 \rightarrow (2.14)$$

It can be shown that the hazard function is decreasing for $\beta < 1$, increasing for $\beta > 1$, and constant when $\beta = 1$.

Problem: 1

The failure time of a certain component has a Weibull distribution with $\beta = 4, \theta = 2000$, and $\gamma = 1000$. Find the reliability of the component and the hazard rate for an operating time of *1500* hours.

Solution: A direct substitution into equation (2.13) yields

$$R(1500) = e^{-\left(\frac{1500-1000}{2000}\right)^4} = 0.996$$

Using equation (2.14), the desired hazard function is given by

$$h(1500) = \frac{4(1500-1000)^{4-1}}{(2000)^4}$$

$$= 3.13 \times 10^{-5} faliures/hour$$

Note that the Rayleigh and exponential distributions are special cases of the Weibull distribution at $\beta = 2, \gamma = 0$, and $\beta = 1, \gamma = 0$, respectively.

For example, when $\beta = 1$ and $\gamma = 0$, the reliability of the Weibull distribution function in equation (2.13) reduces to

$$R(t) = e^{-\frac{t}{\theta}}$$

and the hazard function given in equation (2.14) reduces to $1/\theta$, a constant. Thus, the exponential is a special case of the Weibull distribution. Similarly, when $\gamma = 0$ and $\beta = 2$, the Weibull probability density function becomes the Rayleigh density function. That is

$$f(t) = \frac{2}{\theta} t e^{-\frac{t^2}{\theta}}\ for\ \theta > 0, t \geq 0$$

Other Forms of Weibull Distributions

The Weibull distribution again is widely used in engineering applications. It was originally proposed for representing the distribution of the breaking strength of materials. The Weibull model is very flexible and also has theoretical justification in many applications as a purely empirical model. Another form of Weibull probability density function is for example,

$$f(x) = \lambda\, \gamma\, x^{\gamma-1} e^{-\lambda t^{\gamma}} \quad \rightarrow (2.15)$$

When γ=2 the density function becomes a Rayleigh distribution. It can easily be shown that the mean, variance and reliability of the above Weibull distribution are, respectively, as follows:

$$\text{Mean}\,mean = \lambda^{\frac{1}{\gamma}}\Gamma(1+\frac{1}{\gamma})$$

$$Variance = \lambda^{\frac{2}{\gamma}}\left(\Gamma\left(1+\frac{2}{\gamma}\right)-\left(\Gamma\left(1+\frac{1}{\gamma}\right)\right)^2\right) \quad \rightarrow (2.16)$$

$$Reliabilty = e^{-\lambda t^{\gamma}}$$

Problem;

The time to failure of a part has a Waibull distribution with 1/λ = 250 (measured in 10^5Cycles) and γ=2. Find the part reliability at 106 cycles.

Solution

The part reliability at 10^6 cycles is

$$\mathrm{R}(10^6) = e^{-\frac{(10)^2}{250}} = 0.6703$$

the resulting reliability function is shown in figure 2.4

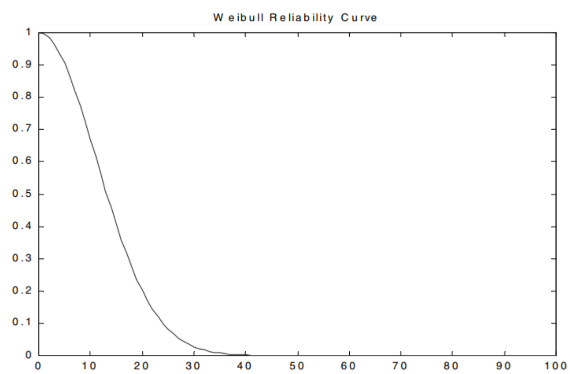


Figure 2.4. Weibull reliability function *vs* time

Bounds on reliability;

Bounds on reliability are a statistical technique used to estimate a range within which the true reliability of a system is likely to fall. This is because determining the exact reliability of a system can be complex and expensive, especially for systems with many components.

There are two main types of bounds on reliability:

- Confidence bounds: These represent the range of values within which the true reliability is likely to fall with a certain level of confidence (e.g., 90% confidence).

- Chebyshev bounds: These provide a guaranteed range for the reliability based on the assumption that the data follows a specific distribution (e.g., normal distribution).
- Bounds on reliability are helpful for engineers and reliability analysts to make informed decisions about system design, maintenance, and testing. By understanding the range of possible reliability values, they can make better trade-offs between cost, performance, and safety

System Mean Time to Failure

Suppose that the reliability function for a system is given by R(t). The expected failure time during which a component is expected to perform successfully, or the system mean time to failure (MTTF), is given by

$$MTTF = \int_0^\infty t\,f(t)dt \quad \rightarrow (1)$$

Substituting

$$f(t) = -\frac{d}{dt}[R(t)]$$

Into equation (1) and performing integration by parts, we obtain

$$MTTF = -\int_0^\infty td\,[R(t)]$$

$$MTTF = [-tR(t)]_0^\infty + \int_0^\infty R(t)\,dt \quad \rightarrow (2)$$

The first term on the right-hand side of equation (2) equals zero at both limits, since the system must fail after a finite amount of operating time. Therefore, we must have $tR(t)\rightarrow 0$ as $t\rightarrow\infty$ This leaves the second term, which equals

$$MTTF = \int_0^\infty R(t)\ dt \ \rightarrow (3)$$

Thus, MTTF is the definite integral evaluation of the reliability function. In general, if λ (t) is defined as the failure rate function, then, by definition, MTTF is not equal to $1/\lambda(t)$.The MTTF should be used when the failure time distribution function is specified because the reliability level implied by the MTTF depends on the underlying failure time distribution. Although the MTTF measure is one of the most widely used reliability calculations, It is also one of the most misused calculations. It has been misinterpreted as “guaranteed minimum lifetime”.

Table 1 Results of a twelve-component life duration test

Component	Time to failure (hours)
1	4510
2	3690
3	3550
4	5280
5	2595
6	3690
7	920
8	3890
9	4320
10	4770
11	3953
12	2750

Using a basic averaging technique, the component MTTF of 3660 hours was estimated. However, one of the components failed after 920 hours. Therefore, it is important to note that the system MTTF denotes the average time to failure. It is neither the failure time that could be expected 50% of the time, nor is it the guaranteed minimum time of system failure. That two failure distributions can have the same MTTF and yet produce different reliability levels. This is illustrated in a case where the MTTFs are equal, but with normal and exponential failure distributions. The normal failure distribution is symmetrical about its mean, thus

$$R\ (\mathrm{MTTF}) = P\ (Z \geq 0) = 0.5$$

Where Z is a standard normal random variable When we compute for the exponential failure distribution recognizing that θ= MTTF, the reliability at the MTTF is

$$\mathrm{R(MTTF)} = \mathrm{e}^{-\frac{\mathrm{MTTF}}{\mathrm{MTTF}}} = 0.368$$

Clearly, the reliability in the case of the exponential distribution is about 74% of that for the normal failure distribution with the same MTTF.

Failure rates and Mean residual life

The concept of failure rate has been beneficial in instructing an imperfect repair model when proper maintenance is performed. The probability density function (*pdf*) and cumulative distribution function (*cdf*) of the lifespan random variable *T*, respectively, are denoted by *f(t)* and *F(t)*. The failure rate function is

$$h(t) = \frac{f(t)}{F(t)}$$

where

$$\bar{F}(t) = 1 - F(t)t \ \geq 0$$

In lifetime experiments, the average additional lifetime of components when they remain up to time t is called mean residual life(MRL) function (Gupta and Kirmani (2000)). The MRL is characterized by

$$(t) = (T - t|T > t), t \geq 0.$$

Since the MRL is closely connected to the failure rate function, some studies examined the monotonicity of MRL concerning the monotonicity of the failure rate function. For a detailed account of MRL, it is recommended to read Guess and Proschan (1988). The basic ageing properties are related by the following illustrations (Deshpande *et al*. (1986) and Rao (1992))

$$\begin{array}{ccccccccc}
\text{IFR} & \Rightarrow & \text{IFRA} & \Rightarrow & \text{NBU} & \Rightarrow & \text{NBUFR} & \Rightarrow & \text{NBUFRA} \\
\Downarrow & & & & \Downarrow & & & & \\
\Downarrow & & & & \text{NBUC} & & & & \\
\Downarrow & & & & \Downarrow & & & & \\
\text{DMRL} & & \Rightarrow & & \text{NBUE} & \Rightarrow & \text{HNBUE} & \Rightarrow & L
\end{array}$$

Unnikrishnan and Vineshkumar (2011) have studied ageing properties in terms of quantile function.

Definition 2.1 (Lai *et al.* (2006))

Assume that the failure rate *h(t)* is a real-valued differentiable function $h(t) : R+ \rightarrow R+$.
Then the
cumulative distribution function *F(t)* is said to be

- Increasing failure rate (IFR) if $h'(t) > 0$ for all t;
- Decreasing failure rate (DFR) if $h'(t) < 0$ for all t;
- Bathtub shaped failure rate (BFR) if $h'(t) < 0$ for $t\epsilon(0, t_0), h'(t_0) = 0$ and $h'(t) > 0$ for $t > t_0$, for a comprehensive account of bathtub failure rate distributions see Rajarshi and Rajarshi (1988);
- Upside-down bathtub shaped failure rate (UBFR) if $h'(t) > 0$ for $t\epsilon(0, t_0), h'(t_0) = 0$ and $h'(t) < 0$ for $t > t_0$.
- Modified bathtub shaped failure rate (MBFR) if h(t) is first increasing and then bathtub shaped;
- Roller-coaster shaped failure rate if there exist n consecutive change points $0 < t_1 < t_2 < \cdots < t_n < \infty$ such that in each interval $[t_{j-i}, t_j], 1 \leq j \leq n + 1$, where $t_0 = 0$,

$t_{n+1} = \infty, h(t)$ is strictly monotone and it has opposite monotonicity in any two adjacent such intervals.

Estimation parameter

Estimating parameters, IFR (Increasing Failure Rate), DFR (Decreasing Failure Rate), and NBUE (Non-increasing Failure Rate) are all interrelated concepts in reliability analysis and lifetime modeling. Here's a breakdown of each:

Parameter Estimation:

* This refers to the process of determining the unknown values of the parameters that define a probability distribution based on observed data. In lifetime modeling, these parameters govern the likelihood of failure over time.

* Common estimation methods include maximum likelihood estimation and method of moments. The choice of method depends on the specific distribution being used.

IFR, DFR Distributions:

* These describe the way the failure rate of a system or component changes over time.

* IFR (Increasing Failure Rate): The failure rate increases with time. This is often seen in systems that wear out or degrade over time.

* DFR (Decreasing Failure Rate): The failure rate decreases with time. This can be due to early failures eliminating weaker units or "burn-in" periods where initial defects are revealed.

Distributions with monotone failure rates have been considered a significant factor because of their practical interest in reliability. Such distributions comprise a massive class. This section deals with mainly the characteristics of IFR and DFR class distributions.

Increasing Failure Rate

In the class of failure rate distributions, increasing failure rate (IFR) distributions have their relevance. (see, for example, Marshall and Proschan (1965), Kvam (2002) and Koutras (2011)). The Weibull and Gamma are two primer distributions that are effectively using in many industries for optimizing system reliability, especially for redundancy allocation problems (RAP) (see, Fyffe *et al.*(1968)). If *F* is IFR, it is equivalent to say that *F* is Polya frequency function of order 2 (PF2). Schoenberg (1951) introduced PF2 and defined by, a PF2 function is a non-negative measurable function *g(x)* defined for all real *x* such that

$$\begin{vmatrix} g(x_1 - y_1) & g(x_1 - y_2) \\ g(x_2 - y_1) & g(x_2 - y_2) \end{vmatrix} \geq 0,$$

Whenever $x_1 < x_2 \, and \, y_1 < y_2$ in addition, $g(x) \neq 0$ for at least two distinct values of *x*. If the density function *f* has PF2 then it would be easy to verify that the distribution has IFR, but the converse is needed not to be true. PF2 has many useful properties including unimodality, closure under convolution, variation diminishing and certain moment properties (see Schoenberg (1951) and Karlin (1961)).

Theorem 3.1.1 (Barlow, Marshall, Proschan (1963))
If *F* and *G* are IFR, then their convolution *H*, given by $H(t) = \int_{-\infty}^{\infty} F(t-x)dG(x)$ is also IFR. Bakouch *et al.* (2014) proposed a model (Binomial-exponential 2) using zero-truncated binomial random variable having IFR,

$$h(x) = \lambda\left(1 - \frac{\theta}{2 - \theta + \lambda\theta x}\right).$$

It is increasing in *x* for $0 \leq \theta \leq 1$ and $\lambda > 0$.
Guilani *et al.* (2016) investigated RAP of a series-parallel system with components having IFR
based on Weibull distribution.
Bugatekin (2017) developed a two-parameter mixed distribution named RL (Rayleigh Logarithmic) distribution with IFR function,

$$h(x) = -\frac{xe^{-\frac{x^2}{2\sigma^2}}P\left(1 - e^{-\frac{x^2}{2\sigma^2}}P\right)^{-1}}{\sigma^2 \, In\left(1 - e^{-\frac{x^2}{2\sigma^2}}P\right)}, \sigma > 0, y\epsilon[0, \infty).$$

Elbarmi *et al.* (2020) provide IFRA model with an estimator of *F* that is uniformly strongly consistent and they derive convergence of the estimator at the point where *F* is IFRA using the arg max theorem.

Decreasing Failure Rate

It is reasonable to assume that the failure rate of the life distribution is increasing (IFR). The wear-out phase easily interprets this fact, implying that the age-old units have a higher chance of failure. However, it appears to be more difficult to explain why a unit's life expectancy is increasing while its failure rate is decreasing. Decreasing failure rate (DFR) distributions may appear in different ways including, when the strength of some metals increases with usage or when an instrument exhibit an ‘infant mortality’ rate during their lifetime. Normally DFR represents the apparatus having betterment with age so that age-old units have less chance of failure.

Theorem 3.2.1 (Barlow, Marshall, Proschan (1963))

F is DFR if and only if the support of F is [0, ∞), and $1 - F(x+y)$is TP2 for $x + y \geq 0$.

• DFR property is not preserved under the reliability conditions of convolution and coherent systems. However, mixtures of DFR distributions are DFR.

• If F (t,∅) is a DFR distribution in t for each $\emptyset\ in\ \Phi$, then

$$G(t) = \int_{\Phi}^{\emptyset} F(t, \emptyset) d\mu(\emptyset)$$

Is DFR where μ is a probability measure in Φ.

Adamidis and Loukas (1998) presented a DFR distribution named Exponential–Geometric distribution obtained as a mixture of the exponential and geometric distribution. Its failure rate

$$h(x) = \beta(1 - pe^{-\beta x})^{-1}, P \in (0,1), \beta > 0, x > 0.$$

is decreasing in x. Finkelstein and Esaulova (2001) observed a mixture of several continuous IFR distributions and analyzed the limiting behavior of the mixture failure rate function. They found the mixture can be DFR and for that the limiting behavior of conditional mean and the conditional variance of the mixing parameters are essential.

Chahkandi and Ganjali (2009) introduced a DFR distribution which is a mixture of powerseries distribution and exponential distribution. Generally, mixtures of exponential distributions are DFR.

Basic Properties of IFR and DFR distributions

Pham (2003) provides some properties of IFR and DFR distributions.
• Suppose X_1 and X_2 are IFR, then $X_1 + X_2$ are also IFR; but DFR property may not be preserved. In the case of mixtures, the opposite happens.
• IFR distributions are preserved under coherent systems.
• Order statistics of an IFR distribution are again IFR; but not necessarily true for DFR distributions.
• Spacing's of DFR distributions gives DFR; but not true in the case of IFR distributions
• The *pdf* of IFR distributions may not be unimodal but a decreasing function for DFR distributions.

In the context of statistical quality control, NBUE stands for "Non-decreasing Failure Rate over Time" or "Non-increasing Failure Rate over Time." It's a property that can be applied to the lifetime distribution of a product or system.

A product or system with an NBUE characteristic demonstrates a decreasing likelihood of failure as it ages and endures use. This indicates that the product or system becomes more reliable over time. This property is desirable in many quality control applications

NBUE Distribution:

* A specific class of lifetime distributions that encompasses both IFR and DFR characteristics. A non-increasing failure rate distribution implies that the failure rate either remains constant or decreases over time.

* NBUE distributions are widely used in reliability analysis because they represent systems that tend to become more reliable or at least not less reliable with increasing age.

Connection between Parameter Estimation and NBUE:

* While there's no single estimation method specific to NBUE distributions, the estimated parameters of a fitted distribution can be used to assess if it belongs to the NBUE class. Certain mathematical properties of the distribution's hazard function can be derived from its parameters and can reveal if the failure rate is non-increasing

In quality control, estimating parameters of failure time distributions is crucial for tasks like:

* Reliability analysis: Assessing how long a product or system can function before failing.

* Warranty setting: Determining appropriate warranty periods based on failure probabilities.

* Preventive maintenance scheduling: Optimizing maintenance intervals to prevent failures.

Understanding failure rate behaviors (IFR, DFR, NBUE) further aids these quality control activities:

* IFR: Products with increasing failure rates (e.g., mechanical components) might require preventive maintenance or replacement after a certain period.

* DFR: Products with decreasing failure rates (e.g., some electronic components) might have a "burn-in" period where early failures occur, followed by a more reliable phase.

* NBUE: This class of distributions (encompassing IFR and DFR) is preferred for quality control as it implies reliability that either stays constant or improves over time.

Estimating parameters of an NBUE distribution allows quality control professionals to:

* Quantify the failure rate: Determine the probability of failure at a specific time.

* Set control limits: Establish thresholds for failure rates to identify potential quality issues.

* Monitor process stability: Track changes in failure rates over time to ensure the manufacturing process remains within acceptable limits.