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Unit-III

Renewal Processes

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UNIT – III

RENEWAL PROCESSES

Introduction to Renewal Processes

A renewal process was introduced as a generalization of Poisson processes. A renewal process is the increasing sequence of random nonnegative numbers S_0, S_1, S_2, . . . gotten by adding i.i.d. positive random variables X_0, X_1, . . . , that is,

$$S_n = S_0 + \sum_{i=1}^{n} X_i$$

When $S_0 = 0$ the renewal process is an ordinary renewal process; when S_0 is a nonnegative random variable the renewal process is a delayed renewal process. In either case, the individual terms S_n of this sequence are called renewals, or sometimes occurrences.

Definition:

A random process $\{N(t), t \geq 0\}$ is a non-negative integer valued stochastic process that registers the successive occurrences of an event during a time interval (0, t], where the time durations between consecutive "events" are positive i. i. d. random variables.

Example:

Consider a person with a phone. Let us associate renewal process for his phone. Let us assume that he will use his phone only for 15 minutes in a day. In that 15 minutes he get any number of calls when he talks with someone through phone that is considered to be dead and it will be renewed only after he ends a calls. Now, at the beginning of the 15 minutes he doesn't gets any call. So at t(0) = 0. There is no renewal say at time 1 minute he gets a call and he talks to that person for 5 minutes. The phone is failed for the 5 minutes and after the immediate end of call the phone is renewed at 5 minutes, say t(1) = 1 renewal.

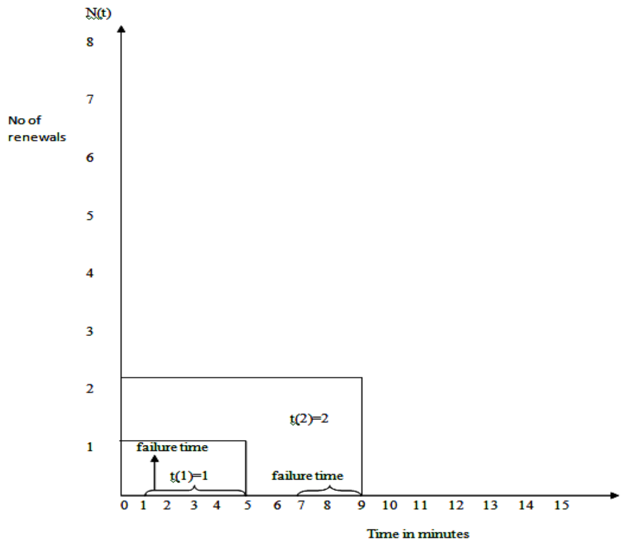
Again we receive a call at the minutes and the call prevails for 2 minutes and at the 9^{th} minutes the second renewal occurs. It is denoted as t(2) = 2.

N(2) = 2

$\sum X_i$ = number of failures and renewal.

= t(1) and t(2) = 2

$\therefore N(t) = 2.$



Renewal Process in Discrete and Continuous Time

The statistical mechanism governing the renewal events can be described in the following manner. Consider the non-negative random variable X, which for the purposes of discussion, is called the failure time of a component. This variable is the length of time between renewal events. The distinction between the continuous time and the discrete time theory is made as follows:

(a) The random variable has a continuous distribution over the range $(0,\infty)$, its distribution being determined by a probability density function, f(x). This is the continuous renewal process.

(b) There is a constant, T, such that the only possible values of X are (T, 2T, ...). The process is determined by its gap length distribution, p(j), which is the probability that X =jT. This latter case is the discrete renewal process.

Renewal Process in Continuous Time Definition:

Let $\{X_n,\ n = 1,2,\ldots.\}$ be a sequence of nonnegative independent random variables. Assume that $P\{X_n = 0\} < 1$ and the random variables are independent and identically distributed with a distribution function F(t). Since X_n is nonnegative it follows that $E(X_n)$ exists and is,

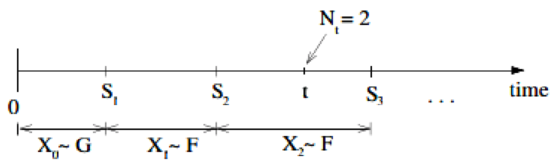
$$E(X_n) = \int_0^\infty x dF(x) = \mu$$

where μ may be infinite. Whenever μ is infinite, 1/μ shall be interpreted as 0.

Renewal Interval

Be $(X_n : n \in N_0)$ a sequence of independent positive random variables, and assume that $(X_n : n \in N)$ are identically distributed. Define the sequence $S = (S_n : n \in N)$ by $S_1 = X_0$ and

$S_{n+1} = S_n + X_n$ for all $n \in N$. The random variable S_n, with $n \in N$, is called the n^{th} renewal time, while the time duration X_n is called the n^{th} renewal interval. Further define the random variable of the number of renewals until time t by $N(t) = \max\{n \in N : S_n \leq t\}$ for all $t \geq 0$ with the convention $\max \emptyset = 0$. Then the continuous time process $N = (N(t) : t \in R_0^+)$ is called a renewal process. The random variable X_0 is called the delay of N. If X_0 and X_1 have the same distribution, then N is called an ordinary renewal process.



Random variables of a renewal process

Renewal Function

The function $M(t) = E\{N(t)\}$ is called the renewal function of process with distribution F. It is crystal clear that,

$$\{N(t) \geq n\} = \{s_n \leq t\} \text{ or } \{N(t) < n\} = \{S_n > t\}$$

Theorem

The distribution N(t) is given by,

$p_n(t) = P\{N(t) = n\} = P\{N(t) \geq n\} - P\{N(t) \geq n+1\} = F_n(t) - F_{n+1}(t)$,

and the expected number of renewals by,

$$M(t) = \sum_{n=1}^{\infty} F_n(t). \qquad \text{----------------- (1)}$$

Proof

Let, $P\{N(t) = n\} = P\{N(t) \geq n\} - P\{N(t) \geq n+1\}$

$$= P\{S_n \leq t\} - P\{S_{n+1} \leq t\} = F_n(t) - F_{n+1}(t)$$

Again,

$$M(t) = E\{N(t)\} = \sum_{n=1}^{\infty} n\, p_n(t)$$

$$= \sum_{n=1}^{\infty} n\, \{F_n(t) - F_{n+1}(t)\}$$

$$= \sum_{n=1}^{\infty} nF_n(t) - \sum_{n=1}^{\infty} nF_{n+1}(t)$$

$$= \sum_{n=1}^{\infty} F_n(t) - \sum_{n=1}^{\infty} (n-1)F_n(t)$$

$$= \sum_{n=1}^{\infty} nF_n(t) = \sum_{n=1}^{\infty} P\{s_n \leq t\}$$

Hence the proof.

Renewal Density:

The derivative *m(t)* of *M(t)* is called the renewal density. We have,

$$m(t) = \lim_{\Delta t \to 0} \frac{P\{one\,or\,more\,renewal\,in(t, t+\Delta t)\}}{\Delta t}$$

$$= \sum_{n=1}^{\infty} \lim_{\Delta t \to 0} \frac{P\{n\,renewal\,occures\,in\,(t, t+\Delta t)\}}{\Delta t}$$

$$= \sum_{n=1}^{\infty} \lim_{\Delta t \to 0} \frac{f_n(t)\,\Delta t + o(\Delta t)}{\Delta t} \quad = \sum_{n=1}^{\infty} f_n(t)$$

{Provided that F(x) is absolutely continues and $F_n'(t) = f_n(t)\}$

$$\sum_{n=1}^{\infty} f_n(t) = \sum_{n=1}^{\infty} F_n'(t) = M'(t)\,.$$

The function m(t) specifies the mean number of renewals to be expected in a narrow interval near t.

Renewal Equation

An integral equation can be obtained for the renewal function

$$m(t) = E\{N(t)\}$$

This gives the expected number of renewals in (0, t].

Theorem

The renewal function *m(t)* is satisfies the equation

$$m(t) = F(t) + \int_0^t M(t-x)dF(x).$$

Proof

By conditional expectation on the duration of the first renewal X_1, we get

$$m(t) = E\{N(t)\}$$

$$\int_0^{\infty} E\{N(t)|X_1 = x\}\, dF(x) \qquad \text{----------------- (2)}$$

Consider $0 \le x \le t$, given that the first renewal occurs at $x(\le t)$.

Then the process starts again at epoch x, the expected number of renewals in the remaining interval of length $(t - x)$, is $E\{N(t - x)\}$

$$E\{N(t)|X_1 = x\} = 1 + E\{N(t-x)\} = 1 + M(t-x)$$

Thus, considering the above the equation, we get

$$m(t) = \int_0^{\infty} \{1 + M(t-x)\} dF(x) \qquad \text{----------------- (3)}$$

$$= F(t) + \int_0^t M(t-x)dF(x)$$

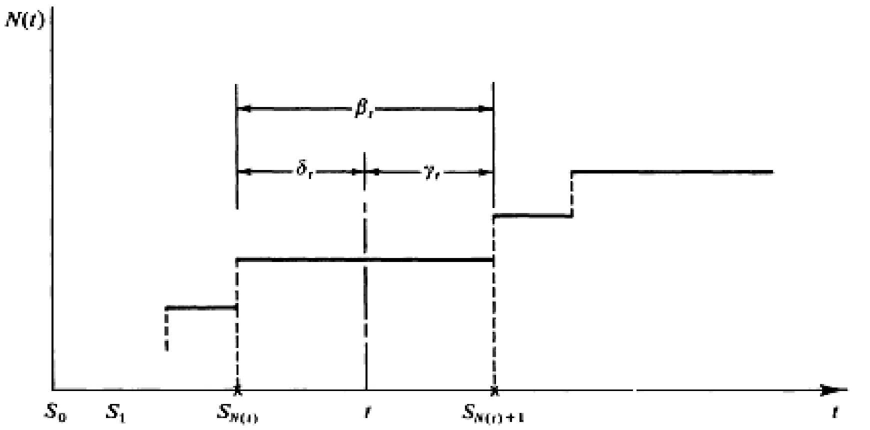
Hence proved.

Application of Renewal equation:

N(t) has Poisson distribution, P[N(t) = k] = $\frac{\mathbf{e^{-\lambda t}\ (\lambda t)^k}}{\mathbf{k!}}$, k= 0 ,1 , 2 , ... with mean M(t) = E[N(t)] = $\lambda t.$

Interest areas

- Excess life
- Current life
- Total life



The excess life γ_t, the current life δ_t, and the total life β_t.

Excess life:

The excess life at time t exceeds x if and only if there are no renewals in the interval (t, t+x). this event has the same probability as that of no intervals (0 , x), it's given by,

$$\gamma_i = S_{N(t)+1} - t$$

Current life:

The current life δ_t of course cannot exceed t, while for x < t , the current life exceeds x if and only if there are no renewal in (t-x , t). It is given by,

$$\delta_t = t - \delta_{N(t)}$$

Total life:

The total life of a product. It's given by,

$$\beta_t = \gamma_t + \delta_t$$

Stopping Time

A stopping time with respect to a sequence of random variables X_1 , X_2 , ... is a random variable T with property that for each t , the occurrence or non-occurrence of the event T = t depends only on the values of X_1 , X_2 , ... , X_t .

Wald's equation

If $X_1, X_2, X_3, \ldots$ be independent identically distributed with finite mean $E(X)$, and if N is a stopping time for $X_1, X_2, \ldots$ such that $E[N] < \infty$, then $E[X_1 + \cdots + X_n] = E[X]\,E[N]$.

Proof

Let $I_n = \begin{matrix} 1, & if, \ N \geq n \\ 0, & Otherwise \end{matrix}$

$$E\left(\sum_{n=1}^{N} X_n\right) = E\left(\sum_{i=1}^{\infty} X_n \,.\, I_n\right)$$

$$= \sum_{n=1}^{\infty} E(X_n . I_n)$$

X_n and I_n are independent.

$$E\left(\sum_{n=1}^{N} X_n\right) = \sum_{n=1}^{\infty} E(X_n)\ \ E(I_n)$$

Where,

$$\sum_{n=1}^{\infty} P(N \geq n) = \sum_{n=1}^{\infty}\sum_{k=n}^{\infty} P(N = n)$$

$$= \sum_{n=1}^{k}\sum_{k=1}^{\infty} P(N = n)$$

$$= \sum_{k=1}^{\infty} k . P(N = n)$$

$$= E(N)$$

Therefore,

$$E\left(\sum_{n=1}^{N} X_n\right) = E(X)E(N)$$

Hence prove.

Renewal theorems

Poisson process with parameter 'a' is a renewal process having exponential inter arrival time X_n with mean $\frac{1}{a}$, we have

$$M(t) = a \times t$$

$$\frac{M(t)}{t} = a = \frac{1}{E(X_n)}$$

In general,

$$\frac{M(t)}{t} \to \frac{1}{\mu}$$

$\mu = E(X_n) < \infty$ as $t \to \infty$ is known as elementary renewal theorem.

Theorem

If probability is one with interval (0, t], we have

$$\mu = E(X_n) \leq \infty$$

Proof

Consider an interval (0, t], we have

$$S_{N(t)} \leq t \leq S_{N(t)+1}$$

Now, this strong law of large numbers for the sequence $\{S_n\}$, so that as $n \to \infty$

$$\frac{S_n}{n} = \frac{X_1 + X_2 + \cdots + X_n}{n} \to E(X_n) = \mu$$

With probability one, again as $t \to \infty, N(t) \to \infty$.

With probability one, $\frac{S_{N(t)}}{N(t)} \to \mu \; as \; t \to \infty$.

Similarly, with probability one,

$$\frac{S_{N(t)+1}}{N(t)} = \frac{S_{N(t)+1}}{N(t)+1} \times \frac{N(t)+1}{N(t)} \to \mu \; as \; t \to \infty.$$

From the three relations, we get with probability one,

$$\frac{N(t)}{t} = \frac{1}{\mu} \; as \; t \to \infty.$$

Hence, proved that for large t, the number of renewals per unit time converges to $\frac{1}{\mu} \left[t = \frac{1}{\mu}\right]$.

Elementary Renewal Theorem

Let M(t) denote mean E[N(t)] of renewal process N(t), then under the hypotheses of basic renewal theorem, we have

$$\frac{M(t)}{t} \to \frac{1}{\mu} \; as \; t \to \infty$$

Where, *M(t) = E[N(t)]* is the renewal function.

Proof

We know that $S_{N(t)+1} > t$ and taking $\mu<\infty$. Therefore, taking expectations on both sides, we have

$$E[S_{N(t)+1}] = E[X_1]\,[M(t)+1]$$

We have,

$$E[S_{N(t)+1}] = \mu[M(t)+1] > t$$

Dividing both sides by μt and taking lim inf on both sides, we get

$$\liminf_{t\to\infty} \frac{M(t)}{t} \geq \frac{1}{\mu} \qquad (1)$$

A truncated random variable argument to show the reverse inequality. We define truncated inter-arrival times $\{\overline{X}_n\}$ as

$$\overline{X}_n = X_n I_{\{X_n \leq M\}} + M\, I_{X_n > M\}}$$

We define $E[\overline{X}_n] = \mu_M$, arrival instants $\{\overline{S}_n\}$ and renewal process $\overline{N}(t)$ for this set of truncated inter-arrival times $\{\overline{X}_n\}$ as

$$\overline{S}_n = \sum_{k=1}^{n} \overline{X}_k \text{ and } \overline{N}(t) = \sup\{n \in N_0 : \overline{S}_n \leq t\}$$

Since $S_n \geq \overline{S}_n$, the number of arrivals would be higher for renewal process $\overline{N}(t)$ with truncated random variables, i.e.

$$N(t) \leq \overline{N}(t) \qquad (2)$$

Further, due to truncation of inter-arrival time, next renewal happens with-in M units of time, i.e.

$$\overline{S}_{\overline{N}(t)+1} \leq t + M$$

Taking expectations on both sides in the above equation, using Wald's lemma for renewal processes, dividing both sides by $t\mu_M$, and taking lim sup on both sides, we obtain

$$\limsup_{t\to\infty} \frac{\overline{M}(t)}{t} \leq \frac{1}{\mu_M}$$

Taking expectations on both sides of (2) and letting M go arbitrary large on RHS, we get

$$\limsup_{t\to\infty} \frac{M(t)}{t} \leq \frac{1}{\mu} \qquad (3)$$

Inequalities (1) and (3) in conjunction imply

$$\lim_{t\to\infty} \frac{M(t)}{t} = \frac{1}{\mu},$$

Hence the proof.

Branching Processes

- A branching process is a type of mathematical object known as a stochastic process, which consists of collections of random variables. The random variables of a stochastic process are indexed by the natural numbers.
- The original purpose of branching processes was to serve as a mathematical model of a population in which each individual in generation produces some random number of individuals in generation, according, in the simplest case, to a fixed probability distribution that does not vary from individual to individual.
- Branching processes are used to model reproduction; for example, the individuals might correspond to bacteria, each of which generates 0, 1, or 2 offspring with some probability in a single time unit.
- Branching processes can also be used to model other systems with similar dynamics, e.g., the spread of surnames in genealogy or the propagation of neutrons in a nuclear reactor.

Assumption

- Probability same for all individuals
- Individuals reproduce independently
- Process starts with a single individual at time 0.

Types of branching processes

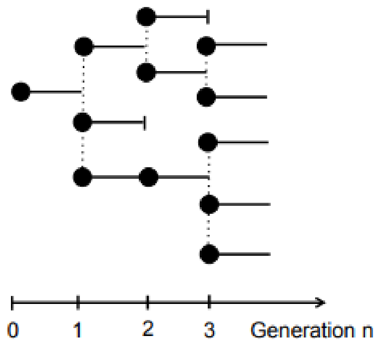
- Discrete time (Galton-Watson branching processes)
- Continuous time,
  - ✓ With exponential lifetime distributions (Markovian branching process), or
  - ✓ General lifetime distributions (age-dependent, Bellman-Harris branching process)
- Single type, or multitype (with finitely or ∞-ly many types)
- Individuals’ reproduction rules may depend on the actual size of the population (population size-dependent branching process)
- Branching processes can undergo catastrophes or live in a random environment.

Galton-Watson Branching Process

The Galton-Watson branching process (or GW-process for short) is the simplest possible model for a population evolving in time. It is based on the assumption that individuals in the population give birth to a number of children independently of each other and all with the same distribution. More precisely, the model can be described as follows:

- We start the process with a single individual, the zero generation of the population.
- This individual gives birth to a random number $X \in N_0$ of children, with $E(X) \in (0, +\infty)$. These children constitute the first generation of the population.
- Each of the individuals in this first generation has children of their own, all of them with the same distribution as X and independently of all the other individuals in the population. These constitute the second generation of the population. This generation in turn gives rise to a third generation of individuals by the same rules as the previous generation, and so on.

A realisation of a GW process through 3 generations starting with a single individual at generation 0:



If Z_n denotes the number of individuals in the n^{th} generation ($n \in N_0$), then Z_n satisfies the recurrence relation

$$Z_{n+1} = \sum_{i=1}^{Z_n} X_i^{(n)}$$

where:

- $Z_0 \equiv 1$ by convention,
- $\{ X_i^{(n)} : i \in N, n \in N_0\}$ is an array of i.i.d. random variables with $X_i^{(n)} \sim X$ for all i, n.

The sequence $(Z_n)_{n \in N_0}$ is what is typically known as the GW-process with offspring distribution X. However, sometimes by a GW-process we shall understand the entire genealogical tree induced by the population, i.e the collection $\{(n, i, X_i^{(n)}) : n \in N_0 , i \in \{1, . . . , Z_n\}\}$, and not just $(Z_n)_{n \in N_0}$. When is it that we mean one or the other will always be clear from the context.

Age and block replacement policies

Age:

In this policy the component is replaced when it completely fails. Excess life is enjoyed here. For example, we use the light bulbs as long as it works, usually even beyond its prescribed life time. The bulb fails to work so we change.

Block:

In this policy if the component fails for the first time at **T.** then at 2T, 3T it is always renewed. The excess life time won't be exercised here. For example, when the expiry date is reached by a medicine, it is always renewed it we throw the pills and buy a new one.

Properties of Generating Functions

Let $\{Z_n\}_{n\in N_0}$ be a branching process and let the generating function of its offspring distribution $\{p_n\}_{n\in N_0}$be given by P(s). Then the generating function of Z_n is the n-fold composition of P with itself, i.e.,

$$P_{Z_n}(s) = \underbrace{P(P(\ldots P(s)\ldots))}_{n\ P's},\ for\ n \geq 1.$$

Proof

For n = 1, the distribution of Z_1 is exactly $\{p_n\}_{n\in N_0}$, so$P_{Z_1} = P(s)$. Suppose that the statement of the proposition holds for some $n \in N$. Then

$$Z_{n+1} = \sum_{i=1}^{Z_n} Z_{i,n}$$

The random sum of Z_n independent random variables with pmf $\{p_n\}_{n\in N_0}$, where the number of summands Z_n is independent of $\{Z_{n,i}\}_{i\in N}$. We have seen that the generating function $P_{Z_{n+1}}$ of Z_{n+1} is a composition of the generating function P(s) of each of the summands and the generating function P_{Z_n} of the random time Z_n. Therefore,

$$P_{Z_{n+1}}(s) = P_{Z_n}(P(s)) = \underbrace{P(P(\ldots P(P(s))\ldots)))}_{n+1\ P's},$$

Hence proved.

Mean and Variance of Z_n

let $\{p_n\}_{n\in N_0}$ be a pmf of the offspring distribution of a branching process $\{Z_n\}_{n\in N_0}$. If $\{p_n\}_{n\in N_0}$ admits an expectation, i.e., if

$$\mu = \sum_{k=0}^{\infty} k p_k < \infty$$

then

$$E[Z_n] = \mu^n. \qquad (1)$$

If the variance of $\{p_n\}_{n \in N_0}$ is also finite, i.e., if

$$\sigma^2 = \sum_{k=0}^{\infty} (k-\mu)^2 p_k < \infty$$

then

$$Var[Z_n] = \sigma^2 \mu^n (1 + \mu + \mu^2 + \cdots + \mu^n)$$

$$= \sigma^2 \mu^n \frac{1-\mu^{n+1}}{1-\mu}, \quad \mu \neq 1$$

$$= \sigma^2 (n+1), \ \ \mu = 1 \qquad (2)$$

Proof:

Since the distribution of Z_1 is just $\{p_n\}_{n \in N_0}$, it is clear that $E[Z_1] = \mu$ and $Var[Z_1] = \sigma^2$. We proceed by induction and assume that the formulas (1) and (2) hold for $n \in N$.

$$P'_{Z_n}(s) = P'_{Z_{n-1}}(P(s))P'(s)$$

$$P'_{Z_n}(1) = P'_{Z_{n-1}}(1)P'(1)$$

$$= E[Z_{n-1}]E[Z_1]$$

$$= \mu^{n-1}\mu = \mu^n$$

Let $Var[Z_1] = \sigma^2$ and $Var(Z_n) = \sigma_n^2$

$$P'_{Z_n}(s) = P'_{Z_{n-1}}(P(s))P'(s)$$

$$P''_{Z_n}(s) = P''_{Z_{n-1}}\big(P(s)\big)P'(s)^2 + P'_{Z_{n-1}}(P(s))P''(s) \qquad (3)$$

Now, P(1) = 1, $P'(1) = \mu$, $P'_{Z_{n-1}}(1) = \mu^{n-1}$ and $P''(1) = \sigma^2 - \mu + \mu^2$.

Also, $\sigma_n^2 = P''_{Z_n}(1) + \mu_n - \mu_n^2$, we have

$$P''_{Z_n}(1) = \sigma_n^2 - \mu^n + \mu^{2n}$$

$$P''_{Z_{n-1}}(1) = \sigma_{n-1}^2 - \mu^{n-1} + \mu^{2n-2}$$

From (3),

$$P''_{Z_n}(1) = P''_{Z_{n-1}}(1)P'(1)^2 + P'_{Z_{n-1}}(1)P''(1)$$

$$\sigma_n^2 - \mu^n + \mu^{2n} = (\sigma_{n-1}^2 - \mu^{n-1} + \mu^{2n-2})\mu^2 + \mu^{n-1}(\sigma^2 - \mu + \mu^2)$$

$$\sigma_n^2 = \mu^2\sigma_{n-1}^2 + \mu^{n-1}\sigma^2$$

Leading to

$$\sigma_n^2 = \mu^{n-1}\sigma^2(1 + \mu + \mu^2 + \cdots + \mu^{n-1})$$

So, we have

$$= \sigma^2\mu^n \frac{1-\mu^{n+1}}{1-\mu}, \quad \mu \neq 1$$

$$= \sigma^2(n+1), \ \ \mu = 1$$

Hence proved.

Ultimate Extinction Probabilities

Let P(X = 0) = p(0) ≠ 0.

Let $\theta_n = P(n^{th}\ generation\ contains\ 0\ individuals)$

$$= P(n^{th}\ generation\ contains\ 0\ individuals)$$

$$\theta_n = P(Z_n = 0) = P_{Z_n}(s)$$

Now, P(extinct by n^{th} generation) = P(extinct by $(n-1)^{th}$) + P(extinct at n^{th}).

So, $\theta_n = \theta_{n-1}$ + P(extinct at n^{th})

$\Rightarrow \theta_n \geq \theta_{n-1}$.

Now, $P_{Z_n}(s) = P[P_{Z_{n-1}}(s)]$

$$P_{Z_n}(0) = P[P_{Z_{n-1}}(0)]$$

$$\theta_n = P[\theta_{n-1}]$$

θ_n is a non-decreasing sequence that is bounded above by 1 (it is a probability). Hence, by the monotone convergence theorem $\lim_{n \to \infty} \theta_n = \theta^*$ exists and $\theta^* \leq 1$.

Now, $\lim_{n \to \infty} \theta_n = \lim_{n \to \infty} \theta_{n-1}$, so θ^* satisfies

$$\theta = P(\theta), \theta \in [0, 1]$$

Consider,

$$P(\theta) = \sum_{x=0}^{\infty} p(x)\theta^x$$

P(0) = p(0) (> 0), and P(1) = 1, also, $P'(1) > 0$ and $\theta > 0, P''(0) > 0$, so P(θ) is a convex increasing function for $\theta \in [0, 1]$ and solutions of $\theta = P(\theta)$ are determined by slope of $P(\theta) at\ \theta = 1$, i.e., by $P'(1) = \mu$.

- If μ < 1 there is one solution at $\theta^* \leq 1$.

  ⇒ extinction is certain.
- If μ > 1 there are two solution: $\theta^* < 1\ and\ \theta^* = 1$. as θ_n is increasing, we want the smaller solution.

  ⇒ extinction is not certain.
- If μ = 1 solution is $\theta^* = 1$.

  ⇒ extinction is certain.

Example of extinction problem:

Consider a parent can produce at most two offspring. The extinction probability in each generation is:

$$d_m = p_0 + p_1 d_{m-1} + p_2(d_{m-1})^2$$

with $d_0 = 0$. For the ultimate extinction probability, we need to find d which satisfies $d = p_0 + p_1 d + p_2 d^2$. Taking as example probabilities for the numbers of offspring produced $p_0 = 0.1$, $p_1 = 0.6$, and $p_2 = 0.3$, the extinction probability for the first 20 generations is as follows:

Generation (1–10)	**Extinction probability**	***Generation (11–20)***	**Extinction probability**
1	0.1	*11*	0.3156
2	0.163	*12*	0.3192
3	0.2058	*13*	0.3221
4	0.2362	*14*	0.3244
5	0.2584	*15*	0.3262
6	0.2751	*16*	0.3276
7	0.2878	*17*	0.3288
8	0.2975	*18*	0.3297
9	0.3051	*19*	0.3304
10	0.3109	*20*	0.331

In this example, we can solve algebraically that $d = 1/3$, and this is the value to which the extinction probability converges with increasing generations.

Distribution of Total Number of Progeny

Let S be total progeny in a branching process $\{Z_n\}$, then

$$S = Z_0 + Z_1$$

Lemma 1

Let there be d types, and let $w^{(i)}(s)$ be the generating function for the total numbers of the various types in all generations, starting with one object of type i. Then the $w^{(i)}(s)$ satisfy the functional equations

$$w^{(i)}(s) = s_i f^{(i)} (w^{(1)}(s), \ldots , w^{(d)}(s)), i = 1, 2, \ldots, d$$

for d = 1, this reduces to

$$w(s) = s f (w(s))$$

Lemma 2

If the branching process starts with i_1 individuals of type *1*, i_2 individuals of type *2*,· · ·, i_d individuals of type d, then the generating function for the total numbers of the various types in all generations is given by

$$w(s) = (w^{(1)}(s))^{(i1)} \ldots (w^{(d)}(s))^{(id)}$$