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Unit-II

States and Chain

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UNIT – II

STATES AND CHAIN

Accessible

State j is accessible from state i, if $P_{ij}(n) > 0$ for some $n \geq 0$.

Communicate

If state j is accessible from state i and state i is accessible from state j, then states i and j are said to communicate. If state i communicate with state j and state j communicates with state k, then state i communicate with state k.

Class

The state may be partitioned into one or more separate classes such that those states that communicate with each other are in the same class.

Path

Given two states of i and j, a **path** from i to j is a sequence of transitions that begins in i and ends in j, such that each transition in the sequence has a positive probability of occurring.

Classification of States

In general m-state chain with states $E_1, E_2, \ldots, E_m$ and transition matrix

$$T = [\,P_{ij}\,]\,;\; 1 \leq i, j \leq m$$

For a homogeneous chain, recollect that P_{ij} is the probability that a transition occurs between E_i and E_j at any step or change of state in the chain. Classify some of the more common types of states which can occur in Markov chains.

- Absorbing state
- Periodic state
- Persistent state
- Transient state
- Ergodic state

Absorbing state

Once entered the state there is no escape from an absorbing state. An absorbing state E_i is characterized by the probabilities

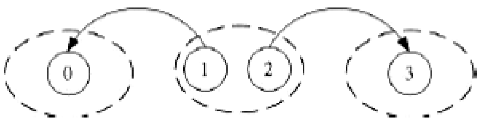
$$P_{ii} = 1, \quad P_{ij} = 0, \qquad i \neq j, \quad j = 1, 2, \cdots, m$$

in the i^{th} row of T .

In other words, a state is said to be an absorbing state if, upon entering this state, the process never will leave this state again. Therefore, state i is an absorbing state if and only if $P_{ii} = 1$.

Example:

$$
\mathbf{P} = \begin{array}{c|cccc}
\text{state} & 0 & 1 & 2 & 3 \\
\hline
0 & 1 & 0 & 0 & 0 \\
1 & 1-p & 0 & p & 0 \\
2 & 0 & 1-p & 0 & p \\
3 & 0 & 0 & 0 & 1
\end{array}
$$



Periodic state

A state *i* is **periodic** with period k > 1 if k is the smallest number such that all paths leading from state *i* back to state *i* have a length that is a multiple of k. If a recurrent state is not periodic, it is referred to as **aperiodic**.

Example

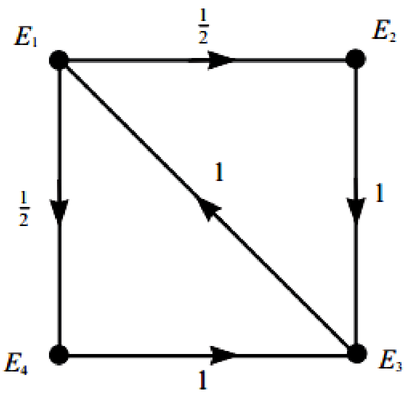
A four-state Markov chain has the transition matrix

$$T = \begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Show that all states have period 3.

Solution

The transition diagram is shown in the figure from which it is clear that all states have period 3. If the chain starts in E1, then returns to E1 are only possible at steps 3, 6, 9, . . . , either through E_2 or E_3.which means that all states are period 3.



Persistent state

Recurrence Time T_{ii} is the first time that the Markov chain returns to state i.

Example

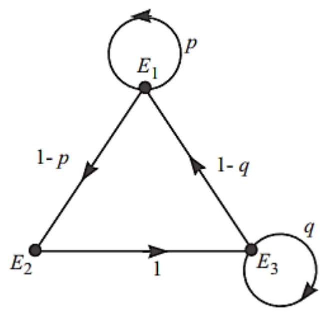
A three-state Markov chain has the transition matrix

$$T = \begin{bmatrix} p & 1-p & 0 \\ 0 & 0 & 1 \\ 1-q & 0 & q \end{bmatrix}$$

where $0 < p < 1$, $0 < q < 1$. Show that the state E_1 is persistent.

Solution

The transition diagram is



Sequence starts in E_1, then it can be seen that first returns to E_1 can be made to E_1 at every step except for n = 2, since after two steps the chain must be in state E_3. From the figure it can be argued that

$$f_1^{(1)} = p, \qquad f_1^{(2)} = 0, \qquad f_1^{(3)} = (1-p)\cdot 1 \cdot (1-q),$$

$$f_1^{(n)} = (1-p).1.q^{n-3}.(1-q), \quad (n \geq 4).$$

The last result for $f_1^{(n)}$ for $n \geq 4$ follows from the following sequence of transitions:

$$E_1\ E_2\ \overbrace{E_3\ E_3\ \cdots\ E_3}^{(n-3)\text{ times}}\ E_1.$$

The probability f_1 that the system returns at least once to E_1 is

$$\begin{aligned} f_1 = \sum_{n=1}^{\infty} f_1^{(n)} &= p + \sum_{n=3}^{\infty}(1-p)(1-q)q^{n-3}, \\ &= p + (1-p)(1-q)\sum_{s=0}^{\infty} q^s, \qquad (s = n-3) \\ &= p + (1-p)\frac{(1-q)}{(1-q)} = 1, \end{aligned}$$

Using the sum formula for the geometric series. Hence $f_1 = 1$, and consequently the state $E_1$is persistent.

Transient state

For a persistent state the probability of a first return at some step in the future is certain. For some states,

$$f_j = \sum_{n=1}^{\infty} f_j^{(n)} < 1$$

which means that the probability of a first return is not certain. Such states are described as transient.

Example

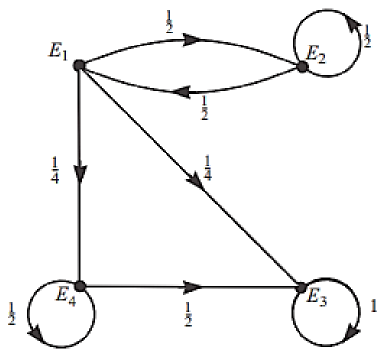
A four-state Markov chain has the transition matrix

$$T = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Show that E_1 is a transient state.

Solution

The transition diagram is



$$f_1^{(1)} = 0, \quad f_1^{(2)} = \tfrac{1}{2} \cdot \tfrac{1}{2} = (\tfrac{1}{2})^2, \quad f_1^{(3)} = (\tfrac{1}{2})^3, \quad f_1^{(n)} = (\tfrac{1}{2})^n.$$

Hence,

$$f_1 = \sum_{n=1}^{\infty} f_1^{(n)} = \sum_{n=2}^{\infty} (\tfrac{1}{2})^n = \tfrac{1}{2} < 1,$$

Implying that E_1 is a transient state. The reason for the transience of E_1 can be seen from the Figure, where transitions from E_3 or E_4 to E_1 or E_2 are not possible.

Ergodic state

An important state which we will return to in the next section is the state which is persistent, non-null, and aperiodic. This state is called Ergodic.

Example

A three-state Markov chain has the transition matrix

$$T = \begin{bmatrix} p & 1-p & 0 \\ 0 & 0 & 1 \\ 1-q & 0 & q \end{bmatrix}$$

where $0 < p < 1$, $0 < q < 1$. Show that the state E_1 is Ergodic.

Solution

The state E_1 was persistent with

$$f_1^{(1)} = p, \qquad f_1^{(2)} = 0, \qquad f_1^{(n)} = (1-p)(1-q)q^{n-3}, \quad (n \geq 3).$$

It follows that its mean recurrence time is

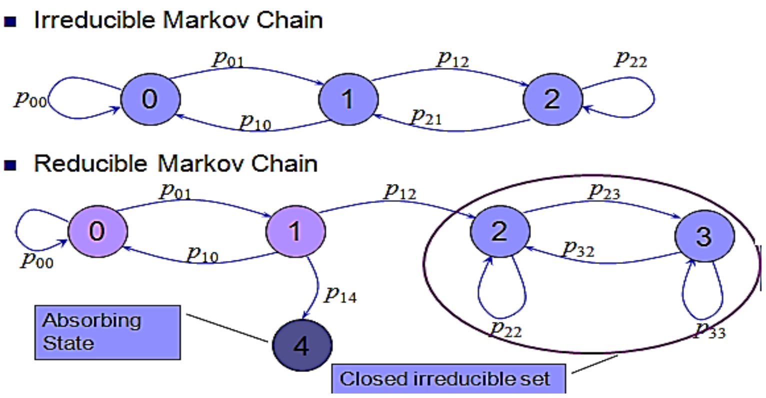
$$\mu_1 = \sum_{n=1}^{\infty} n f_1^{(n)} = p + (1-p)(1-q)\sum_{n=3}^{\infty} n q^{n-3} = \frac{3-2q}{(1-q)^2} < \infty.$$

The convergence of μ_1 implies that E_1 is non-null. Also, the diagonal elements $p_{ii}^{(n)} > 0$ forn ≥ 3 and i = 1, 2, 3, which means that E_1 is aperiodic. Hence from the definition above, E_1 (and E_2 and E_3 also) is Ergodic.

Classification of chains

Irreducible chains

An irreducible chain is one in which every state can be reached or is accessible from every other state in the chain in a finite number of steps. That any state E_j can be reached from any other state E_i means that $p_{ij}^{(n)} > 0$ for some integer n. This isalso referred to as communicating states.



Example

Show that the three-state chain with transition matrix

$$T = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

defines a regular (and hence irreducible) chain

Solution

For the transition matrix T

$$T^2 = \begin{bmatrix} \frac{4}{9} & \frac{1}{9} & \frac{4}{9} \\ 1 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \text{ and } T^3 = \begin{bmatrix} \frac{16}{27} & \frac{4}{27} & \frac{7}{27} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{4}{9} & \frac{1}{9} & \frac{4}{9} \end{bmatrix}$$

Hence $T^3$is a positive matrix, which means that the chain is regular.

Closed sets

A set of states S in a Markov chain is a closed set if no state outside of S is reachable from any state in S, then S is said to be a closed set.

Then $p_{ij} = 0, \; \forall \, E_i \; \in S \; \; and \; \; \forall \, E_j \notin S.$

Ergodic chains

As we have seen, all the states in an irreducible chain belong to the same class. If all states are ergodic, that is, persistent, non-null, and aperiodic, then the chain is described as an ergodic chain.

Example

Show that all states of the chain with transition matrix

$$T = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Show that the chains are Ergodic.

Solution

The chain was shown to be irreducible and regular, which means that all states must be persistent, non-null, and aperiodic. Hence all states are ergodic.

Random Walk

A one-dimensional random walk is a Markov chain whose state space is a finite or infinite subset a, a + 1, . . . , b of the integers, in which the particle, if it is in state i, can in a single transition either stay in i or move to one of the neighbouring states i - 1, i + 1. If the state space is taken as the nonnegative integers, the transition matrix of a random walk has the form

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ i \\ \\ \end{array} \begin{array}{c} \begin{array}{cccccccc} 0 & 1 & 2 & & i-1 & & i & i+1 \end{array} \\ \left\| \begin{array}{cccccccccc} r_0 & p_0 & 0 & \cdots & 0 & \cdots & & & & \\ q_1 & r_1 & p_1 & \cdots & 0 & \cdots & & & & \\ 0 & q_2 & r_2 & \cdots & 0 & \cdots & & & & \\ & & \ddots & & & & & & & \\ & & 0 & & q_i & & r_i & p_i & 0 & \\ & & & \ddots & & & & & & \ddots \end{array} \right\| \end{array}$$

Where, $p_i > 0$, $q_i > 0$, $r_i \geq 0$ and $q_i + r_i + p_i = 1$, i = 1, 2, ... ($i \geq 1$),

$p_0 \geq 0$, $r_0 \geq 0$, $r_0 + p_0 = 1$. Specifically, if $X_n = i$, then for $i \geq 1$,

$$P\{X_{n+1} = i + 1 | X_n = i\} = p_i$$

$$P\{X_{n+1} = 1 - 1 | X_n = i\} = q_i$$

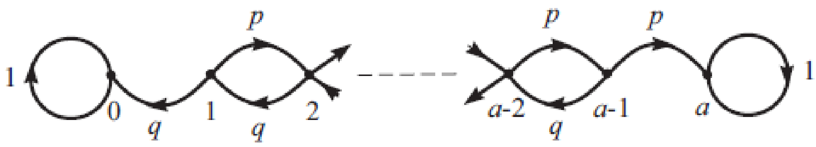
$$P\{X_{n+1} = i | X_n = i\} = r_i$$

with the obvious modifications holding for i = 0.

The designation "random walk" seems apt, since a realization of the process describes the path of a person (suitably intoxicated) moving randomly one step forward or backward.

Suppose that $a+1$ positions are marked out on a straight line and numbered $0, 1, 2, \ldots, a$. A person starts at k where $0 < k < a$. The walk proceeds in such a way that at each step there is a probability p that the walker goes ‘forward’ one place to $k+1$, and a probability $q = 1-p$ that the walker goes ‘back’ one place to $k - 1$. The walk continues until either 0 or a is reached, and then ends. Generally, in a random walk, the position of a walker after having moved n times is known as the state of the walk after n steps or after covering n stages. Thus the walk described above starts at stage k at step 0 and moves to either stage $k - 1$ or stage $k + 1$ after 1 step, and so on. A random walk is said to be symmetric if p = q = ½.

If the walk is bounded, then the ends of the walk are known as barriers, and they may have various properties. In this case the barriers are said to be absorbing, which implies that the walk must end once a barrier is reached since there is no escape. On the other hand, the barrier could be reflecting, in which case the walk returns to its previous state. A useful diagrammatic way of representing random walks is by a transition or process diagram as shown in Figure 1. In a transition diagram the possible stages of the walker can be represented by points on a line. If a transition between two points can occur in one step, then those points are joined by a curve or edge, as shown with an arrow indicating the direction of the walk and a *weighting* denoting the probability of the step occurring. In discrete mathematics or graph theory the transition diagram is known as a directed graph. A walk in the transition diagram is a succession of edges covered without a break. In the following Figure, the closed loops with weightings of 1 at the ends of the walk indicate the absorbing barriers with no escape.



Transition diagram for a random walk with absorbing barriers at each end of the walk

Gambler's Ruin Problem

Consider a game of chance between two players: A, the gambler and B, the opponent. It is assumed that at each play, A either wins one unit from B with probability p or loses one unit to B with probability q = 1 − p. Conversely, B either wins from A or loses to A with probabilities q or p. The result of every play of the game is independent of the results of previous plays. The gambler A and the opponent B each start with a given number of units and the game ends when either player has lost his or her initial stake. What is the probability that the gambler loses all his or her money or wins all the opponent's money, assuming that an unlimited number of plays are possible? This is the classic gambler's ruin problem1. In a simple example of gambler's ruin, each play could depend on the spin of a fair coin, in which case p = q = ½. The word *ruin* is used because if the gambler plays a fair game against a bank or casino with unlimited funds, then the gambler is certain to lose. The problem will be solved by using results from conditional probability, which then leads to a difference equation. There are other questions associated with this problem, such as how many plays are expected before the game finishes. In some games the player might be playing against a casino which has a very large (effectively infinite) initial stake.

Markov Process with discrete state space

Discrete state space Markov Processes has many applicants in day to day processes, such as inventory control in business, queuing systems and reliability theory. Poisson process is a versatile process which represents almost all random processes whose values move on a discrete space. The inter success time or inter-arrival time between two notified events are assumed to be exponential with parameter λ.

Poisson Processes

Poisson is a special kind of Markov process with exponential inter arrival time. It is a stochastic process in continuous time with discrete state space which plays a vital role in modelling real life systems.

Consider a random event such as incoming telephone calls, arrival of customer for services at a counter and occurrence of accidents at a certain places etc.

Let us denote *N(t)* the number of occurrence of the event *E* in an interval of duration *t*. That is *N(t)* denote the number of events *E* occurred up to time epoch t. Then *{N(t): t ≥ 0}* is a counting process with time space R^+.

Let $p_n(t)$ is the probability that the random variable N(t) assumes the value n.

i.e., $p_n(t) = P\{N(t) = n\}$

This probability is a function of time t and $\sum_{n=0}^{\infty} p_n(t) = 1$, where *P{N(t)}* represent probability distribution of the random variable *N(t)* for every value of t.

The family of random variables, *{N(t): t ≥ 0}* is a stochastic process. Now we proceed to show that *N(t)* follows a Poisson distribution with parameter λ, the mean is λt. Hence the stochastic process, *{N(t): t ≥ 0}* is a Poisson process.

Postulates of Poisson Processes

- **Independence**

  The random variable, *(t+h)–N(t)*, the number of occurrences in the interval *(t, t+h)* is independent of the number of occurrences prior to that interval.

- **Homogeneity in time**

  $p_n(t)$ depends only on the length t of the interval and is independent of the position of the interval. That is $p_n(t)$ = Pr {number of occurrence of event E in the interval (t_1, t_1+t)}

- **Regularity**

  In an interval of infinitesimal length h, the probability of exactly one occurrence is *λh + o(h)* and that of more than one occurrence is *o(h)*.

  Here *o(h)* is defined as $\lim_{h \to 0} \frac{o(h)}{h} = 0$.

  In other words, if the interval between *t* and *t + h* is of very short duration *h*, then

$$p_1(h) = \lambda h + o(h)$$

$$\sum_{k=2}^{\infty} p_k(h) = o(h)$$

  Since, $\sum_{n=0}^{\infty} P_n(h) = 1$, It follows that

$$p_0(h) = 1 - \lambda h + o(h) \qquad \text{----------------} \ (1)$$

Theorem

Under the postulates of independence, Homogeneity in time and Regularity, the random variable *N(t)* follows Position distribution with mean λt. That is $p_n(t)$ is given by the Position law:

$$p_n(t) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}, \quad n = 0, 1, 2, \ldots\ldots. \qquad \text{---------------- (2)}$$

Proof

Consider $p_n(t+h)$ for $n \geq 0$:

The n events by epoch t + h can happen in the following mutually exclusive events

$A_1, A_2, A_3, \ldots\ldots A_{n+1}$.

For $n \geq 1$

A_1: n occurrences by epoch t and no occurrence event between t and t + h;

We have,

$$\Pr(A_1) = \Pr\{N(t) = n\}\Pr\{N(h) = 0 | N(t) = n\} \qquad \text{---------------- (3)}$$

$$= p_n(t)\, p_0(h)$$

$$= p_n(t)\,(1 - \lambda h) + o(h)$$

A_2: (n – 1) occurrences by t and 1 occurrences between t and t + h;

We have,

$$\Pr(A_2) = \Pr\{N(t) = n-1\}\Pr\{N(h) = 1 | N(t) = n-1\} \qquad \text{---------------- (4)}$$

$$= p_{n-1}(t)\, p_1(h)$$

$$= p_{n-1}(t)\,(\lambda h) + o(h)$$

For $n \geq 2$

A_3: (n – 2) occurrences by epoch t and 2 occurrences between t and t + h;

We have,

$$\Pr(A_3) = p_{n-2}(t)\{p_2(h)\} \leq p_2(h)$$

Same result holds for $\Pr(A_4), \Pr(A_5), \ldots\ldots\ldots$

Thus we have

$$\sum_{k=2}^{n} \Pr\{A_{k+1}\} \leq \sum_{k=2}^{n} p_k(h) = o(h)$$

And so

$$p_n(t+h) = p_n(t)(1-\lambda h) + p_{n-1}(t)\,(\lambda h) + o(h),\ \ n \geq 1$$

$$\frac{p_n(t+h)-p_n(t)}{h} = -\lambda p_n(t) + \lambda p_{n-1}(t) + \frac{o(h)}{h} \qquad \text{---------------- (5)}$$

taking limit, as h → 0, we get

$$p_n'(t) = -\lambda[p_n(t) - p_{n-1}(t)],\ n \geq 1 \qquad \text{---------------- (6)}$$

For n = 0, we get

$$p_0(t+h) = p_0(t)p_0(h) = p_0(t)(1 - \lambda\text{h}) + \text{o(h)}$$

$$\frac{p_0(t+h) - p_0(t)}{h} = -\lambda p_0(t) + \frac{o(h)}{h}$$

Hence, as h → 0, $p_0'(t) = -\lambda p_0(t)$. ---------------------- (7)

Suppose that the process starts from scratch at time 0, so that N (0) = 0, i. e.

$$p_0(0) = 1$$

$$p_n(0) for\ n \neq 0. \qquad \text{---------------------- (8)}$$

The differential – difference equations (6) and the differential equation (7) together with (8) completely specify the system. Their solutions give the probability distribution $\{p_n(t)\}$ of N(t). The solutions are given by

$$p_n(t) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}, n = 0,1,2, \ldots\ldots$$

Hence the proof.

Problem – 1: The Probability Law of Poisson Processes

Suppose that customers arrive at a Bank according to a Poisson process with a mean rate of a minute. Then the number of customers N(t) arriving in an interval of duration t minutes follows Poisson distribution with mean at. If the rate of arrival is 3 per minute, then in an arrival of 2 minute and 4 minutes, find the probability that the number of customers arriving is:

(i) Exactly 4customers arrive,

(ii) Greater than 4customers arrive,

(iii) Less than 4customers arrive.

Procedure

- To identify the λ and t.
- To calculate the probability law of Poisson processes is

$$p_n(t) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$$

Calculation

Let N(t) denote the number of customers arrived during the interval (0, t).

Then N(t) follows the Poisson distribution.

Here, n = 4, λ = 3 /min and t = 2 /min and 4 /min

The Poisson processes is

$$p_n(t) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$$

(i) Exactly 4 customers arrive

P[Exactly 4 customers arrive during a time interval of two minutes]

Mean = λt = 3 × 2 = 6 /min

$$p_4(6) = \frac{e^{-6}(6)^4}{4!}$$

$$= \frac{3.2125}{4 \times 3 \times 2 \times 1}$$

$$p_4(6) = 0.1339$$

P[Exactly 4 customers arrive during a time interval of four minutes]

Mean = λt = 3 × 4 = 12 /min

$$p_4(12) = \frac{e^{-12}(12)^4}{4!}$$

$$= \frac{0.1274}{4 \times 3 \times 2 \times 1}$$

$$p_4(12) = 0.0053$$

(ii) Greater than 4 customers arrive

P[Greater than 4 customers arrive during a time interval of two minutes]

P{N(2) > 4} = 1 – P{N(2) ≤ 4}

= 1 – {P(N(t) = 0) + P(N(t) = 1) + P(N(t) = 2) + P(N(t) = 3) + P(N(t) = 4)}

$$\mathrm{P}\{\mathrm{N}(\mathrm{t}) > 4\} = \sum_{n=0}^{4} \frac{e^{-6}(6)^4}{4!}$$

$$= 1 - \left\{\frac{e^{-6}(6)^0}{0!} + \frac{e^{-6}(6)^1}{1!} + \frac{e^{-6}(6)^2}{2!} + \frac{e^{-6}(6)^3}{3!} + \frac{e^{-6}(6)^4}{4!}\right\}$$

$$= 1 - e^{-6}\{1 + 6 + 18 + 36 + 54\}$$

= 1 – 0.285

P{N(2) > 4} = 0.715

P[Greater than 4 customers arrive during a time interval of four minutes]

P{N(4) > 4} = 1 – P{N(4) ≤ 4}

= 1 – {P(N(t) = 0) + P(N(t) = 1) + P(N(t) = 2) + P(N(t) = 3) + P(N(t) = 4)}

$$\mathrm{P}\{\mathrm{N}(\mathrm{t}) > 4\} = \sum_{n=0}^{4} \frac{e^{-12}(12)^4}{4!}$$

$$= 1 - \left\{\frac{e^{-12}(12)^0}{0!} + \frac{e^{-12}(12)^1}{1!} + \frac{e^{-12}(12)^2}{2!} + \frac{e^{-12}(12)^3}{3!} + \frac{e^{-12}(12)^4}{4!}\right\}$$

$= 1 - e^{-12}\{1 + 12 + 72 + 288 + 864\}$ = 1 – 0.0076

P{N(4) > 4} = 0.9924

(iii) Less than 4customers arrive

P[Less than 4 customers arrive during a time interval of two minutes]

$$\mathrm{P}\{\mathrm{N}(2) < 4\} = \sum_{n=0}^{3} \frac{e^{-6}(6)^3}{3!}$$

$$= \left\{\frac{e^{-6}(6)^0}{0!} + \frac{e^{-6}(6)^1}{1!} + \frac{e^{-6}(6)^2}{2!} + \frac{e^{-6}(6)^3}{3!}\right\}$$

$$= e^{-6}\{1 + 6 + 18 + 36\}$$

$$\mathrm{P}\{\mathrm{N}(2) < 4\} = 0.1512$$

P[Less than 4 customers arrive during a time interval of four minutes]

$$\mathrm{P}\{\mathrm{N}(4) < \ 4\} = \sum_{n=0}^{3} \frac{e^{-12}(12)^3}{3!}$$

$$= \left\{\frac{e^{-12}(12)^0}{0!} + \frac{e^{-12}(12)^1}{1!} + \frac{e^{-12}(12)^2}{2!} + \frac{e^{-12}(12)^3}{3!}\right\}$$

$$= e^{-12}\{1 + 12 + 72 + 288\}$$

$$\mathrm{P}\{\mathrm{N}(4) < \ 4\} = 0.0023$$

Result

The probability that the number of customers arriving is

(i) Exactly 4 customers arrive

$$p_4(6) = 0.1339$$

$$p_4(12) = 0.0053$$

(ii) Greater than 4 customers arrive

P{N(2) > 4} = 0.715

P{N(4) > 4} = 0.9924

(iii) Less than 4customers arrive

P{N(2) < 4} = 0.1512

P{N(4) < 4} = 0.0023

Problem– 2

A machine goes out of order whenever a component part fails. The failure of this part is in accordance with a Poisson process with mean rate of 1 per week. Then the probability that two weeks have elapsed since the last failure is $e^{-2} = 0.135$, being the probability that time t = 2 weeks, the number of occurrences is 0. Suppose that there are 5 spare parts of the component in an inventory and that the next supply is not due in 10 weeks. What is the probability that the machine will not be out of order in the next 10 weeks?

Solution

Here, $\lambda = 5$ spare parts and t = 2 weeks

Mean = $\lambda t = 5\times 2 = 10$ weeks

The Poisson processes is

$$p_n(t) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$$

The probability that the number of failures in t = 10 weeks will be less than or equal to 5

$$p_n(2) = \sum_{n=0}^{5} \frac{e^{-10}(10)^n}{n!}$$

$$= \left\{\frac{e^{-10}(10)^0}{0!} + \frac{e^{-10}(10)^1}{1!} + \frac{e^{-10}(10)^2}{2!} + \frac{e^{-10}(10)^3}{3!} + \frac{e^{-10}(10)^4}{4!} + \frac{e^{-10}(10)^5}{5!}\right\}$$

$= e^{-10}$ {1 + 10 + 50 + 166.67 + 416.67 + 833.33}

$= e^{-10}$ {1477.67}

= 0.068

The probability that the number of failures in t = 10 weeks will be less than or equal to 5 is 0.068.

Pure Birth Processes:

A natural generalization of the Poisson process is to permit the chance of an event occurring at a given instant of time to depend upon the number of events which have already occurred. An example of this phenomenon is the reproduction of living organisms (and hence the name of the process), in which under certain conditions – sufficient food, no mortality, no migration, etc, the probability of a birth at a given instant is proportional (directly) to the population size at the time.

Here the probability that k events occurs between t and t+h, given that n event occurred by epoch t is given by

$$p_k(h) = p(N(h) = k | N(t) = n) = \begin{cases} \lambda_n(h) + o(h) & k = 1 \\ o(h) & k \geq 2 \\ 1 - \lambda_n(h) + o(h) & k = 0 \end{cases}$$

The equation can be

$$p_n(t+h) = p_n(t)\{1 - \lambda_n \mathrm{h} + p_{n-1}(t)\{\lambda_{n-1}\mathrm{h}\} + \mathrm{o(h)}\}, n \geq 1$$

And taking limits, as $h \to 0$, we have

$$p_1(t) = -\lambda_n p_n(t) + \lambda_{n-1} p_{n-1}(t), \;\; n \geq 1$$

$$p_0(t) = -\lambda_0 p_0(t)$$

This process is called pure births process.

Pure Birth and Death Processes

Consider a pure birth process, where probability that the Number of births between t and t+h is k, given the number of individuals at epoch t is n is given by

$$p(k\,, h \mid n, t) = \begin{cases} \lambda_n(h) + o(h) & k = 1 \\ o(h) & k \geq 2 \\ 1 - \lambda_n(h) + o(h) & k = 0 \end{cases} \qquad \text{--------------- (1)}$$

Probability that the Number of deaths between t and t+h is k, given the number of individuals at epoch t is n is given by

$$p(k\,, h \mid n, t) = \begin{cases} \mu_n(h) + o(h) & k = 1 \\ o(h) & k \geq 2 \\ 1 - \mu_n(h) + o(h) & k = 0 \end{cases} \qquad \text{--------------- (2)}$$

Consider, $A_{ij} : (n - i + j)$ individuals by epoch t, i birth and j death between t and t + h, i, j = 0 , 1.

We have

$$p(A_{00}) = p_n(t)\{1 - \lambda_n \mathrm{h} + \mathrm{o(h)}\}\{1 - \mu_n \mathrm{h} + \mathrm{o(h)}\}$$

$$= p_n(t)\{1 - (\lambda_n + \mu_n)\mathrm{h} + \mathrm{o(h)}\}$$

$$p(A_{10}) = p_{n-1}(t)\{\lambda_{n-1}\mathrm{h} + \mathrm{o(h)}\}\{1 - \mu_{n-1}\mathrm{h} + \mathrm{o(h)}\}$$

$$= p_{n-1}(t)\{\lambda_{n-1}\mathrm{h} + \mathrm{o(h)}\}$$

$$p(A_{01}) = p_{n-1}(t)\{1 - \lambda_{n+1}\mathrm{h} + \mathrm{o(h)}\}\{\mu_{n+1}\mathrm{h} + \mathrm{o(h)}\}$$

$$= p_{n+1}(t)\{\mu_{n+1}\mathrm{h} + \mathrm{o(h)}\}$$

$$p(A_{11}) = p_n(t)\{\lambda_n h + o(h)\}\{\mu_n h + o(h)\} = o(h)$$

Hence, we have, for n ≥ 1,

$$p_n(t+h) = p_n(t)\{1 - (\lambda_n + \mu_n)h + p_{n-1}(t)\{\lambda_{n-1}h\} + p_{n+1}(t)\{\mu_{n+1}h\} + o(h)\} \text{ -------- } (3)$$

$$\frac{p_n(t+h) - p_n(t)}{h} = -(\lambda_n + \mu_n)p_n(t) + \lambda_{n-1}p_{n-1}(t) + \mu_{n+1}p_{n+1}(t) + \frac{o(h)}{h}$$

And taking limits, as $h \to 0$, we have

$$p_n'(t) = -(\lambda_n + \mu_n)p_n(t) + \lambda_{n-1}p_{n-1}(t) + \mu_{n+1}p_{n+1}(t),\ \ n \geq 1 \text{ ----------------------- } (4)$$

For n = 0, we have

$$p_0(t+h) = p_0(t)\{1 - (\lambda_0 h) + o(h)\} + p_1(t)\{1 - \lambda_0 h + o(h)\}\{\mu_n h + o(h)\} \text{----------- } (5)$$

$$= p_0(t) - \lambda_0 h p_0(t) + \{\mu_n h\}\, p_1(t)$$

$$\frac{p_0(t+h) - p_0(t)}{h} = -\lambda_0\, p_0(t) + \mu_1\, p_1(t) + \frac{o(h)}{h}$$

Taken limit as h→ 0, we get

$$p_0'(t) = -(\lambda_0)p_0(t) + \mu_1 p_1(t) \text{ ----------------------- } (6)$$

If at epoch t = 0, there were i (≥ 0) individuals, then the initial condition is

$$p_n(0) = 0$$

$$for\ n \neq i, p_i(0) = 1. \qquad \text{---------------------- } (7)$$

The above equations (4) and (6) are the equations of the birth and death process.

Markov processes with continuous state space

Poisson processes are a real-life process with continuous time with discrete counting state space. But in most of the real-life problems Markov Problems have continuous state space. For example, level of water in a dam over a continuous time space, Life time of a electronic device over a continuous time are Stochastic processes with continuous state space. In mathematical term $\{X(t) : t \in T\}$where $T = (-\infty, \infty)$ and $X(t) \in (-\infty, \infty)$ is a stochastic process with continuous time space and continuous state space.

Weiner process

Consider that a (Brownian) particle performs a random walk such that in a small interval of time of duration Δt, the displacement of the particle to the right or to the left is also of small magnitude Δx, the total displacement X(t) of the particle in time t being x. Suppose that the random variable Z_i denotes the length of the i^{th} step taken by the particle in a small interval of time Δt and that

$$Pr\{Z_i = \Delta x\} = p \; and \; Pr\{Z_i = \Delta x\} = q, \qquad p + q = 1$$

0<p< 1, where *p* is independent of *x and t.*

Suppose that the interval of length t is divided into n equal subintervals of length Δt and that the displacements Z_i, i = 1, … ,n in the n^{th} steps are mutually independent random variables. Then n (Δt) = t and the total displacement X(t) is the sum of n, i.i.d. random variables Z_i, i.e.,

$$X(t) = \sum_{i=1}^{n(t)} Z_i \, , n \equiv n(t) \frac{t}{\Delta t}$$

We have, $E(Z_i) = (p-q)\Delta x \; and \; Var(Z_i) = 4pq\;(\Delta x)^2$.

Hence,$E\{X(t)\} = n\,E(Z_i) = t(p-q)\frac{\Delta x}{\Delta t}$, ------------------- (1)

And $Var\{X(t)\} = n\;var(Z_i) = \frac{4pqr(\Delta x)^2}{\Delta t}$.

To get meaningful result, as $\Delta x \to 0, \Delta t \to 0$, we must have

$\frac{(\Delta x)^2}{\Delta t} \to a\;limit, (p-q) \to a\;multiple\;of\;\Delta x.$ ------------------- (2)

We may suppose, in particular, that in an interval of length t, X(t) has mean value function equal to μt and variance function equal to σ^2t. In other words, we suppose that as Δx → 0, Δt → 0, such a way that (2) are satisfied, and per unit time

$E\{X(t)\} \;\to \mu \; and \; var\{X(t)\} \;\to\; \sigma^2,$ ------------------- (3)

From (1) for t = 1 and (3), we have

$$\frac{(p-q)\Delta x}{\Delta t} \;\to\; \mu;\; \frac{4pqr\;(\Delta x)^2}{\Delta t} \;\to\; \sigma^2. \qquad \text{------------------- (4)}$$

The relations (2) and (4) will be satisfied when

$$\Delta x = \;\sigma(\Delta t)^{\frac{1}{2}} \qquad \text{------------------- (5)}$$

$$p = \frac{1}{2}\left(1 + \frac{\mu(\Delta t)^{\frac{1}{2}}}{\sigma}\right) \; and \; q = \;\frac{1}{2}\left(1 - \frac{\mu(\Delta t)^{\frac{1}{2}}}{\sigma}\right) \qquad \text{------------------- (6)}$$

Now since Z_i are i. i. d. random variables, the sum $\sum_{i=1}^{n(t)} Z_i = X(t)$for large n(t) (=n), is asymptotically normal with mean μt and variance σ^2t.

We may now define a Wiener or a Brownian motion process as follows:

The stochastic process $\{X(t), t \geq 0\}$ is called a Wiener process (or a Wiener Einstein process or a Brownian motion process) with drift μ and variance parameter σ^2, if:

(i) X(t) has independent increments, i. e. for every pair of disjoint intervals of time (s, t) and (u, v), where $s \leq t \leq u \leq v$, the random variables {X(t) – X(s)} and {X(v) – X(u)} are independent.

(ii) Every increment {X(t) – X(s)} is normally distributed with mean μ(t - s) and variance σ^2(t).

Note that (i) implies that Wiener process is a Markov process with independent increments and (ii) implies that a Wiener process is Gaussian.

Stationary processes

Stationarity refers to time invariance of some, or all, of the statistics of a random process, such as mean, autocorrelation, n^{th}-order distribution

We can classify random processes based on many different criteria. One of the important questions that we can ask about a random process is whether it is a stationary process. Intuitively, a random process {X(t), t ∈ j} is stationary if its statistical properties do not change by time. For example, for a stationary process, X(t) and X(t+Δ) have the same probability distributions. In particular, we have

$$F_{X(t)}(x) = F_{X(t+\Delta)}(x), \qquad \text{for all t, t+ } \Delta \in \text{j}$$

More generally, for a stationary process, the joint distribution of $X(t_1)$ and $X(t_2)$ is the same as the joint distribution of $X(t_1+\Delta)$ and $X(t_2+\Delta)$. For example, if you have a stationary process X(t) then

$$P\big((X(t_1), X(t_2)) \in A\big) = P\big((X(t_1+\Delta), X(t_2+\Delta)) \in A\big)$$

For any set $A \in R^2$. In sum, a random process is stationary if *a time shift* ***does not change*** *its statistical properties*. Here is a formal definition of Stationarity of continuous-time processes.

Strict-Sense Stationary:

The stochastic process X(.) is called stationary (or strict-sense stationary (SSS), or strictly stationary) if the joint distribution of any collection of samples depends only on their relative time. That is, for any *k* and any $t_1, t_2, \ldots, t_k$ and any τ, we have

$$p_X(x_1,\ldots,x_k;t_1,\ldots,t_k) = p_X(x_1,\ldots,x_k;t_1-\tau,\ldots,t_k-\tau)$$

If for any τ, where the left side represents the joint density function of the random variables $X_1=X(t_1)$, $X_2=X(t_2),\ldots$, $X_k=X(t_k)$ and the right side corresponds to the joint density function of the random variables $X'_1=X(t_1+\tau)$, $X'_2=X(t_2+\tau),\ldots,X'_k=X(t_k+\tau)$. A process *X(t)* is said to be strict-sense stationary is true for all t_i, $i=1,2,\ldots,k$, $k=1,2,\ldots$ and any τ.

Wide-Sense Stationary:

The process X(.) is said to be wide-sense stationary (WSS) (or weakly stationary) if the mean of the process does not depend on time, and autocorrelation function depends only on the time difference of the two samples. That is,

$$m_X(t) = m_X ; R_X(t_1, t_2) \equiv R_X(t_2 - t_1)$$

Another class of random processes of interest is processes whose description exhibits periodic behaviour. These processes arise in many communications applications, where operations must be repeated periodically.