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Unit-III

Exponential Family of Distribution

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UNIT – III

Most Powerful Test

A statistical test based on best critical region of the test is called most powerful test.

The most powerful test corresponds to the best rejection region '$\subset$' such that the testing a simple null hypothesis H_0: $\theta = \theta_0$ against a simple alternative hypothesis H_1: $\theta = \theta_1$. This test has the largest possible test power $(1 - \beta)$ among all of the size α if it is satisfies the following two conditions

(i) $P[x \in \omega | H_0] = \alpha$

$$\int_{\omega} L_0 dx = \alpha$$

(ii) $P[x \in \omega | H_1] \geq P[x \in \omega_1 | H_1]$

$$(1 - \beta) \geq (1 - \beta_1)$$

Best Critical Region

A critical region w of size α for testing H_0 against H_1 is said to be best critical region (BCR) if w^* *is* any other critical of same size α for which power of $w \geq w^*$ (i.e).

$$1 - P[x \in \bar{w} | H_1] \geq 1 - P[x \in \bar{w} \, | H_1]$$

(or)

$$P[x \in \bar{w} | H_1] \geq P[x \in \bar{w} \, | H_1]$$

Neyman Pearson Lemma

Statement

Let k >0, be a constant and W be a critical region of size α, such that

$$W = \left\{ x \in S : \frac{f(x, \theta_1)}{f(x, \theta_0)} > k \right\}$$

$$W = \left\{ x \in S : \frac{L_1}{L_0} > k \right\} \quad \text{------- (1)}$$

and $$\bar{W} = \left\{ x \in S : \frac{L_1}{L_0} < k \right\} \quad \text{------- (2)}$$

where, L_0 and L_1 are the likelihood functions of the sample observations $x = (x_1, x_2, \ldots, x_n)$ under $H_0$and H_1 respectively. Then W is the most powerful critical region of the test hypothesis $H_0: \theta = \theta_0$against the alternative$H_1: \theta = \theta_1$.

Proof

We are given

$$P(x \in W \mid H_0) = \int_W L_0 \; dx = \alpha$$

The power of the region is

$$P(x \in W \mid H_1) = \int_W L_1 \; dx = 1 - \beta$$

In order to establish the lemma, we have to prove that there exists no other critical region, of size less than or equal to α, which is more powerful than W. Let W_1 be another critical region of size $\alpha_1 \leq \alpha$ and power $1 - \beta_1$ so that we have

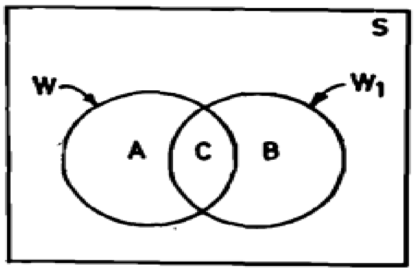
$$P(x \in W_1 \mid H_0) = \int_{W_1} L_0 \; dx = \alpha_1$$

And

$$P(x \in W_1 \mid H_1) = \int_{W_1} L_1 \; dx = 1 - \beta_1$$

Now we have to prove that, $1 - \beta \geq 1 - \beta_1$

Let $W = A \cup C \; and \; W_1 = B \cup C$ (C may be empty, i.e., W and W_1 may be disjoint)



If $\alpha_1 \leq \alpha$, we have

$$\int_{W_1} L_0\, dx \le \int_W L_0\, dx$$

$$\Rightarrow \qquad \int_{B\cup C} L_0\, dx \le \int_{A\cup C} L_0\, dx$$

$$\Rightarrow \qquad \int_B L_0\, dx \le \int_A L_0\, dx$$

$$\Rightarrow \qquad \int_A L_0\, dx \ge \int_B L_0\, dx$$

Since $A \subset W$,

from (1),

$$\Rightarrow \quad \int_A L_1\, dx > K \int_A L_0\, dx \ge k \int_B L_0\, dx$$ (Using equation (7))

Also (2) implies,

$$\frac{L_1}{L_0} \le k \quad \forall\, x \in \overline{W}$$

$$\Rightarrow \qquad \int_{\overline{W}} L_1\, dx \le k \int_{\overline{W}} L_0\, dx$$

This result also holds for any subset of $\overline{W}$, say $\overline{W} \cap W_1 = B$. Hence

$$\int_B L_1\, dx \le k \int_B L_0\, dx \le \int_A L_1\, dx$$

Adding $\int_C L_1\, dx$ to both sides, we get.

$$\int_{W_1} L_1\, dx \le \int_W L_1\, dx$$

$$\Rightarrow \qquad 1-\beta \ge 1-\beta_1$$

Hence the Lemma.

Generalization of Neyman-Pearson Fundamental Lemma

Let $g_0, g_1, \ldots, g_m$ be $(m+1)$ integrable real valued functions defined on χ. Suppose there exists at least one test function $\varphi(x)$ such that,

$$\int_{\mathcal{X}} \varphi(x) g_j(x) dx = c_j,\, j = 1, 2, \ldots, m, \tag{1}$$

where $c_1, \ldots, c_m$ are some known numbers. Let $\varphi_0(x)$ be another test function satisfying (1), such that,

$$\begin{aligned} \varphi_0(x) &= 1 \text{ if } g_0(x) > \sum_{j=1}^{m} k_j g_j(x) \\ &= 0 \text{ if } g_0(x) < \sum_{i=1}^{m} k_j g_j(x) \end{aligned} \tag{2}$$

where $k_1, \ldots, k_m$ are some constants determined appropriately. Then we have,

$$\int_{\mathcal{X}} \varphi_0(x) g_0(x) dx \quad \geq \quad \int_{\mathcal{X}} \varphi(x) g_0(x) dx \tag{3}$$

for all test φ(x) satisfying (1).

Particular case

Suppose $m = 1$, $g_0(x) = p(x, \theta_1)$, $g_1(x) = p(x, \theta_0)$, $k_1 = k$, $c_1 = \alpha$. Then (1) is equivalent to the size condition i.e.

$$\int_{\mathcal{X}} \varphi(x) p(x, \theta_0) dx \quad = \quad \alpha$$

Then the test function $\varphi_0(x)$ the above equation and from (2) we see that

$\varphi_0(x) = 1 \;\; if \; p(x, \theta_1) > kp(x, \theta_0)$

$\varphi_0(x) = 0 \;\; if \; p(x, \theta_1) < kp(x, \theta_0)$

From equation (3) we get

$$\int_{\mathcal{X}} \varphi_0(x) p(x, \theta_1) dx \geq \int_{\mathcal{X}} \varphi(x) p(x, \theta_1) dx$$

Hence $\varphi_0(x)$ is an MP test of size α.

Proof

Let us define a function

$$Q(x) = (\varphi_0(x) - \varphi(x)) \left(g_0(x) - \sum_{i=1}^{m} k_j g_j(x) \right).$$

Then as in the proof of Neyman-Pearson lemma it can be shown that

$$Q(x) \geq 0 \text{ for all} x \in \mathcal{X}.$$

It follows that

$$\int Q(x) dx \quad \geq \quad 0$$

$$\Rightarrow \int_{\mathcal{X}} \varphi_0(x) g_0(x) dx - \int_{\mathcal{X}} \varphi(x) g_0(x) dx \quad \geq \quad \int_{\mathcal{X}} \varphi_0(x) \sum_{j=1}^{m} k_j g_j(x) dx - \int_{\mathcal{X}} \varphi(x) \sum_{j=1}^{m} k_j g_j(x) dx$$

$$
\begin{aligned}
&= \sum_{j=1}^{m} k_j \int_{\mathcal{X}} \varphi_0(x) g_j(x) g_j(x) dx - \sum_{j=1}^{m} k_j \int_{\mathcal{X}} \varphi(x) g_j(x) dx \\
&= \sum_{j=1}^{m} k_j c_j - \sum_{j=1}^{m} k_j c_j \\
&= 0
\end{aligned}
$$

Hence the proof.

Unbiased test

A statistical test of simple null hypothesis against single alternative hypothesis is called unbiased if the power of the test is greater than or equal to single of the test.

Lemma for Unbiased Test

The most powerful test for testing simple H_0 against simple H_1 is always unbiased. Let w be the best critical region of size for H_0: $\theta = \theta_0$ against H_1: $\theta = \theta_1$ from the population. Let $x_1, x_2,...,x_n$ be a random sample from $f(x, \theta)$. Let L_0 and L_1 be the likelihood function H_0 and H_1 respectively.

From Neymann Pearson Lemma (N-P Lemma) for the points inside w, $\frac{L_0}{L_1} \leq k$ and for the point outside w, $\frac{L_0}{L_1} \geq k$ where k is a constant for the points inside w.

$$L_0 \leq k \ L_1$$

$$\int_w L_0 dx \ \leq k \ \int_w L_1 dx$$

$$\alpha \leq k \ (1-\beta) \qquad \text{------------1}$$

For the points outside w or the points inside $\overline{w}$

$$L_0 \geq k \ L_1$$

$$\int_{\overline{w}} L_0 dx \ \geq k \ \int_{\overline{w}} L_1 dx$$

$$1-\alpha \geq k \ \beta \qquad \text{--------------------2}$$

Multiply by equation (1) and (2)

$$k \ (1-\beta)\,(1-\alpha) \geq k \ \beta\,\alpha$$

$$(1-\beta-\alpha+\alpha\beta) \geq \alpha\beta$$

$$1-\alpha-\beta+\alpha\beta-\alpha\beta \geq 0$$

$$(1-\beta) \geq \alpha$$

(i.e.,) power of the test ≥ Significant of the test

Therefore the based on *w* is most powerful and unbiased.

Uniformly Most Powerful Test

A critical region W of size α for testing $H_0:\theta = \theta_0$ against $H_1:\theta \neq \theta_0$ is said to be uniformly most powerful critical region if for every value of $\theta \neq \theta_0$ the power of the critical region W must be greater than or equal to the critical region W must be greater than or equal to power of any other critical region $\overline{W}$ of same size αany test based on uniformly most powerful critical region is called uniformly most powerful test.

Exponential family

An exponential family is a parametric family of distributions whose probability density (or mass) functions satisfy certain properties that make them highly tractable from a mathematical viewpoint.

Parametric family

A parametric family of univariate continuous distributions is said to be an exponential family if and only if the probability density function of any member of the family can be written as

$$f_X(x) = h(x)\exp[\eta(\theta)^{\top} T(x) - A(\theta)]$$

Where,

$h : \mathbb{R} \to \mathbb{R}_+$ is a function that depends only on x;

θ is a $K \times 1$ vector of parameters;

$\eta : \mathbb{R}^K \to \mathbb{R}^L$ is a vector-valued function of the vector of parameters θ;

$T : \mathbb{R} \to \mathbb{R}^L$ is a vector-valued function of x;

$\eta(\theta)^{\top} T(x)$ is the dot product between η and T;

$A : \mathbb{R}^K \to \mathbb{R}$ is a function of θ.

Exponential family of distribution

The Exponential family is a practically convenient and widely used unified family of distributions on finite dimensional Euclidean spaces parameterized by a finite dimensional parameter vector.

- Parameter (θ)
- Sufficient statistic [$T(x)$]
- Natural parameter (η)

- Base measure [$h(x)$]
- Log-partition function [$A(\eta)$]

Examples of exponential family distributions

- Normal distribution
- Exponential distribution
- Gamma distribution
- Poisson distribution
- Bernoulli distribution
- Binomial distribution (with fixed number of trials)
- Multinomial (with fixed number of trials)
- Negative Binomial distribution (with fixed number of failures)

How to build an exponential family

The main steps needed to build an exponential family:

1. we choose a base measure $h(x)$;
2. we choose a vector of sufficient statistics $T(x)$ of dimension $L\times 1$;
3. we write the $L\times 1$ natural parameter as a function $\eta(\theta)$ of a $K\times 1$ parameter θ;
4. we try to find the log-partition function $A(\theta)$ by computing the integral

$$\exp(A(\theta)) = \int_{-\infty}^{\infty} h(x)\exp[\eta(\theta)^{\top} T(x)]dx$$

5. if the log-partition function is finite for some values of η, then we have built a family of distributions, called an exponential family, whose densities are of the form

$$f_X(x) = h(x)\exp[\eta(\theta)^{\top} T(x) - A(\theta)]$$

One Parameter Exponential Family

Exponential families can have any finite number of parameters. For instance, a normal distribution with a known mean is in the one parameter Exponential family, while a normal distribution with both parameters unknown is in the two parameter Exponential family.

Binomial distribution

The family of binomial distributions with probability mass function

$$p_X(x) = 1_{\{x\in\{0,1,\ldots,n\}\}}\frac{n!}{x!(n-x)!}p^x(1-p)^{n-x}$$

We can write the probability mass function as follows:

$$p_X(x) = 1_{\{x\in\{0,1,\dots,n\}\}}\frac{n!}{x!(n-x)!}p^x(1-p)^{n-x}$$
$$= 1_{\{x\in\{0,1,\dots,n\}\}}\frac{n!}{x!(n-x)!}\left(\frac{p}{1-p}\right)^x(1-p)^n$$
$$= 1_{\{x\in\{0,1,\dots,n\}\}}\frac{n!}{x!(n-x)!}\exp\left(x\ln\left(\frac{p}{1-p}\right)\right)\exp(n\ln(1-p))$$

Exponential family for fixed n:

Parameter θ	p
Sufficient statistic $T(x)$	x
Natural parameter η	$\ln\left(\frac{p}{1-p}\right)$
Base measure $h(x)$	$1_{\{x\in\{0,1,\dots,n\}\}}\frac{n!}{x!(n-x)!}$
Log-partition function $A(\eta)$	$-n\ln(1-p)$

Normal distribution

The family of normal distributions with density

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}\right)$$

We can write the density as follows:

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}\right)$$
$$= \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{1}{2\sigma^2}x^2+\frac{\mu}{\sigma^2}x-\frac{\mu^2}{2\sigma^2}\right)$$
$$= \frac{1}{\sqrt{2\pi}}\exp\left(\frac{\mu}{\sigma^2}x-\frac{1}{2\sigma^2}x^2\right)\exp\left(-\frac{\mu^2}{2\sigma^2}-\frac{\ln(\sigma^2)}{2}\right)$$

Exponential family:

Parameter θ	$\begin{bmatrix}\mu & \sigma^2\end{bmatrix}^{\top}$
Sufficient statistic $T(x)$	$\begin{bmatrix}x & x^2\end{bmatrix}^{\top}$
Natural parameter η	$\begin{bmatrix}\frac{\mu}{\sigma^2} & -\frac{1}{2\sigma^2}\end{bmatrix}^{\top}$
Base measure $h(x)$	$\frac{1}{\sqrt{2\pi}}$
Log-partition function $A(\eta)$	$\frac{\mu^2}{2\sigma^2}+\frac{\ln(\sigma^2)}{2}$

Bernoulli distribution

A Bernoulli random variable X assigns probability measure π to the point $x = 1$ and probability measure $1 - \pi$ to $x = 0$. More formally, define ν to be counting measure on $\{0, 1\}$, and define the following density function with respect to ν:

$$\begin{aligned} p(x \mid \pi) &= \pi^x (1-\pi)^{1-x} \\ &= \exp\left\{ \log\left(\frac{\pi}{1-\pi}\right) x + \log(1-\pi) \right\} \end{aligned}$$

Exponential family:

$$\begin{aligned} \eta &= \frac{\pi}{1-\pi} \\ T(x) &= x \\ A(\eta) &= -\log(1-\pi) = \log(1+e^\eta) \\ h(x) &= 1. \end{aligned}$$

Poisson distribution

The probability mass function (i.e., the density respect to counting measure) of a Poisson random variable is given as follows:

$$p(x \mid \lambda) = \frac{\lambda^x e^{-\lambda}}{x!}.$$

$$p(x \mid \lambda) = \frac{1}{x!} \exp\{x \log \lambda - \lambda\}.$$

Exponential family:

$$\begin{aligned} \eta &= \log \lambda \\ T(x) &= x \\ A(\eta) &= \lambda = e^\eta \\ h(x) &= \frac{1}{x!}. \end{aligned}$$

Multiparameter exponential family of distribution

Let $\{P_\theta,\ \theta \in \Theta\}$, $\Theta \subset R^k$, is a k-parameter exponential family if the density/pmf function of $X \sim P_\theta$ is

$$p(x \mid \theta) = h(x) exp[\sum_{j=1}^{k} \eta_j(\theta) T_j(x) - B(\theta)],$$

where $x \in X \subset R^q$, and

$\eta_1, \ldots, \eta_k$ and B are real-valued functions mapping $\Theta \to R$.

$T_1, \ldots, T_k$ and h are real-valued functions mapping $R^q \to R$.

Example

- Two parameter normal distribution
- Two parameter Gamma distribution
- Two parameter inverse Gaussian distribution

Two parameter normal distribution

Suppose $X \sim N(\mu, \sigma^2)$, and we consider both μ, σ to be parameters. If we denote $(\mu, \sigma) = (\theta_1, \theta_2) = \theta$, then parameterized by θ, the density of X is

$$f(x\,|\theta) = \frac{1}{\sqrt{2\pi}\theta_2} e^{-\frac{(x-\theta_1)^2}{2\theta_2^2}} I_{x\in\mathcal{R}} = \frac{1}{\sqrt{2\pi}\theta_2} e^{-\frac{x^2}{2\theta_2^2}+\frac{\theta_1 x}{\theta_2^2}-\frac{\theta_1^2}{2\theta_2^2}} I_{x\in\mathcal{R}}.$$

This is in the two parameter Exponential family with

$$\eta_1(\theta) = -\frac{1}{2\theta_2^2} \qquad T_1(x) = x^2 \qquad \psi(\theta) = \frac{\theta_1^2}{2\theta_2^2} + \log\theta_2$$

$$\eta_2(\theta) = \frac{\theta_1}{\theta_2^2} \qquad T_2(x) = x \qquad h(x) = \frac{1}{\sqrt{2\pi}} I_{x\in\mathcal{R}}$$

The parameter space in the θ parameterization is $\Theta = (-\infty,\infty) \otimes (0,\infty)$.

If we want the canonical form, we let

$$\eta_1 = -\tfrac{1}{2\theta_2^2}, \eta_2 = \tfrac{\theta_1}{\theta_2^2}$$

$$\psi(\eta) = -\tfrac{\eta_2^2}{4\eta_1} - \tfrac{1}{2}\log(-\eta_1).$$

The natural parameter space for (η_1, η_2) is $(-\infty, 0) \otimes (-\infty,\infty)$.

Two Parameter Gamma distribution

If we fix one of the two parameters of a Gamma distribution, then it becomes a member of the one parameter Exponential family. We show in this example that the general Gamma distribution is a member of the two parameter Exponential family. To show this, just observe that with $\theta = (\alpha, \lambda) = (\theta_1, \theta_2)$,

$$f(x\,|\theta) = e^{-\frac{x}{\theta_2}+\theta_1\log x-\theta_1\log\theta_2-\log\Gamma(\theta_1)} \frac{1}{x} I_{x>0}.$$

This is in the two parameter Exponential family with

$$\eta_1(\theta) = -\tfrac{1}{\theta_2} \qquad T_1(x) = x, \qquad \psi(\theta) = \theta_1 \log \theta_2 + \log \Gamma(\theta_1)$$

$$\eta_2(\theta) = \theta_1 \qquad T_2(x) = \log x, \qquad h(x) = \tfrac{1}{x} I_{x>0}$$

The parameter space in the θ parameterization is $(0, \infty) \otimes (0, \infty)$. For the canonical form, use

$$\eta_1 = -\tfrac{1}{\theta_2}, \eta_2 = \theta_1$$

and so, the natural parameter space is $(-\infty, 0) \otimes (0,\infty)$. The natural sufficient statistic is $(X, \log X)$.

Two Parameter Inverse Gaussian distribution

The simple symmetric random walk on R, the time of the r^th return to zero ν_r satisfies the weak convergence result

$$P(\frac{\nu_r}{r^2} \leq x) \to 2[1 - \Phi(\frac{1}{\sqrt{x}})], x > 0, \text{ as } r \to \infty.$$

The density of this limiting CDF is

$$f(x) = \frac{e^{-\frac{1}{2x}} x^{-3/2}}{\sqrt{2\pi}} I_{x>0}$$

This is a special inverse Gaussian distribution. The general inverse Gaussian distribution has the density

$$f(x \,|\theta_1, \theta_2) = \left(\frac{\theta_2}{\pi x^3}\right)^{1/2} e^{-\theta_1 x - \frac{\theta_2}{x} + 2\sqrt{\theta_1 \theta_2}} I_{x>0};$$

The parameter space for $\theta = (\theta_1, \theta_2)$ is $[0,\infty) \otimes (0,\infty)$. The special inverse Gaussian density ascribed to above corresponds to

$$\theta_1 = 0, \theta_2 = \tfrac{1}{2}$$

Locally Most Powerful Test (One sided tests)

Let X be a random variable with p.m.f/ p.d.f.

$$p_\theta(x),\ \theta \in \Theta \subset R.$$

Consider the problem of testing,

$$H_0\colon \theta = \theta_0 \text{ vs. } H_1 : \theta > \theta_0$$

Definition

A test ϕ^0 is called a locally most powerful (LMP) test at size α for testing,

$$H_0: \theta = \theta_0 \text{ vs. } H_1 : \theta > \theta_0$$

if for some $\varepsilon > 0$;

- $\beta_{\emptyset_0}(\theta_0) = \alpha$
- $\beta_{\emptyset_0}(\theta_0) \geq \beta_{\emptyset}(\theta)$ for all $\theta \in (\theta_0, \theta_0 + \varepsilon)$ and whatever ϕ satisfying (i)

Where, $\beta_\phi(\theta) = E_\theta\phi$ denotes the power function of the test ϕ. By Mean value theorem

$$\beta_\phi(\theta) = \beta_\phi(\theta_0) + (\theta - \theta_0)\beta'_\phi(\theta^*),\ \theta_0 < \theta^* < \theta$$

Assumptions

(i) $\beta_\phi(\theta)$ is continuously differentiable in the neighbourhood of θ_0 for every ϕ.

(ii) $\beta'_{\emptyset}(\theta) = \int \emptyset \frac{\partial p_\theta(x)}{\partial \theta} dx$

To minimize $\beta'_\phi(\theta_0)$ subject to, $\beta_\phi(\theta_0) = \int \emptyset p_{\theta_0}(x)dx = \alpha$. Using generalized NP Lemma we get the optimum choice of ϕ as,

$$\emptyset^0 = 1 \ \ if \ \frac{\partial p_\theta(x)}{\partial \theta_0} > kp_{\theta_0}(x)$$

$$\emptyset^0 = \gamma(x) \ \ if \ \frac{\partial p_\theta(x)}{\partial \theta_0} = kp_{\theta_0}(x)$$

$$\emptyset^0 = 0 \ \ if \ \frac{\partial p_\theta(x)}{\partial \theta_0} < kp_{\theta_0}(x)$$

$\Leftrightarrow$

$$\emptyset^0 = 1 \ \ if \ \frac{\partial log\ p_\theta(x)}{\partial \theta_0} > k$$

$$\emptyset^0 = \gamma(x) \ \ if \ \frac{\partial log\ p_\theta(x)}{\partial \theta_0} = k$$

$$\emptyset^0 = 0 \ \ if \ \frac{\partial log\ p_\theta(x)}{\partial \theta_0} < k$$

Locally Most Powerful Test (Two sided tests)

The locally best test may be extended to testing the hypothesis

$$H_0: \theta = \theta_0 \text{ vs. } H_1: \theta \neq \theta_0$$

by specifying the slope of the power function at θ_0 and requiring the second derivative of the power function at θ_0 to be a maximum. Hence we consider only unbiased tests with slope zero at θ_0.

Assumptions

(i) $\beta_\phi(\theta)$ is twice continuously differentiable in the neighbourhood of θ_0 for every ϕ.

(ii) $\beta'_\emptyset(\theta) = \int \emptyset \frac{\partial p_\theta(x)}{\partial \theta} dx$

(iii) $\beta''_\emptyset(\theta) = \int \emptyset \frac{\partial^2 p_\theta(x)}{\partial \theta^2} dx$

By Mean value theorem

$$\beta_\varphi(\theta) = \beta_\varphi(\theta_0) + (\theta - \theta_0)\beta'_\varphi(\theta_0) + \frac{(\theta - \theta_0)^2}{2!}\beta'_\varphi(\theta^\star), \quad min(\theta_0, \theta) < \theta^\star < max(\theta_0, \theta).$$