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Unit-II

Non-parametric Tests

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UNIT – II

Non-Parametric Test

Non-parametric tests are the mathematical methods used in statistical hypothesis testing, which do not make assumptions about the frequency distribution of variables that are to be evaluated. The non-parametric experiment is used when there are skewed data, and it comprises techniques that do not depend on data pertaining to any particular distribution.

The word non-parametric does not mean that these models do not have any parameters. The fact is the characteristics and numbers of parameters are pretty flexible and not predefined. Therefore, these models are called distribution-free models.

Assumption

- The form of the population is unknown.
- The population possesses density function.
- Lower order moments exists i.e μ_1', μ_2' are finite.
- Sample observation are independent and random
- The variable under study is continuous.
- The two populations are identical or the measures of location of two populations are the same.

Applications of Non-Parametric Test

The conditions when non-parametric tests are used are:

- When parametric tests are not satisfied.
- When testing the hypothesis, it does not have any distribution.
- For quick data analysis.
- When unscaled data is available.

Advantages of Non-Parametric Test

The advantages of the non-parametric test are:

- Easily understandable
- Short calculations
- Assumption of distribution is not required
- Applicable to all types of data

Disadvantages of Non-Parametric Test

The disadvantages of the non-parametric test are:

- Less efficient as compared to parametric test
- The results may or may not provide an accurate answer because they are distribution free

Types of Nonparametric Tests

- **1-sample sign test**. Use this test to estimate the median of a population and compare it to a reference value or target value.
- **1-sample Wilcoxon signed rank test**. With this test, you also estimate the population median and compare it to a reference/target value. However, the test assumes your data comes from a symmetric distribution (like the Cauchy distribution or uniform distribution).
- **Friedman test**. This test is used to test for differences between groups with ordinal dependent variables. It can also be used for continuous data if the one-way ANOVA with repeated measures is inappropriate (i.e. some assumption has been violated).
- **Goodman Kruska's Gamma**: a test of association for ranked variables.
- **Kruskal-Wallis test**. Use this test instead of a one-way ANOVA to find out if two or more medians are different. Ranks of the data points are used for the calculations, rather than the data points themselves.
- The **Mann-Kendall Trend Test** looks for trends in time-series data.
- **Mann-Whitney test**. Use this test to compare differences between two independent groups when dependent variables are either ordinal or continuous.
- **Mood's Median test**. Use this test instead of the sign test when you have two independent samples.
- **Correlation Use** when you want to find a correlation between two sets of data.

The following table lists the nonparametric tests and their parametric alternatives:

Nonparametric test	Parametric Alternative
1-sample sign test	One-sample Z-test, One sample t-test
1-sample Wilcoxon Signed Rank test	One sample Z-test, One sample t-test
Friedman test	Two-way ANOVA
Kruskal-Wallis test	One-way ANOVA
Mann-Whitney test	Independent samples t-test
Mood's Median test	One-way ANOVA
Spearman Rank Correlation	Correlation Coefficient

Difference between parametric and non-parametric test

Properties	Parametric Test	Non-parametric test
Assumptions	Yes, assumptions are made.	No, assumptions are not made
Value of central tendency	The mean value is the central tendency	The median value is the central tendency
Correlation	Pearson Correlation	Spearman correlation
Probabilistic Distribution	Normal Probabilistic Distribution	Arbitrary Probabilistic Distribution
Population knowledge	Population knowledge is required	Population knowledge is not required
Used for	Used for finding interval data	Used for finding nominal data
Application	Applicable to variables	Applicable to variables and attributes
Examples	t-test and z-test	Mann-Whitney and Kruskal-wallis

Spearman Correlation

The correlation test measures the strength of the association between two variables. Spearman's Correlation is a statistical measure of measuring the strength and direction of the monotonic relationship between two continuous variables. Therefore, these attributes are ranked or put in the order of their preference. It is denoted by the symbol "rho" (ρ).

Assumption

- Random samples.
- A monotonic association exists between 2 variables.
- Variables are at least ordinal
- Data contains paired samples need variable x and y values, if there is a missing value you need to delete the row.
- Independence of observations.

Advantages of Spearman's Rank Correlation

- This method is easier to understand.
- It is superior for calculating qualitative observations such as the intelligence of people, physical appearance, etc.
- This method is suitable when the series gives only the order of preference and not the actual value of the variable.
- It is robust to the outliers present in the data

- It is designed to capture monotonic relationships between variables. Monotonic relation measures the effect of change in one variable on another variable

Disadvantages of Spearman's Rank Correlation:

- It is not applicable in the case of grouped data.
- It can handle only a limited number of observations or items.
- It Ignores Non-Monotonic Relationships between the variables for example it does not capture other types of relationships, such as curvilinear or nonlinear associations between the variables.
- It only considers the ranks of the data points and ignores the actual magnitude of differences between the values of the variables.
- Converting the data into ranks for Spearman's rank correlation discards the original values of the variables and replaces them with their respective ranks. This transformation may result in a loss of information in the data, especially if the variables of the data have meaningful magnitudes or units.

Formula

Spearman's rank correlation formula is given by,

1. Spearman's rank correlation formula for no tied ranks

$$\rho = 1 - \frac{6\sum d_i^2}{n(n^2-1)}$$

Where,

ρ = Spearman Correlation coefficient

d_i = the difference in paired ranks

n = total number of observation

2. Spearman's rank correlation formula for tied ranks

$$\rho = \frac{\sum (t^3 - t)}{12}$$

Where,

ρ = Spearman Correlation coefficient

t = number of observations in a group tied for a given rank

Test procedure for Spearman's rank correlation

1. Null Hypothesis (Ho)

H_0: There is no significant relationship between two independent variables.

2. Alternative Hypothesis (H_1)

H_1: There is significant relationship between two independent variables.

3. Test statistic

Under the null hypothesis (H_0), the test statistic is

1. Spearman's rank correlation formula for no tied ranks

$$\rho = 1 - \frac{6\sum d_i^2}{n(n^2 - 1)}$$

Where,

ρ = Spearman Correlation coefficient

d_i = the difference in paired ranks

n = total number of observation

2. Spearman's rank correlation formula for tied ranks

$$\rho = \frac{\sum (t^3 - t)}{12}$$

Where,

ρ = Spearman Correlation coefficient

t = number of observations in a group tied for a given rank

4. Level of Significance

The level of significance may be fixed at either 5% or 1%.

5. Critical value

For two sided test, from Spearman's correlation table for (n - 2) degrees of freedom.

6. Inference

If the observed value of the test statistic ρ exceeds the table value of Spearman's correlation, we reject the Null Hypothesis H_o otherwise accept it.

Test for Spearman's Rank correlation

In a marketing survey the price of tea and coffee in a town based on quality was found as shown below. Could you find any relation between tea and coffee price?

Price of tea	88	90	95	70	60	75	50
Price of coffee	120	134	150	115	110	140	100

Procedure

- State the null hypothesis and alternative hypothesis
- State the alpha
- Compute the test statistic

$$\rho = 1 - \frac{6\sum d_i^2}{n(n^2-1)}$$

- To determine the critical value
- Compare the calculated test statistic to the critical value.

Calculation

Hypothesis

H_0: There is no a significant correlation between price of coffee and tea.

H_1: There is a significant correlation between price of coffee and tea.

Level of Significance

$$\alpha = 0.05$$

Test Statistic:

Here, n = 7, then

Price of tea	Rank	Price of coffee	Rank	d	d^2
88	3	120	4	1	1
90	2	134	3	1	1
95	1	150	1	0	0
70	5	115	5	0	0
60	6	110	6	0	0
75	4	140	2	2	4
50	7	100	7	0	0
				Total	**6**

Under the null hypothesis, the test statistic is

$$r = 1 - \frac{6\sum d^2}{n^3 - n}$$

$$r = 1 - \frac{6 \times 6}{7^3 - 7}$$

$$r = 1 - \frac{36}{343 - 7}$$

$$r = 1 - \frac{36}{336}$$

$$r = 1 - 0.1071$$

$$r = 0.893$$

Table Value

The degrees of freedom = (n – 2)

= 7 – 2 = 5df

Table Value = 0.900

Conclusion

Since the calculated value is less than the table value (0.893 < 0.900). We do not reject the null hypothesis. Hence, we conclude that there is no significant correlation between price of coffee and tea, (i.e.,) both coffee and tea's are equal quality.

Chi Square Test for Association of attributes

The Chi-Square Test for Association is used to determine if there is any association between two variables. It is really a hypothesis test of independence. The null hypothesis is that the two variables are not associated, i.e., independent. The alternate hypothesis is that the two variables are associated.

Formula

Chi Square Test for Association of attributes formula is given by,

$$\chi^2 = \sum \frac{(O_i - E_i)^2}{E_i}$$

Where,

O_i = Observed Value

E_i = Expected Value

$$E_i = \frac{(Row\ Total) \times (Column\ Total)}{Total\ Number\ of\ Observations)}$$

Test procedure for Chi Square Test for Association of attributes

1. Null Hypothesis (Ho)

H_0: There is no significant association between two independent variables.

2. Alternative Hypothesis (H_1)

H_1: There is significant association between two independent variables.

3. Test statistic

Under the null hypothesis (H_0), the test statistic is

$$\chi^2 = \sum \frac{(O_i - E_i)^2}{E_i}$$

Where,

O_i = Observed Value

E_i = Expected Value

$$E_i = \frac{(Row\ Total) \times (Column\ Total)}{Total\ Number\ of\ Observations)}$$

4. Level of Significance

The level of significance may be fixed at either 5% or 1%.

5. Critical value

The critical values for the chi-square table depend on the degrees of freedom and the significance level of the test.

6. Inference

If the observed value of the test statistic χ^2 exceeds the table value of chi-square, we reject the Null Hypothesis H_o otherwise accept it.

Test for Chi Square Test for Association of attributes

The table given below shows the data obtained during outbreak of smallpox:

	Attacked (B)	Not Attacked (b)	Total
Vaccinated (A)	31	469	500
Not Vaccinated (a)	185	1315	1500
Total	216	1784	2000

Test the effectiveness of vaccination in preventing the attack from smallpox. Test your result with the help of χ^2 at 5 per cent level of significance.

Procedure

- State the null hypothesis and alternative hypothesis
- State the alpha
- Compute the test statistic

$$\chi^2 = \sum \frac{(O_i - E_i)^2}{E_i}$$

- To determine the critical value
- Compare the calculated test statistic to the critical value.

Calculation

Hypothesis

H_0: The vaccination is not effective in preventing the attack from smallpox.

H_1: The vaccination is effective in preventing the attack from smallpox.

Level of Significance

$\alpha = 0.05$

Test statistic

Under the null hypothesis (H_0), the test statistic is

$$\chi^2 = \sum \frac{(O_i - E_i)^2}{E_i}$$

$$Expectation\ of\ AB = \frac{A \times B}{N}$$

When A represents vaccination and B represents the attack.

$$Expectation\ of\ AB = \frac{500 \times 216}{2000} = 54$$

$$Expectation\ of\ Ab = \frac{500 \times 1784}{2000} = 446$$

$$Expectation\ of\ aB = \frac{1500 \times 216}{2000} = 162$$

$$Expectation\ of\ ab = \frac{1500 \times 1784}{2000} = 1338$$

The expected value is,

	Attacked (B)	Not Attacked (b)	Total
Vaccinated (A)	54	446	500
Not Vaccinated (a)	162185	1338	1500
Total	216	1784	2000

Calculate the chi-square,

Group	O_i	E_i	$(O_i – E_i)$	$(O_i – E_i)^2$	$(O_i – E_i)^2/E_i$
AB	31	54	-23	529	9.796
Ab	469	446	23	529	1.186
aB	185	162	23	529	3.265
Ab	1315	1338	-23	529	0.395
Total					14.642

Chi-square value = 14.642

Table Value

Degrees of freedom = (r-1) (c-1)

= (2-1) (2-1) = 1 df

Table value = 3.841

Conclusion

Since the calculated value is greater than the table value (14.642 > 3.841). We reject the null hypothesis. Hence, we conclude that the vaccination is effective in preventing the attack from smallpox.

Chi-square test for goodness of fit

The Chi-square goodness of fit test is a statistical hypothesis test used to determine whether a variable is likely to come from a specified distribution or not. It is often used to evaluate whether sample data is representative of the full population.

Test Procedure for Chi-square test for goodness of fit

1. Null Hypothesis (Ho)

H_o: The data is good fit for the distribution.

2. Alternative Hypothesis (H_1)

H_1: The data is not good fit for the distribution.

3. Test statistic

Under the null hypothesis (H_0), the test statistic is

$$\chi^2 = \sum \frac{(O_i - E_i)^2}{E_i}$$

Where,

O_i = Observed Value

E_i = Expected Value

$$E_i = \frac{(Row\ Total) \times (Column\ Total)}{Total\ Number\ of\ Observations)}$$

4. Level of Significance

The level of significance may be fixed at either 5% or 1%.

5. Critical value

The critical values for the chi-square table depend on the degrees of freedom and the significance level of the test.

6. Inference

If the observed value of the test statistic χ^2 exceeds the table value of chi-square, we reject the Null Hypothesis H_o otherwise accept it.

Example

Fitting of Poisson distribution for the following data and test for the goodness.

Arrivals	0	1	2	3	4	5	6	7
Frequency	16	30	45	39	27	19	8	6

Procedure

- Define the Null Hypothesis (H_0) and Alternate Hypothesis (H_1) for given data.
- For a given Poisson Distribution Calculate respecting Mean.

$$Mean\ (\lambda) = \frac{\sum fx}{\sum f}$$

- Compute the respective probability using the recurrence relation of Poisson Distribution.

$$P(x) = \frac{e^{-\lambda}\lambda^x}{x!},\ x = 0,1,2,\ldots.$$

- Compute the Expected Frequency [E_i].
- Compute the test Statistic of Chi-Square Test,

$$\chi^2 = \sum_{i=1}^{n} \frac{(O_i - E_i)^2}{E_i}$$

- Taking decision about the hypothesis H_0 with respect to the level of significance and degree of freedom.

Calculation

Hypothesis

H_0: The data is good fit for the Poisson distribution.

H_1: The data is not good fit for the Poisson distribution.

Level of Significance

$\alpha = 0.05$

Test Statistic

Calculate mean (λ).

x	f	fx
0	16	0
1	30	30
2	45	90
3	39	117
4	27	108
5	19	95
6	8	48
7	6	42
Total	190	530

$$\lambda = \frac{\sum_{i=1}^{n} fx}{\sum_{i=1}^{n} f} = \frac{530}{190} = 2.7895$$

For a Poisson distribution the probability mass function is given by,

$$P(x) = \frac{e^{-\lambda}\lambda^x}{x!}, \ x = 0,1,2,\ldots.$$

$$P(0) = \frac{e^{(-2.7895)}(2.7895)^0}{0} = 0.0615$$

$$P(1) = \frac{e^{(-2.7895)}(2.7895)^1}{1!} = 0.1714$$

$$P(2) = \frac{e^{(-2.7895)}(2.7895)^2}{2!} = 0.2391$$

$$P(3) = \frac{e^{(-2.7895)}(2.7895)^3}{3!} = 0.2233$$

$$P(4) = \frac{e^{(-2.7895)}(2.7895)^4}{4!} = 0.1550$$

$$P(5) = \frac{e^{(-2.7895)}(2.7895)^5}{5!} = 0.0865$$

$$P(6) = \frac{e^{(-2.7895)}(2.7895)^6}{6!} = 0.0402$$

$$P(7) = \frac{e^{(-2.7895)}(2.7895)^7}{7!} = 0.0160$$

Calculate the Expected Frequency,

X	F	N . p(x)	Expected Frequency (E_i)
0	16	190 x 0.0615 = 11.68	12
1	30	190 x 0.1714 = 32.57	33
2	45	190 x 0.2391 = 45.43	45
3	39	190 x 0.2233 = 42.23	42
4	27	190 x 0.1550 = 29.45	29
5	19	190 x 0.0865 = 16.43	16
6	8	190 x 0.0402 = 7.63	8
7	6	190 x 0.0160 = 3.04	3

Compute the test Statistic of Chi-Square Test,

$$\chi^2 = \sum_{i=1}^{n} \frac{(O_i - E_i)^2}{E_i}$$

Frequency (O_i)	Expected Frequency (E_i)	$O_i - E_i$	$(O_i - E_i)^2$	$(O_i - E_i)^2/E_i$
16	12	4	16	1.3333
30	33	-3	9	0.2727
45	45	0	0	0
39	42	-3	9	0.2143
27	29	-2	4	0.1379
19	16	3	9	0.5625
8	8	0	0	0
6	3	3	9	3
			Total	5.5207

Table Value

Degree of Freedom = n – p – 1

= 8 – 1 – 1 = 6df

Table Value from Chi-Square Table in 6df = 12.592

Conclusion

Since Calculated Value is less than the Table Value (i.e., 5.5207 < 12.592). So, we do not reject the null Hypothesis. Hence, we conclude that the data is good fit for the Poisson distribution.

Chi-Square test for Homogeneity of Variance

The test of homogeneity expands the test for a difference in two population proportions, which is the two-proportion Z-test we learned in Inference for Two Proportions. We use the two-proportion Z-test when the response variable has only two outcome categories and we are comparing two population subgroups. We use the test of homogeneity if the response variable has two or more categories and we wish to compare two or more population subgroups.

Test Procedure for Chi-square test for Homogeneity of Variance

1. Null Hypothesis (Ho)

H_o: The population has the same proportion of observations.

2. Alternative Hypothesis (H_1)

H_1: The population has not the same proportion of observations.

3. Test statistic

Under the null hypothesis (H_0), the test statistic is

$$\chi^2 = \sum \frac{(O_i - E_i)^2}{E_i}$$

Where,

O_i = Observed Value

E_i = Expected Value

$$E_i = \frac{(Row\ Total) \times (Column\ Total)}{Total\ Number\ of\ Observations)}$$

4. Level of Significance

The level of significance may be fixed at either 5% or 1%.

5. Critical value

The critical values for the chi-square table depend on the degrees of freedom and the significance level of the test.

6. Inference

If the observed value of the test statistic χ^2 exceeds the table value of chi-square, we reject the Null Hypothesis H_o otherwise accept it.

Example

Suppose there is a city of 1,000,000 residents with four neighborhoods: *A*, *B*, *C*, and *D*. A random sample of 650 residents of the city is taken and their occupation is recorded as "white collar", "blue collar", or "no collar". The null hypothesis is that each person's neighborhood of residence is independent of the person's occupational classification. The data are tabulated as:

	A	***B***	***C***	***D***	**total**
White collar	90	60	104	95	349
Blue collar	30	50	51	20	151
No collar	30	40	45	35	150
Total	**150**	**150**	**200**	**150**	**650**

To calculate chi square test for the following data.

Procedure

- State the null hypothesis and alternative hypothesis
- State alpha
- Compute the test statistic

$$\chi^2 = \sum_{i=1}^{n} \frac{(O_i - E_i)^2}{E_i}$$

- To determine the critical value
- Compare the calculated test statistic to the critical value.

Calculation

Hypothesis

H_0: The data is dependent.

H_1: The data is Independent.

Level of Significance

$\alpha = 0.05$

Test statistic

To calculate Expected Frequency,

$$E_{11} = \frac{349 \times 150}{650} = 80.54$$

$$E_{12} = \frac{349 \times 150}{650} = 80.54$$

$$E_{13} = \frac{349 \times 200}{650} = 107.38$$

$$E_{14} = \frac{349 \times 150}{650} = 80.54$$

$$E_{21} = \frac{151 \times 150}{650} = 34.85$$

$$E_{22} = \frac{151 \times 150}{650} = 34.85$$

$$E_{23} = \frac{151 \times 200}{650} = 46.46$$

$$E_{24} = \frac{151 \times 150}{650} = 34.85$$

$$E_{31} = \frac{150 \times 150}{650} = 34.62$$

$$E_{32} = \frac{150 \times 150}{650} = 34.62$$

$$E_{33} = \frac{150 \times 200}{650} = 46.15$$

$$E_{34} = \frac{150 \times 150}{650} = 34.62$$

The expected frequency is

	A	***B***	***C***	***D***
White collar	80.54	80.54	107.38	80.54
Blue collar	34.85	34.85	46.46	34.85
No collar	34.62	34.62	46.15	34.62

To find Chi-Square Statistic,

Observed frequency	Expected Frequency	$(O_i - E_i)$	$(O_i - E_i)^2$	$(O_i - E_i)^2/E_i$
90	80.54	9.46	89.49	1.111
30	34.85	-4.85	23.52	0.675
30	34.62	-4.62	21.34	0.617
60	80.54	-20.54	421.89	5.238
50	34.85	15.15	229.52	6.586
40	34.62	5.38	28.94	0.836
104	107.38	-3.38	11.42	0.106
51	46.46	4.54	20.6116	0.444
45	46.15	-1.15	1.3225	0.029
95	80.54	14.46	209.09	2.596
20	34.85	-14.85	220.52	6.328
35	34.62	0.38	0.1444	0.004
			Total	24.570

$$\chi^2 = \sum_{i=1}^{n} \frac{(O_i - E_i)^2}{E_i} = 24.57$$

Table Value

Degree of Freedom = (3 – 1) (4 – 1)

= 6df

Table Value = 12.592

Conclusion:

Since, Calculated Value is greater than the table vale (i.e.,) **24.57 < 12.592**.We reject the null hypothesis. Hence, we conclude that the data is Independent.

Fisher Exact Test

Fisher's Exact Test is used to determine whether or not there is a significant association between two categorical variables. It is typically used as an alternative to the Chi-Square Test of Independence when one or more of the cells count in a 2×2 table is less than 5.

Formula

The formula for Fisher's Exact Test is,

$$p = \frac{(a+b)!(c+d)!(a+c)!(b+d)!}{a!b!c!d!N!}$$

Where,

a, b, c and d are the individual frequencies of the 2X2 contingency table

N is the total frequency.

Test Procedure for Fisher's Exact Test

1. Null Hypothesis (Ho)

H_o: The two variables are independent.

2. Alternative Hypothesis (H_1)

H_1: The two variables are not independent.

3. Test statistic

Under the null hypothesis (H_0), the test statistic is

$$p = \frac{(a+b)!(c+d)!(a+c)!(b+d)!}{a!b!c!d!N!}$$

Where,

a, b, c and d are the individual frequencies of the 2X2 contingency table

N is the total frequency.

4. Level of Significance

The level of significance may be fixed at either 5% or 1%.

5. Inference

If the observed value of the test statistic p exceeds the level of significance, we do not reject the Null Hypothesis H_o otherwise reject it.

Example

Suppose we want to know whether or not gender is associated with political party preference. We take a simple random sample of 25 voters and survey them on their political party preference. The following table shows the results of the survey:

	Democrat	**Republican**	**Total**
Male	4	9	**13**
Female	8	4	**12**
Total	**12**	**13**	**25**

To find the two-tailed p value, using the Fisher's Exact Test.

Procedure

- State the null hypothesis and alternative hypothesis
- State alpha
- Compute the test statistic

$$p = \frac{(a+b)!\,(c+d)!\,(a+c)!\,(b+d)!}{a!\,b!\,c!\,d!\,N!}$$

- To determine the critical value
- Compare the calculated test statistic to the critical value.

Calculation

Hypothesis

H_0: Gender and political party preference are independent.

H_1: Gender and political party preference are not independent.

Level of Significance

$\alpha = 0.05$

Test statistic

Under the null hypothesis (H_0), the test statistic is

$$p = \frac{(a+b)!\,(c+d)!\,(a+c)!\,(b+d)!}{a!\,b!\,c!\,d!\,N!}$$

$$p = \frac{12!\,13!\,12!\,13!}{4!\,9!\,8!\,4!\,25!}$$

$$= 0.068$$

Conclusion

Since Calculated p value is greater than the level of significance (i.e., 0.068 > 0.05). So, we do not reject the null Hypothesis. Hence, we conclude that the Gender and political party preference are independent.

Run Test

A run test is sequence of letter of one kind followed by a sequence of letters of another kind. The number of letters in a sequence is called length of the run.

For example: xxx/y/xxxxxx/y/x/yyyy

The sequence has 6 runs. The length of third run is 6.

Formula

The test statistic for run test is given by

$$Z = \frac{U - E(U)}{\sqrt{V(U)}}$$

Where,

U = Number of runs in the combined sample

$$E(U) = \left(\frac{2n_1n_2}{n_1 + n_2}\right) + 1$$

$$V(U) = \frac{2n_1n_2(2n_1n_2 - n_1 - n_2)}{(n_1 + n_2)^2(n_1 + n_2 - 1)}$$

Test Procedure for run test

1. Null Hypothesis (Ho)

H_o: The two populations having identical density function.

2. Alternative Hypothesis (H_1)

H_1: The two populations do not have identical density function.

3. Test statistic

Under the null hypothesis (H_0), the test statistic is

$$Z = \frac{U - E(U)}{\sqrt{V(U)}}$$

Where,

U = Number of runs in the combined sample

$$E(U) = \left(\frac{2n_1 n_2}{n_1 + n_2}\right) + 1$$

$$V(U) = \frac{2n_1 n_2 (2n_1 n_2 - n_1 - n_2)}{(n_1 + n_2)^2 (n_1 + n_2 - 1)}$$

4. Level of Significance

The level of significance may be fixed at either 5% or 1%.

5. Critical value

For $\alpha = 0.05$, $Z_{\alpha/2} = 1.96$

6. Inference

If the observed value of the test statistic Z exceeds the table value of Z, we reject the Null Hypothesis H_o otherwise accept it.

Example

The following data relates to two population observations

S-I	10	20	15	25	18	28	23	10	12	14		
S-II	11	13	18	28	30	32	24	27	22	11	12	22

Test whether the samples have come from sample population.

Procedure

- State the null hypothesis and alternative hypothesis
- State alpha
- Compute the test statistic

$$Z = \frac{U - E(U)}{\sqrt{V(U)}}$$

- To determine the critical value
- Compare the calculated test statistic to the critical value.

Calculation

Hypothesis

H_0: The samples are come from sample population.

H_1: The samples are not come from sample population.

Level of Significance

$\alpha = 0.05$

Test Statistic

The runs are,

10 10/ 11 11/12 12/13/14/15/18 18/20/22 22/23/24/25/27/28 28/30/32

Here, U = 16, $n_1 = 10$ and $n_2 = 12$

$$E(U) = \left(\frac{2 \times 10 \times 12}{10 + 12}\right) + 1$$

$$= 11.91$$

$$V(U) = \frac{2 \times 10 \times 12\ (2 \times 10 \times 12 - 10 - 12)}{(10 + 12)^2 (10 + 12 - 1)}$$

$$= 5.15$$

Under the null hypothesis (H_0), the test statistic is

$$Z = \frac{U - E(U)}{\sqrt{V(U)}}$$

$$Z = \frac{16 - 11.91}{\sqrt{5.15}}$$

$$= 1.80$$

Table Value

$Z_{\alpha/2} = 1.96$

Conclusion

Since Calculated Value is greater than the Table Value (i.e., 1.80 > 1.96). So, we reject the null Hypothesis. Hence, we conclude that the samples are not come from sample population.

Test for Randomness

Run test of randomness is a statistical test that is used to know the randomness in data. Run test of randomness is sometimes called the Geary test, and it is a nonparametric test. Run test of randomness is an alternative test to test autocorrelation in the data. Autocorrelation means that the data has correlation with its lagged value. To confirm whether or not the data has correlation with the lagged value, run test of randomness is applied.

Formula

The test statistic for test for randomness is given by

$$Z = \frac{U - E(U)}{\sqrt{V(U)}}$$

Where,

U = Number of runs in the sample

$$E(U) = \left(\frac{n+2}{2}\right)$$

$$V(U) = \frac{n}{4}\left(\frac{n-2}{n-1}\right)$$

Test Procedure for test for randomness

1. Null Hypothesis (Ho)

H_o: The sample is random

2. Alternative Hypothesis (H_1)

H_1: The sample is not random

3. Test statistic

Under the null hypothesis (H_0), the test statistic is

$$Z = \frac{U - E(U)}{\sqrt{V(U)}}$$

Where,

U = Number of rums in the combined sample

$$E(U) = \left(\frac{n+2}{2}\right)$$

$$V(U) = \frac{n}{4}\left(\frac{n-2}{n-1}\right)$$

4. Level of Significance

The level of significance may be fixed at either 5% or 1%.

5. Critical value

For $\alpha = 0.05$, $Z_\alpha = 1.65$

6. Inference

If the observed value of the test statistic Z exceeds the table value of Z, we reject the Null Hypothesis H_o otherwise accept it.

Example

The following are the number of students absent from a college on 24 consecutive days: 29, 25, 31, 28, 30, 28, 33, 31, 35, 29, 31, 33, 35, 28, 36, 30, 33, 26, 30, 28, 32, 31, 38 and 27. Test for randomness at 5% level of significance.

Procedure

- State the null hypothesis and alternative hypothesis
- State alpha
- Compute the test statistic

$$Z = \frac{U - E(U)}{\sqrt{V(U)}}$$

- To determine the critical value
- Compare the calculated test statistic to the critical value.

Calculation

Hypothesis

H_0: The samples are taken at random.

H_1: The samples are not taken at random.

Level of Significance

$\alpha = 0.05$

Test Statistic

The runs are,

25/ 26/ 27/ 28 28 28 28/29 29/ 30 30 30/31 31 31 31/32/ 33 33 33/ 35 35/ 36/ 38

Here, U = 12 and n = 24

$$E(U) = \left(\frac{24+2}{2}\right)$$

= 13

$$V(U) = \frac{24}{4}\left(\frac{24-2}{24-1}\right)$$

= 5.74

Under the null hypothesis (H_0), the test statistic is

$$Z = \frac{U - E(U)}{\sqrt{V(U)}}$$

$$Z = \frac{12 - 13}{\sqrt{5.74}}$$

$= | - 0.42 |$

$= 0.42$

Table Value

$Z_\alpha = 1.65$

Conclusion

Since Calculated Value is less than the Table Value (i.e., 0.42 < 1.65). So, we do not reject the null Hypothesis. Hence, we conclude that the samples are taken at random.

Median Test

The median test is a non-parametric test that is used to test whether two (or more) independent groups differ in central tendency - specifically whether the groups have been drawn from a population with the same median. The null hypothesis is that the groups are drawn from populations with the same median.

Formula

The test statistic for Median test is given by

$$\chi^2_{cal} = \sum_{i=1}^{m}\sum_{j=l}^{n}\left(\frac{(O_{ij} - E_{ij})^2}{E_{ij}}\right) \sim \chi^2_{(m-1)(n-1)}$$

$$\frac{R_i C_j}{N = n_1 + n_2} = E_{ij}$$

Where,

m = Number of rows

n = Number of columns

Test Procedure for Median test

1. Null Hypothesis (Ho)

H_o: The two samples have the same median.

2. Alternative Hypothesis (H_1)

H_1: The two samples do not have the same median.

3. Test statistic

Under the null hypothesis (H_0), the test statistic is

$$\chi^2_{cal} = \sum_{i=1}^{m}\sum_{j=i}^{n}\left(\frac{(O_{ij} - E_{ij})^2}{E_{ij}}\right) \sim \chi^2_{(m-1)(n-1)}$$

$$\frac{R_i C_j}{N = n_1 + n_2} = E_{ij}$$

Where,

m = Number of rows

n = Number of columns

4. Level of Significance

The level of significance may be fixed at either 5% or 1%.

5. Critical value

Find $\chi^2_{(m-1)(n-1)}$ from χ^2 for (m-1)(n-1) degrees of the freedom at level of significance.

6. Inference

If the observed value of the test statistic χ^2 exceeds the table value of χ^2, we reject the Null Hypothesis H_o otherwise accept it.

Example

3 random samples are drawn from 3 population gave the following values if whether the population have the same median.

S-I	1	2	5	7	8	9	3	2			
S-II	2	5	3	8	9	5	2	7	10		
S-III	3	4	2	5	7	8	9	7	11	8	12

Procedure

- State the null hypothesis and alternative hypothesis
- State alpha
- Compute the test statistic

$$\chi^2_{cal} = \sum_{i=1}^{m}\sum_{j=i}^{n}\left(\frac{(O_{ij} - E_{ij})^2}{E_{ij}}\right) \sim \chi^2_{(m-1)(n-1)}$$

- To determine the critical value
- Compare the calculated test statistic to the critical value.

Calculation

Hypothesis

H_0: The three samples have the same median.

H_1: The three samples do not the same median.

Level of Significance

$\alpha = 0.05$

Test Statistic

The median is,

1, 2, 2, 2, 2, 2, 3, 3, 3, 4, 5, 5, 5, ***5, 7***, 7, 7, 7, 8, 8, 8, 8, 9, 9, 9, 10, 11, 12

Median = (5 + 7) / 2

= 6

samples	No. of observation above median	No. of observation below median	Total
1	3	5	8
2	4	5	9
3	7	4	11
Total	14	14	28

$$E_{ij}(3) = \frac{8 \times 14}{28} = 4$$
$$E_{ij}(5) = \frac{8 \times 14}{28} = 4$$
$$E_{ij}(4) = \frac{9 \times 14}{28} = 4.5$$
$$E_{ij}(5) = \frac{9 \times 14}{28} = 4.5$$
$$E_{ij}(7) = \frac{11 \times 14}{28} = 5.5$$
$$E_{ij}(4) = \frac{11 \times 14}{28} = 5.5$$

Observed frequency	Expected Frequency	$(O_i – E_i)^2/E_i$
3	4	0.25
5	4	0.25
4	4.5	0.0556
5	4.5	0.0556
7	5.5	0.4091
4	5.5	0.4091
	Total	1.4294

$$\chi^2 = 1.494$$

Table Value

Degrees of freedom = (m-1) (n-1)

= (3-1) (2-1) = (2) (1) df

$$\chi^2_{(2)\,(1)} = 5.99$$

Conclusion

Since Calculated Value is less than the Table Value (i.e., 1.4294 < 5.99). So, we do not reject the null Hypothesis. Hence, we conclude that the three samples have same median.

Sign Test

The sign test compares the sizes of two groups. It is a non-parametric or "distribution free" test, which means the test doesn't assume the data comes from a particular distribution, like the normal distribution. The sign test is an alternative to a one sample t test or a paired t test. There are two types sign test:

(a) One sample sign test

(b) Two sample sign test

Used for Sign test

- To determine the preference for one product over the other
- Conduct a test for the median of a single population (one sample sign test)
- To perform a test for the median of paired difference using the data from two dependent samples.

One sample sign test

The One-sample Sign Test simply determines a significance test of a hypothesized median value for a single data set. The One sample sign test is a Non Parametric Hypothesis test used to determine whether a statistically significant difference exists between the median of a non-normally distributed continuous data set and a standard. This test basically concerns the median of a continuous population. It is also called the binominal sign test, with p = 0.5.

Formula

The formula for one sample sign test is,

$$P_x = \frac{n!}{x!\,(n-x)!}\, p^x q^{n-x}$$

Where,

P_x = the probability of exactly x events appearing in n trials.

p^x = the expected probability of the event associated with the x term on any given trial.

q^{n-x} = the probability of an event on any given trial q = 1- p

n = the number of events.

x = the number of a given outcome being evaluated

In the application of the sign test, the binomial equation can be used to determine if the number of (+) signs occurs more or less often than the number of (-) signs.

Test Procedure for One sample sign test

1. Null Hypothesis (Ho)

H_o: The difference between medians is zero.

2. Alternative Hypothesis (H_1)

H_1: The difference between medians is not zero.

3. Test statistic

Under the null hypothesis (H_0), the test statistic is

$$P_x = \frac{n!}{x!\,(n-x)!}\, p^x q^{n-x}$$

Where,

P_x = the probability of exactly x events appearing in n trials.

p^x = the expected probability of the event associated with the x term on any given trial.

q^{n-x} = the probability of an event on any given trial q = 1- p

n = the number of events.

x = the number of a given outcome being evaluated

4. Level of Significance

The level of significance may be fixed at either 5% or 1%.

6. Inference

If the observed value of the test statistic p exceeds the level of significance, we do not reject the Null Hypothesis H_o otherwise reject it.

Example

A Bank of America West Palm Beach, FL branch manager shows that the median number of savings account customers per day is 64. A clerk from the same branch claims that it was more than 64. The clerk found the number of savings accounts customers per day data for 10 random days. Can we reject the branch manager's claim at a 0.05 significance level?

Day	1	2	3	4	5	6	7	8	9	10
Customer	60	66	65	70	68	72	46	76	77	75

Procedure

- State the null hypothesis and alternative hypothesis
- State alpha
- Compute the test statistic

$$P_x = \frac{n!}{x!\,(n-x)!}\, p^x q^{n-x}$$

- To determine the critical value
- Compare the calculated test statistic to the critical value.

Calculation

Hypothesis

H_0: Savings account customer median = 64.

H_1: Savings account customer median > 64

Level of Significance

$$\alpha = 0.05$$

Test Statistic

Assign observations less than 64 with a – sign and observations above 64 with a + sign.

Day	1	2	3	4	5	6	7	8	9	10
Customer	60	66	65	70	68	72	46	76	77	75
Sign	-	+	+	+	+	+	-	+	+	+

Total number of + values =8

Total number of – values =2

Here, n = 10, x = 8 and p = 0.5

Under the null hypothesis (H_0), the test statistic is

$$P_x = \frac{n!}{x!\,(n-x)!}\, p^x q^{n-x}$$

$$P_x = \frac{10!}{8!\,(10-8)!} \times 0.5^8 \times 0.5^{10-8}$$

$$= 45 \times 0.0039 \times 0.25$$

$$= 0.0439$$

Conclusion

Since probability Value is less than the level of significance (i.e., $0.0439 < 0.05$). So, we reject the null Hypothesis. Hence, we conclude that the Savings account customer median is greater than 64.

Two sample sign test

The sign test has the null hypothesis that both samples are from the same population. The sign test compares the two dependent observations and counts the number of negative and positive differences. It uses the standard normal distributed z-value to test of significance.

Formula

The test statistic for Median test is given by

$$Z = \frac{2U - n}{\sqrt{n}}$$

Where,

U = Minimum number of signs obtained

n = Sample size

Test Procedure for run test

1. Null Hypothesis (Ho)

H_o: Two populations have identical distribution.

2. Alternative Hypothesis (H_1)

H_1: Two populations have different distribution.

3. Test statistic

Under the null hypothesis (H_0), the test statistic is

$$Z = \frac{2U - n}{\sqrt{n}}$$

Where,

U = Minimum number of signs obtained

n = Sample size

4. Level of Significance

The level of significance may be fixed at either 5% or 1%.

5. Critical value

When $\alpha = 0.05$, $Z_{\alpha/2} = 1.96$.

6. Inference

If the observed value of the test statistic Z exceeds the table value of Z, we reject the Null Hypothesis H_o otherwise accept it.

Example

A random sample of paired observation is given below (10,11), (11,13), (12,10), (13,13), (14,15), (11,14), (12,13), (13,12), (10,8), (10,13), (14,15), (15,17), (15,13), (11,10), (8,9), (9,9), (11,9), (12,14), (13,11), (11, 11). Apply approximately non-parametric test. Test whether there is any gain in B=X-Y.

Procedure

- State the null hypothesis and alternative hypothesis
- State alpha
- Compute the test statistic

$$Z = \frac{2U - n}{\sqrt{n}}$$

- To determine the critical value
- Compare the calculated test statistic to the critical value.

Calculation

Hypothesis

H_0: There is no gain in B=X-Y.

H_1: There is gain in B=X-Y.

Level of Significance

$$\alpha = 0.05$$

Test Statistic

X	Y	Sign	X	Y	Sign
10	11	-	14	15	-
11	13	-	15	17	-
12	10	+	15	13	+
13	13	0	11	10	+
14	15	-	8	9	-
11	14	-	9	9	0
12	13	-	11	9	+
13	12	+	12	14	-
10	8	+	13	11	+
10	13	-	11	11	0

Reduced Sample Size=Total Number signed observations-Non-Signed Observations

=20 – 3=17

Under the null hypothesis (H_0), the test statistic is

$$Z = \frac{2U - n}{\sqrt{n}}$$

$$Z = \frac{2(7) - 17}{\sqrt{17}}$$

$= |-0.7276|$

$= 0.7276$

Table Value

$Z_{\alpha/2} = 1.96$

Conclusion

Since Calculated Value is less than the Table Value (i.e., 0.7276 < 1.96). So, we do not reject the null Hypothesis. Hence, we conclude that there is no gain in B = X - Y.

Wilcoxon's Signed Rank Test

The **Wilcoxon signed rank test** is a non-parametric test to compare dependent samples t-test data. When the word "non-parametric" is used in stats, it doesn't quite mean that you know nothing about the population. It usually means that you know the population data does not have a normal distribution. The Wilcoxon signed rank test should be used if the differences between pairs of data are non-normally distributed.

Assumptions

Two slightly different versions of the test exist:

- The **Wilcoxon signed rank test** compares your sample median against a hypothetical median.
- The **Wilcoxon matched-pairs signed rank test** computes the difference between each set of matched pairs and then follows the same procedure as the signed rank test to compare the sample against some median.

Formula

The formula for **Wilcoxon signed rank test** is

$$Z = \frac{T - \frac{n(n+1)}{4}}{\sqrt{\frac{n(n+1)(2n+1)}{24}}}$$

Where,

T = Minimum Sum of the Rank Value

n= Sample Size

Test Procedure for Wilcoxon's Signed Rank Test

1. Null Hypothesis (Ho)

H_0: The difference between the paired observations in the population is zero.

2. Alternative Hypothesis (H_1)

H_1: The difference between the paired observations is not equal to zero.

3. Test statistic

Under the null hypothesis (H_0), the test statistic is

$$Z = \frac{T - \frac{n(n+1)}{4}}{\sqrt{\frac{n(n+1)(2n+1)}{24}}}$$

Where,

T = Minimum Sum of the Rank Value

n= Sample Size

4. Level of Significance

The level of significance may be fixed at either 5% or 1%.

5. Critical value

When $\alpha = 0.05$, $Z_{\alpha/2} = 1.96$

6. Inference

If the observed value of the test statistic Z exceeds the table value of Z, we reject the Null Hypothesis H_o otherwise accept it.

Example

An experiment is conducted to judge the effect of brand name on quality perception. 16 subjects are recruited for the purpose and are asked to taste and compare two samples of product on a set of scale items judged to be ordinal. The following data are obtained:

Pair	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Brand A	73	43	47	53	58	47	52	58	38	61	56	56	34	55	65	75
Brand B	51	41	43	41	47	32	24	58	43	53	52	57	44	57	40	68

Test the hypothesis, using Wilcoxon matched-pairs test, that there is no difference between the perceived qualities of the two samples. Use 5% level of significance.

Procedure

- State the null hypothesis and alternative hypothesis
- State alpha
- Compute the test statistic

$$Z = \frac{T - \frac{n(n+1)}{4}}{\sqrt{\frac{n(n+1)(2n+1)}{24}}}$$

- To determine the critical value
- Compare the calculated test statistic to the critical value.

Calculation

Hypothesis

H_0: There is no difference between the perceived quality of two samples.

H_1: There is difference between the perceived quality of two samples.

Level of Significance

$\alpha = 0.05$

Test Statistic

Using Wilcoxon matched-pairs test, we work out the value of the test statistic T as under:

Pair	Brand A	Brand B	d_i	Rank of d_i	Sign +	Sign -
1	73	51	22	13	13	-
2	43	41	2	2.5	2.5	-
3	47	43	4	4.5	4.5	-
4	53	41	12	11	11	-
5	58	47	11	10	10	-
6	47	32	15	12	12	-
7	52	24	28	15	15	-
8	58	58	0	-	-	-
9	38	43	-5	6	-	6
10	61	53	8	8	8	-
11	56	52	4	4.5	4.5	-
12	56	57	-1	1	-	1
13	34	44	-10	9	-	9
14	55	57	-2	2.5	-	2.5
15	65	40	25	14	14	-
16	75	68	7	7	7	-
				Total	101.5	18.5

$W^- = 18.5$

$W^+ = 101.5$

Under the null hypothesis (H_0), the test statistic is

$$Z = \frac{T - \frac{n(n+1)}{4}}{\sqrt{\frac{n(n+1)(2n+1)}{24}}}$$

Here, T = 18.5 and n = 15

$$Z = \frac{18.5 - \frac{15(15+1)}{4}}{\sqrt{\frac{15(15+1)(2\times 15+1)}{24}}}$$

$$Z = \frac{18.5 - 60}{\sqrt{310}}$$

$$Z = \frac{-41.5}{17.61}$$

$Z = |-2.36|$

$= 2.36$

Table Value

$Z_{\alpha/2} = 1.96$

Conclusion

Since Calculated Value is greater than the Table Value (i.e., 2.36 > 1.96). So, we reject the null Hypothesis. Hence, we conclude that there is difference between the perceived quality of two samples.

Mann-Whitney U Test

Mann-Whitney u-Test is a non-parametric test used to test whether two independent samples were selected from population having the same distribution. Another name for the Mann-Whitney U Test is Wilcoxon Rank Sum Test.

Assumptions

Mann-Whitney U test is a non-parametric test, so it does not assume any assumptions related to the distribution of scores. There are, however, some assumptions that are assumed

- The sample drawn from the population is random.
- Independence within the samples and mutual independence is assumed. That means that an observation is in one group or the other (it cannot be in both).
- Ordinal measurement scale is assumed.

Formula

The Mann-Whitney U test formula is

$$U = n_1 n_2 + \frac{n_2(n_2+1)}{2} - \sum_{i=n_1}^{n_2} R_i$$

Where,

U = Mann-Whitney U test
n_1 = Sample Size One
n_2 = Sample Size Two
R_i = Rank of the sample Size

Test Procedure for Mann-Whitney U Test

1. Null Hypothesis (Ho)

H_o: The populations have the same density function.

2. Alternative Hypothesis (H_1)

H_1: The populations do not have the same density function

3. Test statistic

Under the null hypothesis (H_0), the test statistic is

$$U = n_1 n_2 + \frac{n_2(n_2+1)}{2} - \sum_{i=n_1}^{n_2} R_i$$

Where,

U = Mann-Whitney U test

n_1 = *Sample Size One*

n_2 = *Sample Size Two*

R_i = *Rank of the sample Size*

4. Level of Significance

The level of significance may be fixed at either 5% or 1%.

5. Critical value

Find Mann-Whitney U table for (n_1, n_2) degrees of the freedom at level of significance.

6. Inference

If the observed value of the test statistic U exceeds the table value of Mann-Whitney U, we reject the Null Hypothesis H_0 otherwise accept it.

Example

The values in one sample are 53, 38, 69, 57, 46, 39, 73, 48, 73, 74, 60 and 78. In another sample they are 44, 40, 61, 52, 32, 44, 70, 41, 67, 72, 53 and 72. Test at the 5% level the hypothesis that they come from populations with the same mean. Apply U-test.

Procedure

- State the null hypothesis and alternative hypothesis
- State alpha
- Compute the test statistic

$$U = n_1 n_2 + \frac{n_2(n_2+1)}{2} - \sum_{i=n_1}^{n_2} R_i$$

- To determine the critical value
- Compare the calculated test statistic to the critical value.

Calculation

Hypothesis

H_0: The population has same mean.

H_1: The population does not have same mean.

Level of Significance

$\alpha = 0.05$

Test Statistic

Using Mann-Whitney U test, we work out the value of the test statistic R as under:

Size of sample	Rank	Name of related Sample	Size of sample	Rank	Name of related Sample
32	1	B	57	13	A
38	2	A	60	14	A
39	3	A	61	15	B
40	4	B	67	16	B
41	5	B	69	17	A
44	6.5	B	70	18	B
44	6.5	B	72	19.5	B
46	8	A	72	19.5	B
48	9	A	73	21.5	A
52	10	B	73	21.5	A
53	11.5	B	74	23	A
53	11.5	A	78	24	A

The ranks assigned to sample one $R_1$is,

R_1= 2 + 3 + 8 + 9 + 11.5 + 13 + 14 + 17 + 21.5 + 21.5 + 23 + 24 = 167.5

The ranks assigned to sample two R_2 is,

R_2 = 1 + 4 + 5 + 6.5 + 6.5 + 10 + 11.5 + 15 + 16 + 18 + 19.5 + 19.5 = 132.5

we have $n_1 = 12$ and $n_2 = 12$

Under the null hypothesis (H_0), the test statistic is

$$U = n_1 n_2 + \frac{n_2(n_2+1)}{2} - \sum_{i=n_1}^{n_2} R_i$$

The test statistic for the Mann Whitney U Test is denoted **U** and is the ***smaller*** of U_1 and U_2, defined below.

$$U_1 = n_1 n_2 + \frac{n_2(n_2+1)}{2} - R_1$$

$$U_1 = (12 \times 12) + \frac{12(12+1)}{2} - 167.5$$

= 144 + 78 – 167.5

= 54.5

$$U_2 = n_1 n_2 + \frac{n_2(n_2+1)}{2} - R_2$$

$$U_1 = (12 \times 12) + \frac{12(12+1)}{2} - 132.5$$

$$= 144 + 78 - 132.5$$

$$= 89.5$$

Therefore, U = 54.5.

Table Value

The degrees of freedom, (n_1,n_2):

df = (12, 12)

Table Value = 37

Conclusion

Since Calculated Value is greater than the Table Value (i.e., 54.5 > 32). So, we reject the null Hypothesis. Hence, we conclude that the population does not have same mean.

Kolmogorov-Smirvov Two Sample Test

Kolmogorov-Smirvov test is used for testing whether there is a significance difference between an observed and theoretical frequency. Kolmogorov-Smirvov test is another measure of the goodness of fit of a theoretical frequency distribution.

Formula

The formula for Kolmogorov-Smirvov Test is,

$$KS = Max|F_o - F_e|$$

Where,

F_o = Probability of Observed frequency

F_e = Probability of Expected frequency

Test Procedure for Kolmogorov-Smirvov Test

1. Null Hypothesis (Ho)

H_o: There is no significance difference between an observed and theoretical frequency.

2. Alternative Hypothesis (H_1)

H_1: There is a significance difference between an observed and theoretical frequency.

3. Test statistic

Under the null hypothesis (H_0), the test statistic is

$$KS = Max|F_o - F_e|$$

Where,

F_o = Probability of Observed frequency

F_e = Probability of Expected frequency

4. Level of Significance

The level of significance may be fixed at either 5% or 1%.

5. Critical value

The critical values for the KS in a Kolmogorov-Smirvov test depend on the degrees of freedom and the significance level of the test.

6. Inference

If the observed value of the test statistic KS exceeds the table value of Kolmogorov-Smirvov, we reject the Null Hypothesis H_o otherwise accept it.

Example

Compute Kolmogorov-Smirvov test for the following data:

Class	51-60	61-70	71-80	81-90	91-100
Observed frequency	30	100	440	500	130
Expected frequency	40	170	500	390	100

Procedure

- State the null hypothesis and alternative hypothesis
- State alpha
- Compute the test statistic

$$KS = Max|F_o - F_e|$$

- To determine the critical value
- Compare the calculated test statistic to the critical value.

Calculation

Hypothesis

H_0: There is no significance difference between an observed and expected frequency.

H_1: There is a significance difference between an observed and expected frequency.

Level of Significance

$\alpha = 0.05$

Test Statistic

The observed cumulative distribution functions of the two samples

Observed frequency	Cumulative Observed frequency	F_o	Expected frequency	Cumulative Expected frequency	F_e	$\lvert F_o - F_e \rvert$
30	30	0.025	40	40	0.033	0.008
100	130	0.108	170	210	0.175	0.067
440	570	0.475	500	710	0.592	0.117
500	1070	0.892	390	1100	0.917	0.025
130	1200	1.000	100	1200	1.000	0.000

Under the null hypothesis (H_0), the test statistic is

$$KS = Max|F_o - F_e|$$

$$= 0.117$$

Table Value

The degrees of freedom = n

= 5

Table Value = 0.510

Conclusion

Since Calculated Value is less than the Table Value (i.e., 0.117 < 0.510). So, we do not reject the null Hypothesis. Hence, we conclude that there is no significance difference between an observed and expected frequency.

Kruskal-Wallis Test

The Kruskal-Wallis H test is a rank-based nonparametric test that can be used to determine if there are statistically significant differences between two or more groups of an independent variable on a continuous or ordinal dependent variable. It is considered the nonparametric alternative to the one-way ANOVA, and an extension of the Mann-Whitney U test to allow the comparison of more than two independent groups.

Assumptions

There are certain assumptions in the Kruskal-Wallis test.

- It is assumed that the observations in the data set are independent of each other.
- It is assumed that the distribution of the population should not be necessarily normal and the variances should not be necessarily equal.
- It is assumed that the observations must be drawn from the population by the process of random sampling.

Formula

The formula for Kruskal-Wallis Test is,

$$H = \frac{12}{n(n+1)} \sum \frac{R_j^2}{n_j} - 3(n+1)$$

Where,

$\sum \frac{R_j^2}{n_j}$ = sum of squares of all groups

n = Sample Size

Test Procedure for Kruskal-Wallis Test

1. Null Hypothesis (Ho)

H_o: There is no difference between the sample means.

2. Alternative Hypothesis (H_1)

H_1: There is difference between the sample means.

3. Test statistic

Under the null hypothesis (H_0), the test statistic is

$$H = \frac{12}{n(n+1)} \sum \frac{R_j^2}{n_j} - 3(n+1)$$

Where,

$\sum \frac{R_j^2}{n_j}$ = sum of squares of all groups

n = Sample Size

4. Level of Significance

The level of significance may be fixed at either 5% or 1%.

5. Critical value

find $\chi^2_{(k-1)}$ from χ^2 for (k-1) degrees of the freedom at level of significance.

6. Inference

If the observed value of the test statistic H exceeds the table value of χ^2, we reject the Null Hypothesis H_o otherwise accept it.

Example

Use the Kruskal-Wallis test at 5% level of significance to test the null hypothesis that a professional bowler performs equally well with the four bowling balls, given the following results:

	Bowling Results in Five Games				
With Ball No. A	271	282	257	248	262
With Ball No. B	252	275	302	268	276
With Ball No. C	260	255	239	246	266
With Ball No. D	279	242	297	270	258

Procedure

- State the null hypothesis and alternative hypothesis
- State alpha
- Compute the test statistic

$$H = \frac{12}{n(n+1)} \sum \frac{R_j^2}{n_j} - 3(n+1)$$

- To determine the critical value
- Compare the calculated test statistic to the critical value.

Calculation

Hypothesis

H_0: Bowler performs equally well with the four bowling balls.

H_1: Bowler performs not equally well with the four bowling balls.

Level of Significance

$\alpha = 0.05$

Test Statistic

To apply the H test or the Kruskal-Wallis test to this problem, we begin by ranking all the given figures from the highest to the lowest, indicating besides each the name of the ball as under:

	Bowling Results in Five Games					Total
With Ball No. A	271	282	257	248	262	
Rank	7	3	14	17	11	**52**
With Ball No. B	252	275	302	268	276	
Rank	16	6	1	9	5	**37**
With Ball No. C	260	255	239	246	266	
Rank	12	15	20	18	10	**75**
With Ball No. D	279	242	297	270	258	
Rank	4	19	2	8	13	**46**

Under the null hypothesis (H_0), the test statistic is

$$H = \frac{12}{n(n+1)}\sum \frac{R_j^2}{n_j} - 3(n+1)$$

$$H = \frac{12}{20(20+1)}\left\{\frac{52^2}{5} + \frac{37^2}{5} + \frac{75^2}{5} + \frac{46^2}{5}\right\} - 3(20+1)$$

$$H = (0.02857 \times 2362.8) - 63$$

$$H = 67.51 - 63$$

$$H = 4.51$$

Table Value

The degrees of freedom, (Number of Groups Minus 1): (k - 1)

df = 4 – 1 = 3df

Table Value = 7.815

Conclusion

Since Calculated Value is less than the Table Value (i.e., 4.51 < 7.815). So, we do not reject the null Hypothesis. Hence, we conclude that the Bowler performs equally well with the four bowling balls.

Friedman Test

Friedman Test is a non-parametric test alternative to the one-way ANOVA with repeated measures. It tries to determine if subjects changed significantly across occasions/conditions. For example: - Problem-solving ability of a set of people is the same or different in Morning, Afternoon, Evening. It is used to test for differences between groups when the dependent variable is ordinal. This test is particularly useful when the sample size is very small.

Elements of Friedman Test

- One group that is measured on three or
  more blocks of measures overtime/experimental conditions.
- One dependent variable which can be Ordinal, Interval or Ratio.

Assumptions of Friedman Test

- The group is a random sample from the population
- No interaction between blocks (rows) and treatment levels (columns)
- The one group that is measured on three or more different occasions
- Data should be at least an ordinal or continuous
- The samples are do not need to be normally distributed

Formula

The formula for Friedman test is,

$$F_R = \frac{12}{nk(k+1)} \sum R_i^2 - 3n(K+1)$$

Where,

n = total number of subjects/participants.

k = total number of blocks to be measured.

R_i = sum of ranks of all subjects for a block i

Test Procedure for Friedman Test

1. Null Hypothesis (Ho)

H_o: There is no significant difference between the means of three or more groups.

2. Alternative Hypothesis (H_1)

H_1: There is significant difference between the means of three or more groups.

3. Test statistic

Under the null hypothesis (H_0), the test statistic is

$$F_R = \frac{12}{nk(k+1)}\sum R_i^2 - 3n(K+1)$$

Where,

n = total number of subjects/participants.

k = total number of blocks to be measured.

R_i = sum of ranks of all subjects for a block i

4. Level of Significance

The level of significance may be fixed at either 5% or 1%.

5. Critical value

Find $\chi^2_{(k-1)}$ from χ^2 for (k-1) degrees of the freedom at level of significance.

6. Inference

If the observed value of the test statistic F exceeds the table value of χ^2, we reject the Null Hypothesis H_o otherwise accept it.

Example

7 random people were given 3 different drugs and for each person, the reaction time corresponding to the drugs were noted. Test the claim at the 5% significance level that all the 3 drugs have the same probability distribution.

	Drug A	Drug B	Drug C
1	1.24	1.50	1.62
2	1.71	1.85	2.05
3	1.37	2.12	1.68
4	2.53	1.87	2.62
5	1.23	1.34	1.51
6	1.94	2.33	2.86
7	1.72	1.43	2.86

Procedure

- State the null hypothesis and alternative hypothesis
- State alpha
- Compute the test statistic

$$F_R = \frac{12}{nk(k+1)}\sum R_i^2 - 3n(K+1)$$

- To determine the critical value
- Compare the calculated test statistic to the critical value.

Calculation

Hypothesis

H_0: All three drugs have the same probability distribution.

H_1: At least two of them differ from each other.

Level of Significance

$\alpha = 0.05$

Test Statistic

Under the null hypothesis (H_0), the test statistic is

$$F_R = \frac{12}{nk(k+1)} \sum R_i^2 - 3n(K+1)$$

Assign Ranks for the drugs corresponding to each person and find the sum.

Ranks will be in ascending order.

	Drug A	Rank	Drug B	Rank	Drug C	Rank
1	1.24	1	1.50	2	1.62	3
2	1.71	1	1.85	2	2.05	3
3	1.37	1	2.12	3	1.68	2
4	2.53	2	1.87	1	2.62	3
5	1.23	1	1.34	2	1.51	3
6	1.94	1	2.33	2	2.86	3
7	1.72	2	1.43	1	2.86	3
Total		9		13		20

Here, n = 7 and k = 3

$$F_R = \frac{12}{7\times3(3+1)}\{9^2 + 13^2 + 20^2\} - 3 \times 7 \times 4$$

$$F_R = \frac{12}{84} \times 650 - 84$$

$$F_R = 92.8584$$

$$F_R = 8.85$$

Table Value

The degrees of freedom, (Number of Groups Minus 1):

df = 3 – 1 = 2df

Table Value = 5.991

Conclusion

Since Calculated Value is greater than the Table Value (i.e., 8.85 > 5.991). So, we reject the null Hypothesis. Hence, we conclude that all the three drugs do not have the same probability distribution.

McNemar's Test

McNemer test is one of the important nonparametric tests often used when the data happen to be nominal and relate to two related samples. As such this test is especially useful with before-after measurement of the same subjects.

Assumptions

The McNemar test has three assumptions that have to be met before running the test:

- One categorical dependent variable with two categories (i.e., a dichotomous variable) and one categorical independent variable with two related groups.
- The two groups of your dependent variable must be mutually exclusive. This means that no groups can overlap: a participant can only be in one of the two groups!
- The cases (e.g., participants) are a random sample from the population of interest.

Formula

The test statistic under McNemer Test is

Before	After	
	Yes	No
Yes	a	b
No	c	d

$$\chi^2 = \frac{(b-c)^2}{(b+c)} \sim 1df$$

Example

In a certain before-after experiment the responses obtained from 1000 respondents, when classified, gave the following information:

Before treatment	After treatment	
	Favorable response	Unfavorable response
Favorable response	300	200
Unfavorable response	100	400

Test at 5% level of significance, whether there has been a significant change in people's attitude before and after the concerning experiment.

Procedure

- State the null hypothesis and alternative hypothesis
- State alpha
- Compute the test statistic

$$\chi^2 = \frac{(b-c)^2}{(b+c)} \sim 1df$$

- To determine the critical value
- Compare the calculated test statistic to the critical value.

Calculation

Hypothesis

H_0: There is no significant change in people's attitude before and after the concerning experiment.

H_1: There is a significant change in people's attitude before and after the concerning experiment.

Level of Significance

$\alpha = 0.05$

Test Statistic

Under the null hypothesis (H_0), the test statistic is

$$\chi^2 = \frac{(b-c)^2}{(b+c)} \sim 1df$$

Before treatment	After treatment	
	Favorable response	Unfavorable response
Favorable response	300 (a)	200 (b)
Unfavorable response	100 (c)	400 (d)

$$\chi^2 = \frac{(200-100)^2}{(200+100)}$$

$$\chi^2 = \frac{(100)^2}{(300)}$$

$$= 33.33$$

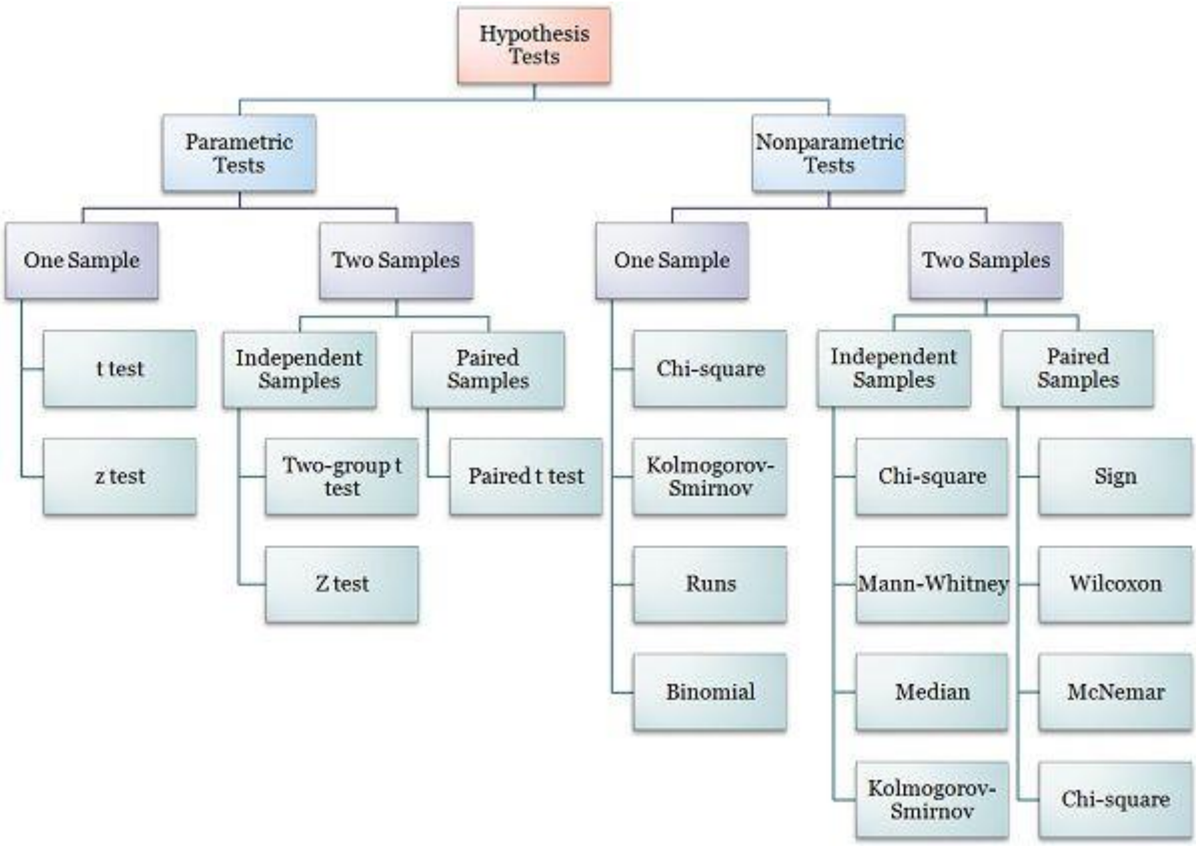
Table Value

The degrees of freedom = 1

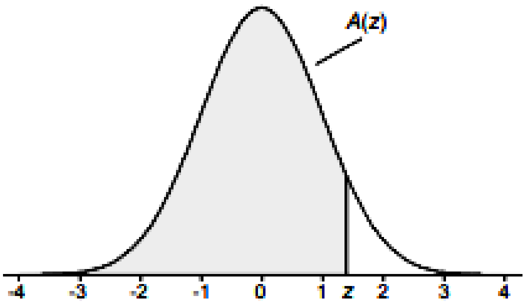
Table Value = 3.84

Conclusion

Since Calculated Value is greater than the Table Value (i.e., 33.33 < 3.84). So, we reject the null Hypothesis. Hence, we conclude that there is a significant change in people's attitude before and after the concerning experiment.



Cumulative Standardized Normal Distribution



$A(z)$ is the integral of the standardized normal distribution from $-\infty$ to z (in other words, the area under the curve to the left of z). It gives the probability of a normal random variable not being more than z standard deviations above its mean. Values of z of particular importance:

z	$A(z)$	
1.645	0.9500	Lower limit of right 5% tail
1.960	0.9750	Lower limit of right 2.5% tail
2.326	0.9900	Lower limit of right 1% tail
2.576	0.9950	Lower limit of right 0.5% tail
3.090	0.9990	Lower limit of right 0.1% tail
3.291	0.9995	Lower limit of right 0.05% tail

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.6	0.9998	0.9998	0.9999							

t Distribution: Critical Values of *t*

Degrees of freedom	Two-tailed test: / One-tailed test:	10% / 5%	5% / 2.5%	2% / 1%	1% / 0.5%	0.2% / 0.1%	0.1% / 0.05%
		Significance level					
1		6.314	12.706	31.821	63.657	318.309	636.619
2		2.920	4.303	6.965	9.925	22.327	31.599
3		2.353	3.182	4.541	5.841	10.215	12.924
4		2.132	2.776	3.747	4.604	7.173	8.610
5		2.015	2.571	3.365	4.032	5.893	6.869
6		1.943	2.447	3.143	3.707	5.208	5.959
7		1.894	2.365	2.998	3.499	4.785	5.408
8		1.860	2.306	2.896	3.355	4.501	5.041
9		1.833	2.262	2.821	3.250	4.297	4.781
10		1.812	2.228	2.764	3.169	4.144	4.587
11		1.796	2.201	2.718	3.106	4.025	4.437
12		1.782	2.179	2.681	3.055	3.930	4.318
13		1.771	2.160	2.650	3.012	3.852	4.221
14		1.761	2.145	2.624	2.977	3.787	4.140
15		1.753	2.131	2.602	2.947	3.733	4.073
16		1.746	2.120	2.583	2.921	3.686	4.015
17		1.740	2.110	2.567	2.898	3.646	3.965
18		1.734	2.101	2.552	2.878	3.610	3.922
19		1.729	2.093	2.539	2.861	3.579	3.883
20		1.725	2.086	2.528	2.845	3.552	3.850
21		1.721	2.080	2.518	2.831	3.527	3.819
22		1.717	2.074	2.508	2.819	3.505	3.792
23		1.714	2.069	2.500	2.807	3.485	3.768
24		1.711	2.064	2.492	2.797	3.467	3.745
25		1.708	2.060	2.485	2.787	3.450	3.725
26		1.706	2.056	2.479	2.779	3.435	3.707
27		1.703	2.052	2.473	2.771	3.421	3.690
28		1.701	2.048	2.467	2.763	3.408	3.674
29		1.699	2.045	2.462	2.756	3.396	3.659
30		1.697	2.042	2.457	2.750	3.385	3.646
32		1.694	2.037	2.449	2.738	3.365	3.622
34		1.691	2.032	2.441	2.728	3.348	3.601
36		1.688	2.028	2.434	2.719	3.333	3.582
38		1.686	2.024	2.429	2.712	3.319	3.566
40		1.684	2.021	2.423	2.704	3.307	3.551
42		1.682	2.018	2.418	2.698	3.296	3.538
44		1.680	2.015	2.414	2.692	3.286	3.526
46		1.679	2.013	2.410	2.687	3.277	3.515
48		1.677	2.011	2.407	2.682	3.269	3.505
50		1.676	2.009	2.403	2.678	3.261	3.496
60		1.671	2.000	2.390	2.660	3.232	3.460
70		1.667	1.994	2.381	2.648	3.211	3.435
80		1.664	1.990	2.374	2.639	3.195	3.416
90		1.662	1.987	2.368	2.632	3.183	3.402
100		1.660	1.984	2.364	2.626	3.174	3.390
120		1.658	1.980	2.358	2.617	3.160	3.373
150		1.655	1.976	2.351	2.609	3.145	3.357
200		1.653	1.972	2.345	2.601	3.131	3.340
300		1.650	1.968	2.339	2.592	3.118	3.323

F Distribution: Critical Values of *F* (5% significance level)

ν_2 \ ν_1	1	2	3	4	5	6	7	8	9	10	12	14	16	18	20
1	161.45	199.50	215.71	224.58	230.16	233.99	236.77	238.88	240.54	241.88	243.91	245.36	246.46	247.32	248.01
2	18.51	19.00	19.16	19.25	19.30	19.33	19.35	19.37	19.38	19.40	19.41	19.42	19.43	19.44	19.45
3	10.13	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81	8.79	8.74	8.71	8.69	8.67	8.66
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96	5.91	5.87	5.84	5.82	5.80
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77	4.74	4.68	4.64	4.60	4.58	4.56
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06	4.00	3.96	3.92	3.90	3.87
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64	3.57	3.53	3.49	3.47	3.44
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35	3.28	3.24	3.20	3.17	3.15
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14	3.07	3.03	2.99	2.96	2.94
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02	2.98	2.91	2.86	2.83	2.80	2.77
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90	2.85	2.79	2.74	2.70	2.67	2.65
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80	2.75	2.69	2.64	2.60	2.57	2.54
13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71	2.67	2.60	2.55	2.51	2.48	2.46
14	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65	2.60	2.53	2.48	2.44	2.41	2.39
15	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59	2.54	2.48	2.42	2.38	2.35	2.33
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54	2.49	2.42	2.37	2.33	2.30	2.28
17	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49	2.45	2.38	2.33	2.29	2.26	2.23
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46	2.41	2.34	2.29	2.25	2.22	2.19
19	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42	2.38	2.31	2.26	2.21	2.18	2.16
20	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39	2.35	2.28	2.22	2.18	2.15	2.12
21	4.32	3.47	3.07	2.84	2.68	2.57	2.49	2.42	2.37	2.32	2.25	2.20	2.16	2.12	2.10
22	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34	2.30	2.23	2.17	2.13	2.10	2.07
23	4.28	3.42	3.03	2.80	2.64	2.53	2.44	2.37	2.32	2.27	2.20	2.15	2.11	2.08	2.05
24	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30	2.25	2.18	2.13	2.09	2.05	2.03
25	4.24	3.39	2.99	2.76	2.60	2.49	2.40	2.34	2.28	2.24	2.16	2.11	2.07	2.04	2.01
26	4.22	3.37	2.98	2.74	2.59	2.47	2.39	2.32	2.27	2.22	2.15	2.09	2.05	2.02	1.99
27	4.21	3.35	2.96	2.73	2.57	2.46	2.37	2.31	2.25	2.20	2.13	2.08	2.04	2.00	1.97
28	4.20	3.34	2.95	2.71	2.56	2.45	2.36	2.29	2.24	2.19	2.12	2.06	2.02	1.99	1.96
29	4.18	3.33	2.93	2.70	2.55	2.43	2.35	2.28	2.22	2.18	2.10	2.05	2.01	1.97	1.94
30	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21	2.16	2.09	2.04	1.99	1.96	1.93
35	4.12	3.27	2.87	2.64	2.49	2.37	2.29	2.22	2.16	2.11	2.04	1.99	1.94	1.91	1.88
40	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12	2.08	2.00	1.95	1.90	1.87	1.84
50	4.03	3.18	2.79	2.56	2.40	2.29	2.20	2.13	2.07	2.03	1.95	1.89	1.85	1.81	1.78
60	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04	1.99	1.92	1.86	1.82	1.78	1.75
70	3.98	3.13	2.74	2.50	2.35	2.23	2.14	2.07	2.02	1.97	1.89	1.84	1.79	1.75	1.72
80	3.96	3.11	2.72	2.49	2.33	2.21	2.13	2.06	2.00	1.95	1.88	1.82	1.77	1.73	1.70
90	3.95	3.10	2.71	2.47	2.32	2.20	2.11	2.04	1.99	1.94	1.86	1.80	1.76	1.72	1.69
100	3.94	3.09	2.70	2.46	2.31	2.19	2.10	2.03	1.97	1.93	1.85	1.79	1.75	1.71	1.68
120	3.92	3.07	2.68	2.45	2.29	2.18	2.09	2.02	1.96	1.91	1.83	1.78	1.73	1.69	1.66
150	3.90	3.06	2.66	2.43	2.27	2.16	2.07	2.00	1.94	1.89	1.82	1.76	1.71	1.67	1.64
200	3.89	3.04	2.65	2.42	2.26	2.14	2.06	1.98	1.93	1.88	1.80	1.74	1.69	1.66	1.62
250	3.88	3.03	2.64	2.41	2.25	2.13	2.05	1.98	1.92	1.87	1.79	1.73	1.68	1.65	1.61
300	3.87	3.03	2.63	2.40	2.24	2.13	2.04	1.97	1.91	1.86	1.78	1.72	1.68	1.64	1.61
400	3.86	3.02	2.63	2.39	2.24	2.12	2.03	1.96	1.90	1.85	1.78	1.72	1.67	1.63	1.60
500	3.86	3.01	2.62	2.39	2.23	2.12	2.03	1.96	1.90	1.85	1.77	1.71	1.66	1.62	1.59
600	3.86	3.01	2.62	2.39	2.23	2.11	2.02	1.95	1.90	1.85	1.77	1.71	1.66	1.62	1.59
750	3.85	3.01	2.62	2.38	2.23	2.11	2.02	1.95	1.89	1.84	1.77	1.70	1.66	1.62	1.58
1000	3.85	3.00	2.61	2.38	2.22	2.11	2.02	1.95	1.89	1.84	1.76	1.70	1.65	1.61	1.58

Critical Values of the Mann-Whitney U
(Two-Tailed Testing)

n_2	α	n_1																	
		3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
3	.05	--	0	0	1	1	2	2	3	3	4	4	5	5	6	6	7	7	8
	.01	--	0	0	0	0	0	0	0	0	1	1	1	2	2	2	2	3	3
4	.05	--	0	1	2	3	4	4	5	6	7	8	9	10	11	11	12	13	14
	.01	--	--	0	0	0	1	1	2	2	3	3	4	5	5	6	6	7	8
5	.05	0	1	2	3	5	6	7	8	9	11	12	13	14	15	17	18	19	20
	.01	--	--	0	1	1	2	3	4	5	6	7	7	8	9	10	11	12	13
6	.05	1	2	3	5	6	8	10	11	13	14	16	17	19	21	22	24	25	27
	.01	--	0	1	2	3	4	5	6	7	9	10	11	12	13	15	16	17	18
7	.05	1	3	5	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34
	.01	--	0	1	3	4	6	7	9	10	12	13	15	16	18	19	21	22	24
8	.05	2	4	6	8	10	13	15	17	19	22	24	26	29	31	34	36	38	41
	.01	--	1	2	4	6	7	9	11	13	15	17	18	20	22	24	26	28	30
9	.05	2	4	7	10	12	15	17	20	23	26	28	31	34	37	39	42	45	48
	.01	0	1	3	5	7	9	11	13	16	18	20	22	24	27	29	31	33	36
10	.05	3	5	8	11	14	17	20	23	26	29	33	36	39	42	45	48	52	55
	.01	0	2	4	6	9	11	13	16	18	21	24	26	29	31	34	37	39	42
11	.05	3	6	9	13	16	19	23	26	30	33	37	40	44	47	51	55	58	62
	.01	0	2	5	7	10	13	16	18	21	24	27	30	33	36	39	42	45	48
12	.05	4	7	11	14	18	22	26	29	33	37	41	45	49	53	57	61	65	69
	.01	1	3	6	9	12	15	18	21	24	27	31	34	37	41	44	47	51	54
13	.05	4	8	12	16	20	24	28	33	37	41	45	50	54	59	63	67	72	76
	.01	1	3	7	10	13	17	20	24	27	31	34	38	42	45	49	53	56	60
14	.05	5	9	13	17	22	26	31	36	40	45	50	55	59	64	67	74	78	83
	.01	1	4	7	11	15	18	22	26	30	34	38	42	46	50	54	58	63	67
15	.05	5	10	14	19	24	29	34	39	44	49	54	59	64	70	75	80	85	90
	.01	2	5	8	12	16	20	24	29	33	37	42	46	51	55	60	64	69	73
16	.05	6	11	15	21	26	31	37	42	47	53	59	64	70	75	81	86	92	98
	.01	2	5	9	13	18	22	27	31	36	41	45	50	55	60	65	70	74	79
17	.05	6	11	17	22	28	34	39	45	51	57	63	67	75	81	87	93	99	105
	.01	2	6	10	15	19	24	29	34	39	44	49	54	60	65	70	75	81	86
18	.05	7	12	18	24	30	36	42	48	55	61	67	74	80	86	93	99	106	112
	.01	2	6	11	16	21	26	31	37	42	47	53	58	64	70	75	81	87	92
19	.05	7	13	19	25	32	38	45	52	58	65	72	78	85	92	99	106	113	119
	.01	3	7	12	17	22	28	33	39	45	51	56	63	69	74	81	87	93	99
20	.05	8	14	20	27	34	41	48	55	62	69	76	83	90	98	105	112	119	127
	.01	3	8	13	18	24	30	36	42	48	54	60	67	73	79	86	92	99	105

Critical Values of the Wilcoxon Signed Ranks Test

n	Two-Tailed Test		One-Tailed Test	
	α = .05	α = .01	α = .05	α = .01
5	--	--	0	--
6	0	--	2	--
7	2	--	3	0
8	3	0	5	1
9	5	1	8	3
10	8	3	10	5
11	10	5	13	7
12	13	7	17	9
13	17	9	21	12
14	21	12	25	15
15	25	15	30	19
16	29	19	35	23
17	34	23	41	27
18	40	27	47	32
19	46	32	53	37
20	52	37	60	43
21	58	42	67	49
22	65	48	75	55
23	73	54	83	62
24	81	61	91	69
25	89	68	100	76
26	98	75	110	84
27	107	83	119	92
28	116	91	130	101
29	126	100	140	110
30	137	109	151	120

Critical Values of Spearman's Rank Correlation Coefficient R_s

	Nominal α					
n	0.10	0.05	0.025	0.01	0.005	0.001
4	1.000	1.000	-	-	-	-
5	0.800	0.900	1.000	1.000	-	-
6	0.657	0.829	0.886	0.943	1.000	-
7	0.571	0.714	0.786	0.893	0.929	1.000
8	0.524	0.643	0.738	0.833	0.881	0.952
9	0.483	0.600	0.700	0.783	0.833	0.917
10	0.455	0.564	0.648	0.745	0.794	0.879
11	0.427	0.536	0.618	0.709	0.755	0.845
12	0.406	0.503	0.587	0.678	0.727	0.818
13	0.385	0.484	0.560	0.648	0.703	0.791
14	0.367	0.464	0.538	0.626	0.679	0.771
15	0.354	0.446	0.521	0.604	0.654	0.750
16	0.341	0.429	0.503	0.582	0.635	0.729
17	0.328	0.414	0.488	0.566	0.618	0.711
18	0.317	0.401	0.472	0.550	0.600	0.692
19	0.309	0.391	0.460	0.535	0.584	0.675
20	0.299	0.380	0.447	0.522	0.570	0.662
21	0.292	0.370	0.436	0.509	0.556	0.647
22	0.284	0.361	0.425	0.497	0.544	0.633
23	0.278	0.353	0.416	0.486	0.532	0.621
24	0.271	0.344	0.407	0.476	0.521	0.609
25	0.265	0.337	0.398	0.466	0.511	0.597
26	0.259	0.331	0.390	0.457	0.501	0.586
27	0.255	0.324	0.383	0.449	0.492	0.576
28	0.250	0.318	0.375	0.441	0.483	0.567
29	0.245	0.312	0.368	0.433	0.475	0.558

CHI-SQUARE TABLE

df	$\chi^2_{.995}$	$\chi^2_{.990}$	$\chi^2_{.975}$	$\chi^2_{.950}$	$\chi^2_{.900}$	$\chi^2_{.100}$	$\chi^2_{.050}$	$\chi^2_{.025}$	$\chi^2_{.010}$	$\chi^2_{.005}$
1	0.000	0.000	0.001	0.004	0.016	2.706	3.841	5.024	6.635	7.879
2	0.010	0.020	0.051	0.103	0.211	4.605	5.991	7.378	9.210	10.597
3	0.072	0.115	0.216	0.352	0.584	6.251	7.815	9.348	11.345	12.838
4	0.207	0.297	0.484	0.711	1.064	7.779	9.488	11.143	13.277	14.860
5	0.412	0.554	0.831	1.145	1.610	9.236	11.070	12.833	15.086	16.750
6	0.676	0.872	1.237	1.635	2.204	10.645	12.592	14.449	16.812	18.548
7	0.989	1.239	1.690	2.167	2.833	12.017	14.067	16.013	18.475	20.278
8	1.344	1.646	2.180	2.733	3.490	13.362	15.507	17.535	20.090	21.955
9	1.735	2.088	2.700	3.325	4.168	14.684	16.919	19.023	21.666	23.589
10	2.156	2.558	3.247	3.940	4.865	15.987	18.307	20.483	23.209	25.188
11	2.603	3.053	3.816	4.575	5.578	17.275	19.675	21.920	24.725	26.757
12	3.074	3.571	4.404	5.226	6.304	18.549	21.026	23.337	26.217	28.300
13	3.565	4.107	5.009	5.892	7.042	19.812	22.362	24.736	27.688	29.819
14	4.075	4.660	5.629	6.571	7.790	21.064	23.685	26.119	29.141	31.319
15	4.601	5.229	6.262	7.261	8.547	22.307	24.996	27.488	30.578	32.801
16	5.142	5.812	6.908	7.962	9.312	23.542	26.296	28.845	32.000	34.267
17	5.697	6.408	7.564	8.672	10.085	24.769	27.587	30.191	33.409	35.718
18	6.265	7.015	8.231	9.390	10.865	25.989	28.869	31.526	34.805	37.156
19	6.844	7.633	8.907	10.117	11.651	27.204	30.144	32.852	36.191	38.582
20	7.434	8.260	9.591	10.851	12.443	28.412	31.410	34.170	37.566	39.997
21	8.034	8.897	10.283	11.591	13.240	29.615	32.671	35.479	38.932	41.401
22	8.643	9.542	10.982	12.338	14.041	30.813	33.924	36.781	40.289	42.796
23	9.260	10.196	11.689	13.091	14.848	32.007	35.172	38.076	41.638	44.181
24	9.886	10.856	12.401	13.848	15.659	33.196	36.415	39.364	42.980	45.559
25	10.520	11.524	13.120	14.611	16.473	34.382	37.652	40.646	44.314	46.928
26	11.160	12.198	13.844	15.379	17.292	35.563	38.885	41.923	45.642	48.290
27	11.808	12.879	14.573	16.151	18.114	36.741	40.113	43.195	46.963	49.645
28	12.461	13.565	15.308	16.928	18.939	37.916	41.337	44.461	48.278	50.993
29	13.121	14.256	16.047	17.708	19.768	39.087	42.557	45.722	49.588	52.336
30	13.787	14.953	16.791	18.493	20.599	40.256	43.773	46.979	50.892	53.672
40	20.707	22.164	24.433	26.509	29.051	51.805	55.758	59.342	63.691	66.766
50	27.991	29.707	32.357	34.764	37.689	63.167	67.505	71.420	76.154	79.490
60	35.534	37.485	40.482	43.188	46.459	74.397	79.082	83.298	88.379	91.952
70	43.275	45.442	48.758	51.739	55.329	85.527	90.531	95.023	100.425	104.215
80	51.172	53.540	57.153	60.391	64.278	96.578	101.879	106.629	112.329	116.321
90	59.196	61.754	65.647	69.126	73.291	107.565	113.145	118.136	124.116	128.299
100	67.328	70.065	74.222	77.929	82.358	118.498	124.342	129.561	135.807	140.169

Critical values for Friedman's two-way analysis of Variance by ranks

	$k=3$		$k=4$		$k=5$		$k=6$	
b	0.05	0.01	0.05	0.01	0.05	0.01	0.05	0.01
2	-	-	6.000	-	7.600	8.000	9.143	9.714
3	6.000	-	7.400	9.000	8.533	10.13	9.857	11.76
4	6.500	8.000	7.800	9.600	8.800	11.20	10.29	12.71
5	6.400	8.400	7.800	9.960	8.960	11.68	10.49	13.23
6	7.000	9.000	7.600	10.20	9.067	11.87	10.57	13.62
7	7.143	8.857	7.800	10.54	9.143	12.11		
8	6.250	9.000	7.650	10.50	9.200	12.30		
9	6.222	9.556	7.667	10.73	9.244	12.44		
10	6.200	9.600	7.680	10.68				
11	6.545	9.455	7.691	10.75				
12	6.500	9.500	7.700	10.80				
13	6.615	9.385	7.800	10.85				
14	6.143	9.143	7.714	10.89				
15	6.400	8.933	7.720	10.92				
16	6.500	9.375	7.800	10.95				
17	6.118	9.294	7.800	11.05				
18	6.333	9.000	7.733	10.93				
19	6.421	9.579	7.863	11.02				
20	6.300	9.300	7.800	11.10				
21	6.095	9.238	7.800	11.06				
22	6.091	9.091	7.800	11.07				
23	6.348	9.391						
24	6.250	9.250						