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Unit-II

Random Sampling of Multivariate Normal Distribution

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UNIT – II

RANDOM SAMPLING OF MULTIVARIATE NORMAL DISTRIBUTION

Random Sampling from multivariate normal distribution

To generate a random sample from a multivariate normal distribution, use the following formula:

$$X = \mu + Z \times \Sigma^{\frac{1}{2}}$$

where:

- X is the random sample
- μ is the mean vector
- Z is a vector of independent standard normal random variables
- Σ is the covariance matrix

Maximum likelihood estimators of the parameters of multivariate normal distribution

When a distribution such as the multivariate normal is assumed to hold for a population, estimates of the parameters are often found by the method of maximum likelihood. This technique is conceptually simple: The observation vectors $x_1, x_2, \cdots, x_n$ are considered to be known, and values of μ and Σ are sought that maximize the joint density of the *x*'s, called the likelihood function. For the multivariate normal, the maximum likelihood estimates of μ and Σ are

(i) Maximum likelihood estimate of the mean vector

Let $x_1, x_2, \cdots, x_n$ be a random sample of size n (> p) from $N_p(\mu, \Sigma)$. The likelihood function

$$\phi = f(x_1)\, f(x_2) \cdots f(x_n) = \frac{1}{(2\pi)^{np/2} |\Sigma|^{n/2}} \exp\left[-\frac{1}{2} \sum_{\alpha=1}^{n} (x_\alpha - \mu)^T \Sigma^{-1} (x_\alpha - \mu) \right]$$

$$\log \phi = -\frac{np}{2} \log(2\pi) - \frac{n}{2} \log|\Sigma| - \frac{1}{2} \sum_{\alpha=1}^{n} (x_\alpha - \mu)^T \Sigma^{-1} (x_\alpha - \mu)$$

Differentiating with respect to μ and equating to zero

$$\frac{\partial \log \phi}{\partial \mu} = 0 = 0 - 0 - \frac{1}{2}\sum_{\alpha=1}^{n} 2\Sigma^{-1}(\mu - x_\alpha)$$

$$= \Sigma^{-1}\sum_{\alpha=1}^{n}(\mu - x_\alpha) = 0$$

$$\hat{\mu} = \frac{1}{n}\sum_{\alpha=1}^{n} x_\alpha = \bar{x}$$

(ii) Maximum likelihood estimate of variance covariance matrix

Let $\Sigma^{-1} = (\sigma_{ij})$ and $\Sigma = (\sigma_{ij})$ then the $\left|\Sigma^{-1}\right| = \sigma^{i1}\Sigma^{i1} + \cdots + \sigma^{ip}\Sigma^{ip}$ where Σ^{ij} is the cofactor of σ^{ij} in Σ^{-1}, therefore, $\frac{\Sigma^{ij}}{\left|\Sigma^{-1}\right|} = (i\ j)^{th}$ elements of $(\Sigma^{-1})^{-1} = (i\ j)^{th}$ elements of $\Sigma = \sigma_{ii}$.

Now, the logarithm of the likelihood function is

$$\log \phi = -\frac{np}{2}\log(2\pi) + \frac{n}{2}\log\left|\Sigma^{-1}\right| - \frac{1}{2}\sum_{\alpha=1}^{n}(x_\alpha - \mu)^T \Sigma^{-1}(x_\alpha - \mu)$$

$$= -\frac{np}{2}\log(2\pi) + \frac{n}{2}\log(\sigma^{i1}\Sigma^{i1} + \cdots + \sigma^{ij}\Sigma^{ij} + \cdots + \sigma^{ip}\Sigma^{ip}) - \frac{1}{2}\sum_{\alpha}\sum_{i,j}\sigma^{ij}(x_{i\alpha} - \mu_i)(x_{j\alpha} - \mu_j)$$

Since, $x'Ax = \sum_{i,j} a_{ij}x_i x_j$

Differentiating with respect to σ^{ij} and equating to zero, we get

$$\frac{\partial \log \phi}{\partial \sigma^{ij}} = 0 = \frac{n}{2}\frac{\Sigma^{ij}}{\left|\Sigma^{-1}\right|} - \frac{1}{2}\sum_{\alpha=1}^{n}(x_{i\alpha} - \mu_i)(x_{j\alpha} - \mu_j), \text{ because } \frac{\partial}{\partial x}\log f(x) = \frac{f'(x)}{f(x)}$$

$$\frac{n}{2}\sigma_{ij} = \frac{1}{2}\sum_{\alpha=1}^{n}(x_{i\alpha} - \mu_i)(x_{j\alpha} - \mu_j)$$

$$\hat{\sigma}_{ij} = \frac{1}{n}\sum_{\alpha=1}^{n}(x_{i\alpha} - \hat{\mu}_i)(x_{j\alpha} - \hat{\mu}_j)$$

Hence, $\hat{\Sigma} = \frac{A}{n}$.

Distribution of sample mean vector

For the distribution of $\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$ distinguish two cases:

(a) When $\bar{x}$ is based on a random sample $x_1, x_2, \cdots, x_n$ from a multivariate normal distribution $N_p(\mu, \Sigma)$, the $\bar{x}$ is $N_p(\mu, \Sigma/n)$.

(b) When $\bar{x}$ is based on a random sample $x_1, x_2, \cdots, x_n$ from a non-normal multivariate population with mean vector μ and covariance matrix Σ, then for large n, $\bar{x}$ is approximately $N_p(\mu, \Sigma/n)$.

More formally, this result is known as the multivariate central limit theorem: If $\bar{x}$ is the mean vector of a random sample $x_1, x_2, \cdots, x_n$ from a population with mean vector μ and covariance matrix Σ, then as $n \to \infty$, the distribution of $\sqrt{n}(\bar{x} - \mu)$ approaches $N_p(0, \Sigma)$.

Sample dispersion mean vector

The sample dispersion (or sample covariance) of the sample mean vector from a multivariate normal distribution is given by:

$$S = \frac{1}{n-1}(x_i - \bar{x})\Sigma(x_i - \bar{x})'$$

- S is the sample covariance matrix
- x_i is the i^{th} observation
- $\bar{x}$ is the sample mean vector
- n is the sample size

The sample covariance matrix S has the following properties:

- It is an unbiased estimator of the population covariance matrix Σ.
- It is a consistent estimator of Σ (i.e., it converges to Σ as $n \to \infty$).
- It follows a Wishart distribution (a multivariate extension of the chi-squared distribution).

The Wishart distribution has the following properties:

- It is a multivariate distribution
- It has two parameters: n-1 (degrees of freedom) and Σ (scale matrix)
- It is used to model the distribution of sample covariance matrices

Necessary condition for a quadratic form to be distributed as chi-square

Given $x_1^{(1)}, x_2^{(1)}, \cdots, x_\alpha^{(1)}, \cdots, x_{n_1}^{(1)}$ be a random sample from $N_p(\mu^{(1)}, \Sigma)$ and $x_1^{(2)}, x_2^{(2)}, \cdots, x_\alpha^{(2)}, \cdots, x_{n_2}^{(2)}$ from $N_p(\mu^{(2)}, \Sigma)$. $H_0 : \mu^{(1)} = \mu^{(2)}$, then the test statistic is

$$\frac{n_1 n_2}{n_1 + n_2}(\bar{x}^{(1)} - \bar{x}^{(2)})' \Sigma^{-1} (\bar{x}^{(1)} - \bar{x}^{(2)}) \sim \chi_p^2.$$

Proof

We know that,

$\bar{x}^{(1)} \sim N_p\left(\mu^{(1)}, \frac{\Sigma}{n_1}\right)$ and $\bar{x}^{(2)} \sim N_p\left(\mu^{(2)}, \frac{\Sigma}{n_2}\right)$. Further $\bar{x}^{(1)}$ and $\bar{x}^{(2)}$ are independent and $(\bar{x}^{(1)} - \bar{x}^{(2)}) \sim N_p\left[\mu^{(1)} - \mu^{(2)}, \left(\frac{1}{n_1} + \frac{1}{n_2}\right)\Sigma\right]$.

Make the transformation (nonsingular)

$$(\bar{x}^{(1)} - \bar{x}^{(2)}) = Cy \Rightarrow y = C^{-1}(\bar{x}^{(1)} - \bar{x}^{(2)})$$

Since C is a non-singular matrix such that

$C'\Sigma^{*-1}C = I$, where $\Sigma^* = \frac{n_1 + n_2}{n_1 n_2}\Sigma = CC'$, and $Ey = C^{-1}E(\bar{x}^{(1)} - \bar{x}^{(2)}) = 0$ under H_0.

$$E_y = E(y - Ey)(y - Ey)' = C^{-1}E(\bar{x}^{(1)} - \bar{x}^{(2)})(\bar{x}^{(1)} - \bar{x}^{(2)})'C^{-1'}$$

$$= C^{-1}\Sigma^* C^{-1'} = (C'\Sigma^{*-1}C)^{-1} = I$$

Therefore,

$$y \sim N_p(0, I), i.e., y_i \sim N(0,1), \text{ for all } i = 1, 2, \ldots, p.$$

Now,

$$\frac{n_1 n_2}{n_1 + n_2}(\bar{x}^{(1)} - \bar{x}^{(2)})' \Sigma^{-1} (\bar{x}^{(1)} - \bar{x}^{(2)}) = (\bar{x}^{(1)} - \bar{x}^{(2)})' (CC')^{-1} (\bar{x}^{(1)} - \bar{x}^{(2)})$$

$$[C^{-1}(\bar{x}^{(1)} - \bar{x}^{(2)})]'[C^{-1}(\bar{x}^{(1)} - \bar{x}^{(2)})] = y'y = \sum_{i=1}^{p} y_i^2 \sim \chi_p^2.$$

Let $\chi_p^2(\alpha)$ be the number such that $\Pr\left[\chi_p^2 \geq \chi_p^2(\alpha)\right] = \alpha$, then for testing H_0, we use the critical region

$$\frac{n_1 n_2}{n_1 + n_2}(\bar{x}^{(1)} - \bar{x}^{(2)})' \Sigma^{-1} (\bar{x}^{(1)} - \bar{x}^{(2)}) \sim \chi_p^2(\alpha).$$

Hence proved.

Sufficient condition for a quadratic form to be distributed as chi-square

The joint density function of $x_1, \ldots, x_n$ with $x_\alpha \sim N_p(\mu, \Sigma)$ is

$$\frac{1}{(2\pi)^{np/2}|\Sigma|^{n/2}}\exp\left[-\frac{1}{2}\sum_{\alpha=1}^{n}(x_\alpha-\mu)^T\Sigma^{-1}(x_\alpha-\mu)\right]$$

Consider

$$\sum_{\alpha=1}^{n}(x_\alpha-\mu)^T\Sigma^{-1}(x_\alpha-\mu) = tr\sum_{\alpha=1}^{n}(x_\alpha-\mu)^T\Sigma^{-1}(x_\alpha-\mu)$$

$$= tr\sum_{\alpha=1}^{n}\Sigma^{-1}(x_\alpha-\mu)^T(x_\alpha-\mu)$$

$$= tr\Sigma^{-1}\sum_{\alpha=1}^{n}(x_\alpha-\mu)^T(x_\alpha-\mu).$$

We can write,

$$\sum_{\alpha=1}^{n}(x_\alpha-\mu)^T(x_\alpha-\mu) = \sum_{\alpha}\left[(x_\alpha-\bar{x})+(\bar{x}-\mu)\right]\left[(x_\alpha-\bar{x})+(\bar{x}-\mu)\right]^T$$

$$= \sum_{\alpha=1}^{n}\left[(x_\alpha-\bar{x})(x_\alpha-\bar{x})^T+(x_\alpha-\bar{x})(\bar{x}-\mu)^T+(\bar{x}-\mu)(x_\alpha-\bar{x})^T+(\bar{x}-\mu)(\bar{x}-\mu)^T\right]$$

$$= \sum_{\alpha=1}^{n}(x_\alpha-\bar{x})(x_\alpha-\bar{x})^T+\left\{\sum_{\alpha}(x_\alpha-\bar{x})\right\}(\bar{x}-\mu)^T+(\bar{x}-\mu)\sum_{\alpha}(x_\alpha-\bar{x})^T+n(\bar{x}-\mu)(\bar{x}-\mu)^T$$

$$= A+n(\bar{x}-\mu)(\bar{x}-\mu)^T, \text{ because } \sum_{\alpha}(x_\alpha-\bar{x}) = \sum_{\alpha}x_\alpha - n\bar{x} = 0.$$

Thus the density of $x_1, \ldots, x_n$ can be written as

$$\frac{1}{(2\pi)^{np/2}|\Sigma|^{n/2}}\exp\left[-\frac{1}{2}tr\Sigma^{-1}\left(A+n(\bar{x}-\mu)(\bar{x}-\mu)^T\right)\right]$$

$$= \frac{1}{(2\pi)^{np/2}|\Sigma|^{n/2}}\exp\left[-\frac{1}{2}\left(n(\bar{x}-\mu)^T\Sigma^{-1}(\bar{x}-\mu)+tr\Sigma^{-1}A\right)\right]$$

$$= \frac{1}{(2\pi)^{np/2}|\Sigma|^{n/2}}\exp\left[-\frac{1}{2}n(\bar{x}-\mu)^T\Sigma^{-1}(\bar{x}-\mu)\right]\times\frac{1}{(2\pi)^{np/2}|\Sigma|^{n/2}}\exp\left[-\frac{1}{2}tr\Sigma^{-1}A\right].$$

Thus, $\bar{x}$ and $\frac{1}{n}A$ form a sufficient set of statistics for μ and Σ. If Σ is known, $\bar{x}$ is a sufficient statistic for μ. However, if μ is known $\frac{1}{n}A$ is not a sufficient statistic for Σ, but $\frac{1}{n}\sum_{\alpha}(x_\alpha-\mu)(x_\alpha-\mu)^T$ is a sufficient statistic for Σ.

Inference concerning the sample mean vector when covariance matrix is known

Given a random sample $x_1, x_2, , x_\alpha, , x_n$ from $N_p(\mu, \Sigma)$. The hypothesis of interest is $H_0 : \mu = \mu_0$, where μ_0 is a specified vector, then, under H_0, the test statistic is $n(\bar{x}-\mu_0)^T\Sigma^{-1}(\bar{x}-\mu_0) \sim \chi_p^2.$

Proof

Let C be a non-singular matrix such that

$C'\Sigma^{*-1}C = I$ and $CC' = \Sigma^* = \Sigma/n$. Make the transformation

$(\bar{x}-\mu_0) = C_y \Rightarrow y = C^{-1}(\bar{x}-\mu_0)$, and

$Ey = C^{-1}E(\bar{x}-\mu_0) = C^{-1}(\mu_0-\mu_0) = 0$, under H_0.

$$\Sigma_y = E(y-Ey)(y-Ey)' = C^{-1}E(\bar{x}-\mu_0)(\bar{x}-\mu_0)'C^{-1'}$$

$$C^{-1}\Sigma^* C^{-1'} = (C'\Sigma^{*^{-1}}C)^{-1} = I.$$

Therefore,

$$y \sim N_p(0, I), i.e., y_i \sim N(0,1), \text{ for all } i = 1, 2, \ldots, p.$$

Now

$$n(\bar{x}-\mu_0)^T\Sigma^{-1}(\bar{x}-\mu_0) = (\bar{x}-\mu_0)^T(CC')^{-1}(\bar{x}-\mu_0)$$

$$= \left[C^{-1}(\bar{x}-\mu_0)\right]^T\left[C^{-1}(\bar{x}-\mu_0)\right] = y^T y = \sum_{i=1}^{p} y_i^2 \sim \chi_p^2.$$

Let $\chi_p^2(\alpha)$ be the number such that $\Pr\left[\chi_p^2 \geq \chi_p^2(\alpha)\right] = \alpha$, then for testing H_0, we use the critical region

$$n(\bar{x}-\mu_0)^T\Sigma^{-1}(\bar{x}-\mu_0) \geq \chi_p^2(\alpha).$$

For computational purpose use

$(\bar{x}-\mu_0) = d$, then solve

$\Sigma\lambda = d$ (by Doolittle method)

$\Rightarrow \lambda = \Sigma^{-1}d$ and

$nd'\lambda = n(\bar{x}-\mu_0)^T\Sigma^{-1}(\bar{x}-\mu_0).$

Note: If H_0 is not true, then

$Ey = C^{-1}E(\bar{x}-\mu_0) = C^{-1}(\mu-\mu_0) = \delta$ (say), $\Sigma_y = I$, and

$n(\bar{x}-\mu_0)^T\Sigma^{-1}(\bar{x}-\mu_0) = \sum_{i=1}^{p} y_i^2 \sim \chi^2_{p,\sum_i \delta_i^2}$, where

$\sum_{i=1}^{p}\delta_i^2 = \delta\,\delta^T = n(\mu-\mu_0)'\Sigma^{-1}(\mu-\mu_0)$ = non-centrality parameter.

The confidence region for μ is the set of possible values of μ satisfying $n(\bar{x}-\mu_0)^T\Sigma^{-1}(\bar{x}-\mu_0) \leq \chi_p^2(\alpha)$, this has confidence coefficient $1-\alpha$.

<u>**SCL PROBLEMS**</u>

1. Method of obtaining Variance, Covariance and Correlation Matrix

Find variance, covariance and correlation matrix from the following information.

Sample No.	***1***	***2***	***3***	***4***	***5***	***6***	***7***	***8***	***9***	***10***
X_1	24	27	28	23	25	32	33	24	23	25
X_2	41	44	44	40	43	49	52	47	44	45
X_3	55	52	56	50	51	56	48	55	56	55
X_4	36	39	38	30	37	34	31	32	38	35

Sample No.	***11***	***12***	***13***	***14***	***15***	***16***	***17***	***18***	***19***	***20***
X_1	24	23	26	23	27	21	34	24	26	32
X_2	41	47	46	41	43	37	42	56	53	44
X_3	56	48	56	55	43	57	56	56	53	48
X_4	38	32	36	36	39	28	33	38	31	32

Procedure

- Mean Vector = $\begin{bmatrix} \bar{X}_1 \\ \bar{X}_2 \\ \bar{X}_3 \\ \bar{X}_4 \end{bmatrix}$
- $\sigma_{ij} = \sigma_i^2 = \dfrac{\sum X_i^2}{n} - \bar{X}_i^2$
- $\sigma_{ij} = \dfrac{\sum X_i X_j}{n} - \bar{X}_i \bar{X}_j$
- $r_{ij} = \dfrac{\sigma_{ij}}{\sigma_i \sigma_j}$

Calculation

Mean Vector

$$\bar{X}_1 = \frac{524}{20} = 26.2 \qquad \bar{X}_2 = \frac{899}{20} = 44.95$$

$$\bar{X}_3 = \frac{1062}{20} = 53.1 \qquad \bar{X}_4 = \frac{693}{20} = 34.65$$

$$\text{Mean Vector} = \begin{bmatrix} 26.2 \\ 44.95 \\ 53.1 \\ 34.65 \end{bmatrix}$$

Sample No.	X_1	X_2	X_3	X_4	X_1^2	X_2^2	X_3^2	X_4^2	$X_1 X_2$	$X_1 X_3$	$X_1 X_4$	$X_2 X_3$	$X_2 X_4$	$X_3 X_4$
1	24	41	55	36	576	1681	3025	1296	984	1320	864	2255	1476	1980
2	27	44	52	39	729	1936	2704	1521	1188	1404	1053	2288	1716	2028
3	28	44	56	38	784	1936	3136	1444	1232	1568	1064	2464	1672	2128
4	23	40	50	30	529	1600	2500	900	920	1150	690	2000	1200	1500
5	25	43	51	37	625	1849	2601	1369	1075	1275	925	2193	1591	1887
6	32	49	56	34	1024	2401	3136	1156	1568	1792	1088	2744	1666	1904
7	33	52	48	31	1089	2704	2304	961	1716	1584	1023	2496	1612	1488
8	24	47	55	32	576	2209	3025	1024	1128	1320	768	2585	1504	1760
9	23	44	56	38	529	1936	3136	1444	1012	1288	874	2464	1672	2128
10	25	45	55	35	625	2025	3025	1225	1125	1375	875	2475	1575	1925
11	24	41	56	38	576	1681	3136	1444	984	1344	912	2296	1558	2128
12	23	47	48	32	529	2209	2304	1024	1081	1104	736	2256	1504	1536
13	26	46	56	36	676	2116	3136	1296	1196	1456	936	2576	1656	2016
14	23	41	55	36	529	1681	3025	1296	943	1265	828	2255	1476	1980
15	27	43	43	39	729	1849	1849	1521	1161	1161	1053	1849	1677	1677
16	21	37	57	28	441	1369	3249	784	777	1197	588	2109	1036	1596
17	34	42	56	33	1156	1764	3136	1089	1428	1904	1122	2352	1386	1848
18	24	56	56	38	576	3136	3136	1444	1344	1344	912	3136	2128	2128
19	26	53	53	31	676	2809	2809	961	1378	1378	806	2809	1643	1643
20	32	44	48	32	1024	1936	2304	1024	1408	1536	1024	2112	1408	1536
Total	**524**	**899**	**1062**	**693**	**13998**	**40827**	**56676**	**24223**	**23648**	**27765**	**18141**	**47714**	**31156**	**36816**

Covariance Matrix

$$\sigma_{11} = \sigma_1^2 = \frac{13998}{20} - (26.2)^2 = 13.46$$

$$\sigma_{22} = \sigma_2^2 = \frac{40827}{20} - (44.95)^2 = 20.85$$

$$\sigma_{33} = \sigma_3^2 = \frac{56676}{20} - (53.1)^2 = 14.19$$

$$\sigma_{44} = \sigma_4^2 = \frac{24223}{20} - (34.65)^2 = 10.53$$

$$\sigma_{12} = \frac{23648}{20} - (26.2 \times 44.95) = 4.71$$

$$\sigma_{13} = \frac{27765}{20} - (26.2 \times 53.1) = -2.97$$

$$\sigma_{14} = \frac{18141}{20} - (26.2 \times 34.65) = -0.78$$

$$\sigma_{23} = \frac{47714}{20} - (44.95 \times 53.1) = -1.145$$

$$\sigma_{24} = \frac{31156}{20} - (44.95 \times 34.65) = 0.2825$$

$$\sigma_{34} = \frac{36816}{20} - (53.1 \times 34.65) = 0.885$$

$$\text{Covariance Matrix} = \begin{bmatrix} 13.46 & 4.71 & -2.97 & -0.78 \\ 4.71 & 20.85 & -1.145 & 0.2825 \\ -2.97 & -1.145 & 14.19 & 0.885 \\ -0.78 & 0.2825 & 0.885 & 10.53 \end{bmatrix}$$

Correlation Matrix

$$r_{12} = \frac{\sigma_{12}}{\sigma_1 \sigma_2} = \frac{4.71}{\sqrt{13.46} \times \sqrt{20.85}} = 0.2812$$

$$r_{13} = \frac{\sigma_{13}}{\sigma_1 \sigma_3} = \frac{-2.97}{\sqrt{13.46} \times \sqrt{14.19}} = -0.2149$$

$$r_{14} = \frac{\sigma_{14}}{\sigma_1 \sigma_4} = \frac{-0.781}{\sqrt{13.46} \times \sqrt{10.53}} = -0.0655$$

$$r_{23} = \frac{\sigma_{23}}{\sigma_2 \sigma_3} = \frac{-1.145}{\sqrt{20.85} \times \sqrt{14.19}} = -0.0666$$

$$r_{24} = \frac{\sigma_{24}}{\sigma_2 \sigma_4} = \frac{0.2825}{\sqrt{20.85} \times \sqrt{10.53}} = 0.0191$$

$$r_{34} = \frac{\sigma_{34}}{\sigma_3 \sigma_4} = \frac{0.885}{\sqrt{14.19} \times \sqrt{10.53}} = 0.0724$$

$$\text{Correlation Matrix} = \begin{bmatrix} 1 & 0.2812 & -0.2149 & -0.0655 \\ 0.2812 & 1 & -0.0666 & 0.0191 \\ -0.2149 & -0.0666 & 1 & 0.0724 \\ -0.0655 & 0.0191 & 0.0724 & 1 \end{bmatrix}$$

Result

- $\text{Mean Vector} = \begin{bmatrix} 26.2 \\ 44.95 \\ 53.1 \\ 34.65 \end{bmatrix}$

- $\text{Covariance Matrix} = \begin{bmatrix} 13.46 & 4.71 & -2.97 & -0.78 \\ 4.71 & 20.85 & -1.145 & 0.2825 \\ -2.97 & -1.145 & 14.19 & 0.885 \\ -0.78 & 0.2825 & 0.885 & 10.53 \end{bmatrix}$

- $\text{Correlation Matrix} = \begin{bmatrix} 1 & 0.2812 & -0.2149 & -0.0655 \\ 0.2812 & 1 & -0.0666 & 0.0191 \\ -0.2149 & -0.0666 & 1 & 0.0724 \\ -0.0655 & 0.0191 & 0.0724 & 1 \end{bmatrix}$

2. Gauss Doolittle's Method

Solve the following linear equation using Gauss Doolittle method:

$12X_1 + 5X_2 - 7X_3 + 2X_4 = 27$
$5X_1 + 3X_2 + X_3 - 4X_4 = 14$
$- 7X_1 + X_2 + 9X_3 + 6X_4 = 46$
$2X_1 - 4X_2 + 6X_3 + 2X_4 = 24$

Procedure

- Doolittle method for Upper(U) and Lower (L) decomposition

$A = L \times U$

$$\text{Where, } L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ l_{12} & 1 & 0 & 0 \\ l_{13} & l_{23} & 1 & 0 \\ l_{14} & l_{24} & l_{34} & 1 \end{bmatrix} \text{ and } U = \begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ 0 & u_{22} & u_{23} & u_{24} \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{bmatrix}$$

- The given linear equation can be written n matrix form, $AX = B$.
- After converting the diagonal elements to zero, we obtain the solution for linear equations.

Calculation

$12X_1 + 5X_2 - 7X_3 + 2X_4 = 27$ - - - - - (1)

$5X_1 + 3X_2 + X_3 - 4X_4 = 14$ - - - - - (2)

$- 7X_1 + X_2 + 9X_3 + 6X_4 = 46$ - - - - - (3)

$2X_1 - 4X_2 + 6X_3 + 2X_4 = 24$ - - - - - (4)

Now converting given equation into matrix form

$$\begin{bmatrix} 12 & 5 & -7 & 2 \\ 5 & 3 & 1 & -4 \\ -7 & 1 & 9 & 6 \\ 2 & -4 & 6 & 2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = \begin{bmatrix} 27 \\ 14 \\ 46 \\ 24 \end{bmatrix}$$

Now

$$A = \begin{bmatrix} 12 & 5 & -7 & 2 \\ 5 & 3 & 1 & -4 \\ -7 & 1 & 9 & 6 \\ 2 & -4 & 6 & 2 \end{bmatrix}, \quad X = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 27 \\ 14 \\ 46 \\ 24 \end{bmatrix}$$

Doolittle method for Upper(U) and Lower (L) decomposition

Let, $A = L \times U$

$$\begin{bmatrix} 12 & 5 & -7 & 2 \\ 5 & 3 & 1 & -4 \\ -7 & 1 & 9 & 6 \\ 2 & -4 & 6 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ l_{12} & 1 & 0 & 0 \\ l_{13} & l_{23} & 1 & 0 \\ l_{14} & l_{24} & l_{34} & 1 \end{bmatrix} \times \begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ 0 & u_{22} & u_{23} & u_{24} \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{bmatrix}$$

$$= \begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ l_{21}u_{11} & l_{21}u_{12}+u_{22} & l_{21}u_{13}+u_{23} & l_{21}u_{14}+u_{24} \\ l_{31}u_{11} & l_{31}u_{12}+l_{32}u_{22} & l_{31}u_{13}+l_{32}u_{31}+u_{33} & l_{31}u_{14}+l_{32}u_{24}+u_{34} \\ l_{41}u_{11} & l_{41}u_{12}+l_{42}u_{22} & l_{41}u_{13}+l_{42}u_{23}+l_{43}u_{33} & l_{41}u_{14}+l_{42}u_{24}+u_{34}+u_{44} \end{bmatrix}$$

Here, $u_{11} = 12, u_{12} = 5, u_{13} = -7$ *and* $u_{14} = 2$. Then,

$l_{21}u_{11} = 5 \Rightarrow l_{21} \times 12 = 5 \Rightarrow l_{21} = \frac{5}{12}$

$l_{21}u_{12} + u_{22} = 3 \Rightarrow \frac{5}{12} \times 5 + u_{22} = 3$ $\qquad u_{22} = \frac{11}{12}$

$l_{21}u_{13} + u_{23} = 1 \Rightarrow \frac{5}{12} \times -7 + u_{23} = 1$ $\qquad u_{23} = \frac{47}{12}$

$l_{21}u_{14} + u_{24} \Rightarrow \frac{5}{12} \times 2 + u_{24} = -4$ $\qquad u_{24} = \frac{-29}{6}$

$l_{31}u_{11} = -7 \Rightarrow l_{31} \times 12 = -7 \Rightarrow l_{31} = \frac{-7}{12}$

$l_{31}u_{12} + l_{32}u_{22} = 1 \Rightarrow \frac{-7}{12} \times 5 + l_{32} \times \frac{11}{12} = 1$ $\qquad l_{32} = \frac{47}{11}$

$l_{31}u_{13} + l_{32}u_{31} + u_{33} = 9 \Rightarrow \frac{-7}{12} \times (-7) + \frac{47}{11} \times \frac{47}{11} + u_{33} = 9$ $\qquad u_{33} = -\frac{130}{11}$

$l_{31}u_{14} + l_{32}u_{24} + u_{34} = 6 \Rightarrow \frac{-7}{12} \times 2 + \frac{47}{11} \times \frac{-29}{6} + u_{34} = 6$ $\qquad u_{34} = \frac{306}{11}$

$l_{41}u_{11} = 2 \Rightarrow l_{41} \times 12 = 2 \Rightarrow l_{41} = \frac{1}{6}$

$l_{41}u_{12} + l_{42}u_{22} = -4 \Rightarrow \frac{1}{6} \times 5 + l_{42} \times \frac{11}{12} = -4$ $\qquad l_{42} = \frac{-58}{11}$

$l_{41}u_{13} + l_{42}u_{23} + l_{43}u_{33} = 6 \Rightarrow \frac{1}{6} \times (-7) + \frac{-58}{11} \times \frac{47}{12} + l_{43} \times \frac{-130}{11} = 6;$ $\quad l_{43} = \frac{-153}{65}$

Now, $AX = B$ and $A = L \times U$

Let, $UX = Y$, then $LY = B$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.4167 & 1 & 0 & 0 \\ -0.5833 & 4.2727 & 1 & 0 \\ 0.1667 & -5.277 & -2.358 & 1 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{bmatrix} = \begin{bmatrix} 27 \\ 14 \\ 46 \\ 24 \end{bmatrix}$$

$Y_1 = 27$

$0.416Y_1 + Y_2 = 14$

$-0.5833Y_1 + 4.272Y_2 + Y_3 = 46$

$0.1667Y_1 - 5.2727Y_2 - 2.3538Y_3 + Y_4 = 26$

Now, use forward substitution method

(1) $Y_1 = 27$

(2) $Y_2 = 2.75$

(3) $Y_3 = 50$

(4) $Y_4 = 151.69$

Now, $UX = Y$

$$\begin{bmatrix} 12 & 5 & -7 & 2 \\ 0 & 0.9967 & 3.9167 & -4.83 \\ 0 & 0 & -11.8182 & 27.81 \\ 0 & 0 & 0 & 41.6613 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = \begin{bmatrix} 27 \\ 2.75 \\ 50 \\ 151.69 \end{bmatrix}$$

$12X_1 + 5X_2 - 7X_3 + 2X_4 = 27$

$0.9967X_2 + 3.9167\,X_3 - 4.83X_4 = 2.75$

$-11.818X_3 + 27.81X_4 = 50$

$41.6613X_4 = 151.69$

Now, use back substitution method

$41.6613X_4 = 151.69$

$X_4 = 3.641$

$-11.818X_3 + 27.81 \times 3.641 = 50$

$X_3 = 4.3397$

$0.9967X_2 + 3.9167\,(4.3397) - 4.83(3.641) = 2.75$

$X_2 = 3.6558$

$12X_1 + 5(3.6558) - 7(4.3397) + 2(3.641) = 27$

$X_1 = 2.6514$

Result

The solution for linear equation using Gauss Doolittle method:

$X_1 = 2.6514$

$X_2 = 3.6558$

$X_3 = 4.3397$

$X_4 = 3.641$