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Unit-I

Linear programming problems

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UNIT - I

Introduction to operations Research.

Origin :-

Operation Research came into existence and gained prominence during the world war II in Britian with the establishment of team of Scientists to study the strategic and tactical problems of various military operations

By applying in the fields of Industry, trade, Agriculture, planning etc., it is also equally useful for irrigation / Agriculture, administrators etc..

The use of operational Research has no limited to Britain only. India was one of the few first Countries who started using O.R Regional Research Lab located at Hyderabad during 1949.

At the Same time one more unit was set up in defence Science Lab. In 1955, Operations Research Society of India was formed

Today, O.R became a professional discipline and studied as a popular subject in management Institutes.

Definition :-

operations Research can be defined of two words, operations means some action applied in any area of interest and research imply some organized process of getting and analyzing Information.

* O.R is a Scientific method of providing executive departments with a quantitative analytical and objective basis for decisions.

* O.R is the application of Scientific methods, techniques & tools to problems Involving the operations of System as to Provide these in Control of the operations with optimum Solution.

* O.R is a management activity pursued in a complementary ways, one half by free and bold exercise of common sense untrammeled by any routine & other half by application of pre-created method & techniques.

Advantages of operations Research :-

* It helps decision maker to take better and quick decisions. To evaluate the riskes & results of all the alternate decisions. so it improves the quality of decisions & make effective.

* It helps in preparing future managers, as it Provides in depth knowledge about a particular action.

* O.R. develop models, which provide logical & Systematic approach for understanding solving and Controlling a problem.

* It helps users in optimum use of resources For examples; Linear programming techniques in O.R. Suggest must effective methods and efficient ways of optimality.

* It helps . Suggest alternative solutions for the Same optimum profit if the management wants so.

Limitations of O.R :-

* Model is abstraction of real life situations and not the reality.

* Assumptions need to be made about the nature and the importance of some factors in order to construct an O.R model.

* Validity of any model with regard to Corresponding operations can only be verified by carrying out the experiments and observing relevant data.

* Construct of models require experts from various discipline.
* The help of computers is required for the large number of computation for such problems which discourage small companies and other organisation.

Scope of O.R :-

1) In Agriculture :- with the explosion of population and consequent shortage of food, every country is facing the problem of optimum allocation of land to various crops in accordance with the climate condition.

2) In defence operations :- Since second world war O.R have been used for Defence operations with the aim of obtaining maximum gain with minimum effects.

3) In finance :- In these modern times, government of every country or every organisation wants to introduce such type of planning / policies regarding their finance and accounting which optimize capital investment, determine optimal replacement strategies, apply cash flow analysis for long range capital investments, formulate credict policies, credit risk.

4) In Marketing :- A marketing administrator has to face many problems like production selection, formulation of competitive & distribution strategies selection of advertising media with respect to cost & time. finding the optimal no. of salesman finding optimum time to launch a product.

5) In personnel Mangement :- Every organization wants to make selection of personnel on minimum salary. It needs to find the best combination of workers in different categories with respect to costs, skills, age and no. of jobs. It also needs to frame recruitment policies, assign jobs to machines or workers.

6) In LIC : OR Techniques can be fruitfully applied in LIC offices as it enables the policy makers to decide the premium rates for various modes of policies.

7) In Research & Development :- In determination of the areas of concentration of research and development. It also helps in project selection.

Methodology of operations Research :-

Formula the problem
Construct a mathematical model
Acquire the input data
Derive the solution from the model
Validate the model
Establish Control over the solution
Impliment the final Research.

Application of operations Research :

operations Research has successfully entered many different areas of a research in defence, Government, Service organization and industry.

Some applications of operations research in the functional area of management.

* Finance, Budjecting and Investment.
* Marketing
* Physical Distribution
* Purchasing, procurement & Explanation
* Personnel
* Production
* Research and Development.

Linear programming problem (LPP) :

Definition :.

Linear programming is a technique part determining an optimum schedule of interdependents activities in view of the available resources.

Linear programming is a method of optimizing operations with some constraints.

The main objective of linear programming is to maximize or minimize the numerical value.

Mathematical formulation of the problem :.

The procedure of formulation of LPP is as follows :.

Step : 01

Study the given situation to find the key decision to be made.

Step : 02

Identify the variables involved and designate them by symbols x_j $(j = 1, 2 ... n)$

Step : 03

State the feasible alternatives which generally are $x_j \geq 0$, $\forall$ $j = 1, 2, ...$

Step : 04

Identify the constraints in the problem and express them as linear inequalities or equations, left handside, of which are linear function of the decision variables.

Step : 05

Identify the objective function and express it as a linear function of the decision variables.

General linear programming problem :

The linear programming problem involving more than two values may be expressed has follows

maximize (or) minimize.

$z = c_1 x_1 + c_2 x_2 + \dots + c_n x_n$

Subject to constraints

$a_{11} x_1 + a_{12} x_2 + \dots + a_{1n} x_n \leq$ (or) $=$ (or) $\geq b_1$

$a_{21} x_1 + a_{22} x_2 + \dots + a_{2n} x_n \leq$ (or) $=$ (or) $\geq b_2$

$\vdots \qquad \vdots \qquad \vdots$

$a_{m1} x_1 + a_{m2} x_2 + \dots + a_{mn} x_n \leq$ (or) $=$ (or) $\geq b_m$.

and the non-negativity restrictions.

$x_1, x_2, \dots x_n \geq 0$.

where all c's, b's, a's are known constants.

Definitions :.

Solutions :

A set of values $x_1, x_2 \dots x_n$ which satisfies a contraints of the linear programming problems is called its solution.

Feasible Solution :.

Any solution to a linear programming problem which satisfies the non-negativity restriction of the linear programming problems is called its feasible solution.

optimum Solution :.

Any feasible solution which optimize the objective function of LPP is called its optimum solution or optimal solution.

Slack Variables :-

If the Constraints of the general LPP be

$$\sum_{j=1}^{n} a_{ij} x_j < b_i \; (i=1,2 \ldots k) \quad \text{---} (1)$$

Then the non-negative variables s_i which are introduce to convert the inequalities.

(i) to the equalities

$$\sum_{j=1}^{n} a_{ij} x_{ij} + s_i = b_i \; (i=1,2 \ldots k)$$

are called slack variables.

Surplus Variables !-

If the Constraints of a generally LPP be

$$\sum_{j=1}^{n} a_{ij} x_j \geq b_i \; (i=1,2 \ldots k) \text{ ——— } (2)$$

Then the non-negative variables s_i which are introduce to convert the inequalities (i) to the equalities

$$\sum_{j=1}^{n} a_{ij} x_j - s_i = b_i \; (i=1,2 \ldots k)$$

are called Surplus Variables.

The Standard form :

The general LPP is of the form maximize

$$z = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n.$$

Subject to Constraints

$$a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n = b_1$$

$$a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n = b_2$$

$$\vdots \qquad\qquad \vdots$$

$$a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} a_n = b_m$$

and

$$x_1, x_2, \ldots, x_n \geq 0.$$

Characteristics of the standard form :-

* The objective function is maximization type.
* All constraints are expressed as equation.
* RHS of each constraints are non-negative.
* All variables are non-negative.

Note :-

The minimization of a function $f(x)$ is equivalent to the maximization of the negative expression of these function.

$$\text{Min } f(x) = -\text{Max}[F(x)].$$

$$\text{Min } z = -\text{Max}[-z]$$

For example,

$$\text{Min } z = c_1 x_1 + c_2 x_2$$

$$\text{Max } (-z) = -c_1 x_1 - c_2 x_2.$$

1 Problem : 01

A manufacture produces two types of models M_1 and M_2 Each model of the type M_1 requires 4 hours of grainding and 2 hours of polishing where as, each model of type M_2 requires 2 hours of grainding and 5 hours of polishing. The manufactures have 2 grainders and 3 polichers. Each gainder works 40 hours a week and each polichers works for 60 hours a week. profit an M_1 model is Rs. 3 and an model M_2 & Rs. 4. whatever the produced in a week is sold in the market. How should the manufacture. allocate his production capacity to the two types of models so that he may ~~als~~ make the maximum profit in a week ?

Solution :-

Decision variables :-

Let x_1 and x_2 be the no. of units of M_1 and M_2 model

Objective function :-

Since the profit on both the models are given we have to maximise the profit viz

$$\max z = 3x_1 + 4x_2.$$

Constraints :

There are two constraints one for grainding and the other for polishing No. of hrs available on each graindor for one week is 40 hrs. There are two graindors.

Hence the manufactures does not have more than $2 \times 40 = 80$ hrs of grainding.

M_1 requires 4 hrs of grinding and
M_2 requires 2 hrs of grinding

The ~~gainding~~ grainding constraints is given by,

$$4x_1 + 2x_2 \leq 80$$

Since there are 3 polishers the available time for Polishing in a week is given by $3 \times 60 = 180$.

M_1 requires 2 hrs of polishing and
M_2 requires 5 hrs of polishing.

Hence we have

$$2x_1 + 5x_2 \leq 180$$

Finally, we have

$$\max z = 3x_1 + 4x_2$$

Subject to Constraints.

$$4x_1 + 2x_2 \leq 80 \quad \text{——} \quad (1)$$

$$2x_1 + 5x_2 \leq 180 \quad \text{——} \quad (2)$$

$$x_1, x_2 \geq 0$$

Graphical method :-

Procedure of Solving LPP by graphical method :-

The steps involved in graphical method are as follows :

Step : 01

Consider each inequality constraints as equations.

Step : 02.

Plot each equation on the graph as each will geometrically represents a straight line.

Step : 03

Mark the region. If the inequality constraints corresponding to that line is ≤ then the region below the line typing in the first quadrant shaded. For the inequality constraints ≥ , The region above the line in the first quadrant is shaded . the points lying in common region will satisfied all the constraints simultaneously. The common region thus obtained is called the feasible region.

Step : 04

Assign an orbitary value say 0 for the objective functions.

Step : 05

Draw the straight line to represents the objective functions with the orbitary value.

Step : 06

Stretch the objective function line till the extreme points of the feasible region. In the maximization case this line will stop for these from the origin and passing through atleast one corner of the feasible region.

Step : 07

Find the co-ordinates of the extream points selected in step 6 and find the maximum or minimum value of z.

GRAPHICAL METHOD

Problem : 01

Solve the following LPP by graphical method

Minimize $z = 20x_1 + 10x_2$.

Subject to constraints :

$$x_1 + 2x_2 \leq 40$$
$$3x_1 + x_2 \geq 30$$
$$4x_1 + 3x_2 \geq 60$$
$$x_1, x_2 \geq 0$$

Soln :-

Given $z = 20x_1 + 10x_2$.

$x_1 + 2x_2 = 40$ —— ①

Put $x_1 = 0$

$2x_2 = 40$

$x_2 = 20$

x_1	0	40
x_2	20	0

Put $x_2 = 0$

$x_1 = 40$

Point = (40, 20)

$3x_1 + x_2 = 30$ —— ②

Put $x_1 = 0$

$x_2 = 30$

Put $x_2 = 0$

$3x_1 = 30$

$x_1 = 10$

x_1	0	10
x_2	30	0

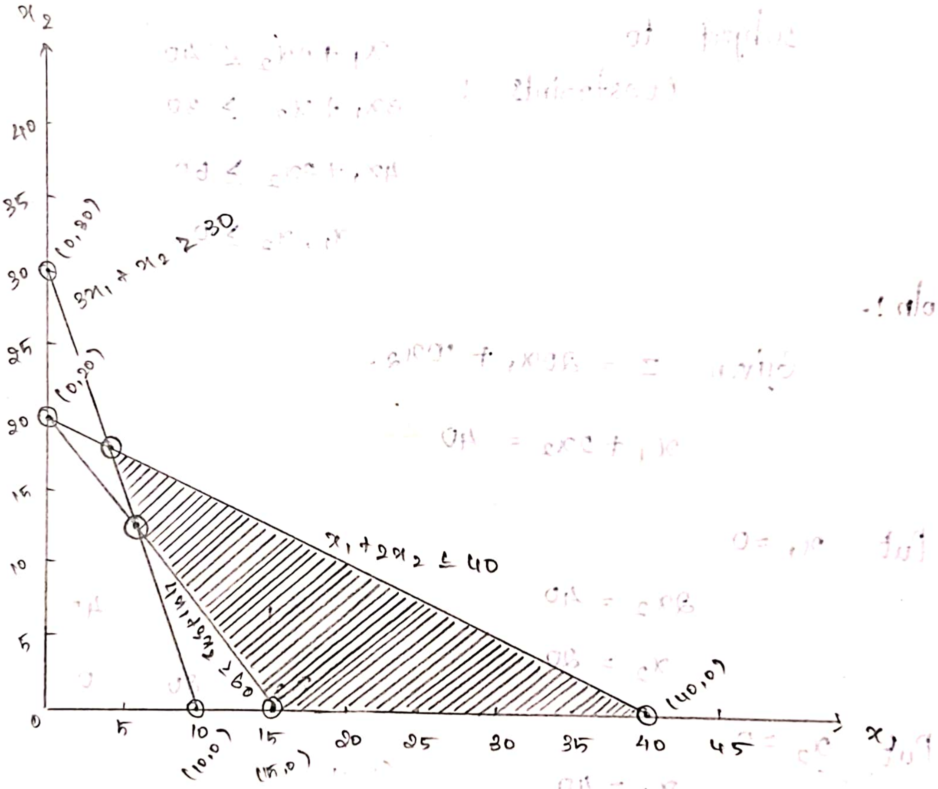
Point = (10, 30)

$4x_1 + 3x_2 = 60$ —— (3)

Put $x_1 = 0$	put $x_2 = 0$
$3x_2 = 60$	$4x_1 = 60$
$x_2 = 20$	$x_1 = 15$

x_1	0	15
x_2	20	0

Point (15, 20)



To Find C, D.

$$x_1 + 2x_2 = 40$$

$$(2) \times 2 \Rightarrow \quad 6x_1 + 2x_2 = 60$$

$$-5x_1 = -20$$

$$x_1 = 4$$

Sub $x_1 = 4$ in ①

$$4 + 2x_2 = 40$$

$$2x_2 = 36$$

$$x_2 = 18$$

Point $C(4, 18)$

② × 3 ⇒ $9x_1 + 3x_2 = 90$

$$4x_1 + 3x_2 = 60$$

$$5x_1 = 30$$

$$x_1 = 6$$

Sub $x_1 = 6$ in eq,n ②

$$3x_1 + x_2 = 30$$

$$18 + x_2 = 30$$

$$x_2 = 12$$

Point $D(6, 12)$

Corner points	Value of $Z = 20x_1 + 10x_2$.
$A(15, 0)$	300
$B(40, 0)$	400
$C(4, 18)$	260
$D(6, 12)$	240

∴ The minimum value of Z occurs at $D(6, 12)$

Hence the ~~optional~~ optimal Solution is :

$x_1 = 6$, $x_2 = 12$

(3) Find the maximum Value of $z = 5x_1 + 7x_2$

Subject to constraints ~~$x_1 + x_2 \le 4$~~.

$$x_1 + x_2 \le 4$$

$$3x_1 + 8x_2 \le 24$$

$$10x_1 + 7x_2 \le 35$$

$$x_1, x_2 \ge 0.$$

Given,

$$\text{Max } z = 5x_1 + 7x_2$$

$$x_1 + x_2 = 4 \quad \text{---} \; (1)$$

$$3x_1 + 8x_2 = 24 \quad \text{---} \; (2)$$

$$10x_1 + 7x_2 = 35 \quad \text{---} \; (3)$$

(1) $\rightarrow$ $x_1 + x_2 = 4$

Put $x_1 = 0$	Put $x_2 = 0$
$x_2 = 4$	$x_1 = 4$

Point $(4, 4)$

x_1	0	4
x_2	4	0

(2) $\rightarrow$ $3x_1 + 8x_2 = 24$

Put $x_1 = 0$	Put $x_2 = 0$
$8x_2 = 24$	$3x_1 = 24$
$x_2 = 3$	$x_1 = 8$

Point $(8, 3)$

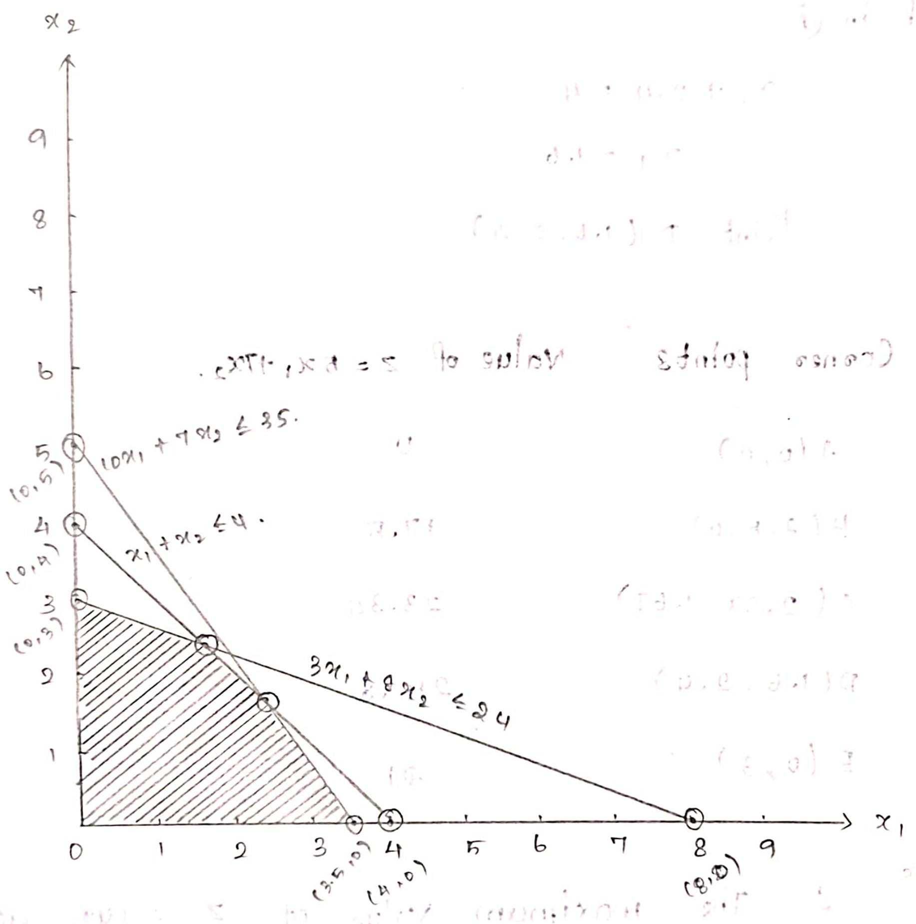
x_1	0	8
x_2	3	0

③ $\rightarrow$ $10x_1 + 7x_2 = 35$

Put $x_1 = 0$	Put $x_2 = 0$
$7x_2 = 35$	$10x_1 = 35$
$x_2 = 5$	$x_1 = 3.5$

Point $(3.5, 5)$

x_1	0	3.5
x_2	5	0



To find C, D

$x_1 + x_2 = 4$

$10x_1 + 7x_2 = 35$

① $\times 10 \Rightarrow$ $10x_1 + 10x_2 = 40$

$10x_1 + 7x_2 = 35$

$3x_2 = 5$

$x_2 = 1.67$

Sub x_2 in ①

$x_1 + 1.67 = 4$

$x_1 = 2.33$

Point $(2.33, 1.67)$

③ $\times 1 \Rightarrow$ $3x_1 + 3x_2 = 12$

$3x_1 + 8x_2 = 24$

$-5x_2 = -12$

$x_2 = 2.4$

Sub in ①

$x_1 + 2.4 = 4$

$x_1 = 1.6$

Point $D(1.6, 2.4)$

Corner points	Value of $z = 5x_1 + 7x_2$.
$A(0,0)$	0
$B(3.5,0)$	17.5
$C(2.33, 1.67)$	23.34
$D(1.6, 2.4)$	24.8
$E(0,3)$	21

$\therefore$ The maximum value of z occurs at $D(1.6, 2.4)$

Hence the ~~optional~~ optimal solution is :

$x_1 = 1.6$, $x_2 = 2.4$.

Artificial Variables :-

The artificial variables are introduced for the limited purpose of obtaining an initial solution when constraints of the type $\geq$ or $=$.

Defn :-

when we use surplus variables to convert inequlities into equations, then to obtain basic matrix as identity matrix, we used artificial variable in each constraints.

The summary of the extra variables to be added in the given LPP to convert it into standard form is given in the following table :

Type of Constraints	Extra Variable	operation	Coefficient of extra variable in the objective functions	
			Max. z	Min. z
$\leq$	slack variable	added	0	0
$\geq$	Surplus variable artificial variable	Subtracted added	0 $-M$	0 $+M$
$=$	Artificial variable	added	$-M$	$+M$

Basic Feasible Solution :-

A feasible solution to a general LPP which is also basic solution, i.e., all basic variables assume non-negative values is called basic feasible solution.

~~Generally,~~ basic feasible solutions are of two types :-

(i) Degenerate Basic Feasible Solution :-

A basic solution to the system of equations is called degenerate if one or more of the basic variables become equal to zero.

ii) Non-degenerate Basic Feasible Solution :-

A basic solution is called non-degenerate if values of m basic variables are non-zero and positive.

Optimal Basic Feasible Solution :

Any basic feasible solution which optimize (maximize or minimize) the objective function of a general LPP is called optimal basic feasible solution.

Unbounded Solution :-

A solution which can increase or decrease the value of the objective function of an LPP indefinitely is said to be an unbounded solution.

Feasible Region :-

The common region formed by all the constraints and non-negative restrictions of an LPP is called feasible region.

Infeasible Region :-

The region common to all constraints in which all the decision variables are negative is called infeasible region.

Convex Region :-

If the line segment joining any two arbitrary points of the region lies entirely within the region, then this region is said to be convex region.

Simplex method :-

Terminology and Notations :-

The general form of an LPP is given below :

$$\text{Max. } z = C \cdot x$$

Subject to $Ax = b, x \geq 0$

where,

$$A = [a_{ij}]_{m \times N}, \quad x = (x_1, x_2 \dots, x_n, \dots x_N)_{N \times 1}$$

$$C = (c_1, c_2 \dots c_n, 0, 0, \dots, 0)_{1 \times N}$$

and $$b = [b_1, b_2, \dots b_m]_{m \times 1}$$

Now, different symbolic representation is given in the following table :

S.No	Name	Representation	Description
1.	Basic matrix	$B = \{\beta_1, \beta_2 \dots \beta_m\}$	A non-singular submatrix B of order mxn whose column vectors are m no. of linearly independent columns selected from A.
2.	Basic variables	$x_{B_1}, x_{B_2}, \dots x_{B_m}$	The variable corresponding to $\beta_1, \beta_2, \dots, \beta_m$ are called basic variables.
3.	Basic Feasible Solution	$x_B = B^{-1} \cdot b$	$x_B = [x_{B_1}, x_{B_2}, \dots x_{B_m}] = B^{-1} \cdot b$
4.	Non-basic Variables	$x_{B_{m_i}}, i > m$	The variables, other than basic, are called non-basic variables.
5.	Coefficient of basic ~~Variables~~	$C_{B_i} :, i = 1$ to m	Corresponding to any x_B, C_B will represent the row vector containing the constants $C_{B_1}, C_{B_2} \dots C_{B_m}$.

Simplex Algorithm :-

To find the optimal solution of the given linear programming problem, we use the following steps :

Step : 01

General Steps :-

(i) If the given problem is of minimization, first convert it into the maximization problem by multiplying both sides of the objective function by -1 and put $-z = z^*$. Remember that if v is the maximum value of z^* then $-v$ will be the minimum value of z.

(ii) The RHS of each of the constraints should be non-negative. If there is any constraints for which b_i is negative then multiply this constraints by -1 to convert it into positive values.

Step : 02

Check whether all $b_i\,(i = 1, 2, \dots m)$ are positive. If any b_i is negative then multiply the inequation of the constraint by -1 so as to get all b_i to be positive.

Step : 03

Express the problem in standard form by introducing slack / surplus variables to convert the inequality constraints into equations.

Step : 04

obtain an initial basic feasible solution to the problem in the form $x_B = B^{-1}b$ and put it in the first column of the simplex table. Form the initial simplex table as given below.

		$C_j =$	C_1	C_2	C_3		...		0	0	...	0
C_B	S_B	X_B	X_1	X_2	X_3	X_4	...	X_n	S_1	S_2	...	S_m
C_{B_1}	S_1	b_1	a_{11}	a_{12}	a_{13}	a_{14}	...	a_{1n}	1	0	...	0
C_{B_2}	S_2	b_2	a_{21}	a_{22}	a_{23}	a_{24}	...	a_{2n}	1	0	...	0

Step : 05

Compute the net evaluations $z_j - C_j$ by using the relation $z_j - C_j = C_B(a_j - c_j)$ Examine the sign of $z_j - C_j$.

(i) If all $z_j - C_j \geq 0$, then the initial basic feasible solution X_B is an optimum basic feasible solution.

(ii) If atleast one $z_j - C_j < 0$, then proceed to next step as the solution is not optimal.

Step : 06

(To find the entering variable (i.e)., key column)

If there are more than one negative $z_j - C_j$, choose the most negative of them. Let it be $z_r - C_r$ for some $j = r$. This gives the entering variable x_r and is indicated by an arrow at the bottom of the r^{th} column. If there are more than one variables having the same most negative $z_j - C_j$ then any one of them can be selected arbitrarily as the entering variable.

(i) If all $a_{ir} \leq 0$ $(i = 1, 2 \dots m)$ then there is an unbounded solution to the given problem.

(ii) ~~If~~ atleast one $a_{ir} > 0$ $(i = 1, 2 \dots m)$ then the corresponding vector x_r enters the basis

Step: 07

(To find the leaving Variable or key row).

Compute the ratio (x_{Bi} / a_{ir}, $a_{ir} > 0$).

If the minimum of these ratios be x_{Bi}/a_{kr}, then choose the variable x_K to leave the basis called the key row and the element at the intersection of key row and key column is called the key element.

Step: 08

Form a new basis by dropping the leaving variable and introducing the entering variable along with the associated value under C_B column. Convert the leading element to unity by dividing the key equation by the key element and all other elements in its column to zero by using Gauss Elimination on the formula.

$$\text{New element} = \text{old element} - \left[\frac{\text{Product of elements in keyrow and column}}{\text{key element}}\right]$$

Step: 09

Go to step (5) and repeat the procedure until either an optimum solution is obtained or there is an indication of unbounded Solution.

Operations Research:

Developed during II world war.
Finding the optimal solution for a given problem.
Linear programming Models (LP)

* Graphical method.
* Simplex method
* Duality method
* Assignment problems
* Transportation problems.

Example :-

A furniture dealer deals in two items viz, tables and chairs. He has ₹ 10,000 to invest and a space to store almost 60 pieces. A table costs him ₹ 500 and a chair of ₹ 200. He can sell a table at profit of ₹ 50 and a chair at a profit of ₹ 15. Assume that he can sell all the items that he buys. Formulate the problem as an LPP, so that he can maximize profit.

Soln :-

$$\max z = 50x_1 + 15x_2 \quad (\text{Decision Variable } x_1 \;\&\; x_2)$$

Constraints :- $500x_1 + 200x_2 \le 10{,}000 \quad (5x_1 + 2x_2 \le 100)$

$$x_1 + x_2 \le 60$$

$$x_1, x_2 \ge 0$$

(5) Maximize $Z = 5x_1 + 4x_2$

Subject to constrain $6x_1 + 4x_2 \leq 24$

$x_1 + 2x_2 \leq 6$

$-x_1 + x_2 \leq 1$

$x_2 \leq 2$

$x_1, x_2 \geq 0.$

Given that,

$Z = 5x_1 + 4x_2$

$\Rightarrow 6x_1 + 4x_2 = 24.$ —— (1)

Put $x_1 = 0$ | Put $x_2 = 0$

$4x_2 = 24$ | $6x_1 = 24$

$x_2 = 6$ | $x_1 = 4$

$\boxed{(0, 6)}$ | $\boxed{(4, 0)}$

$\Rightarrow x_1 + 2x_2 = 6$ —— (2)

Put $x_1 = 0$ | Put $x_2 = 0$

$2x_2 = 6$ | $x_1 = 6$

$x_2 = 3$ | $\boxed{(6, 0)}$

$\boxed{(0, 3)}$

$\Rightarrow -x_1 + x_2 = 1$ —— (3)

Put $x_1 = 0$ | Put $x_2 = 0$

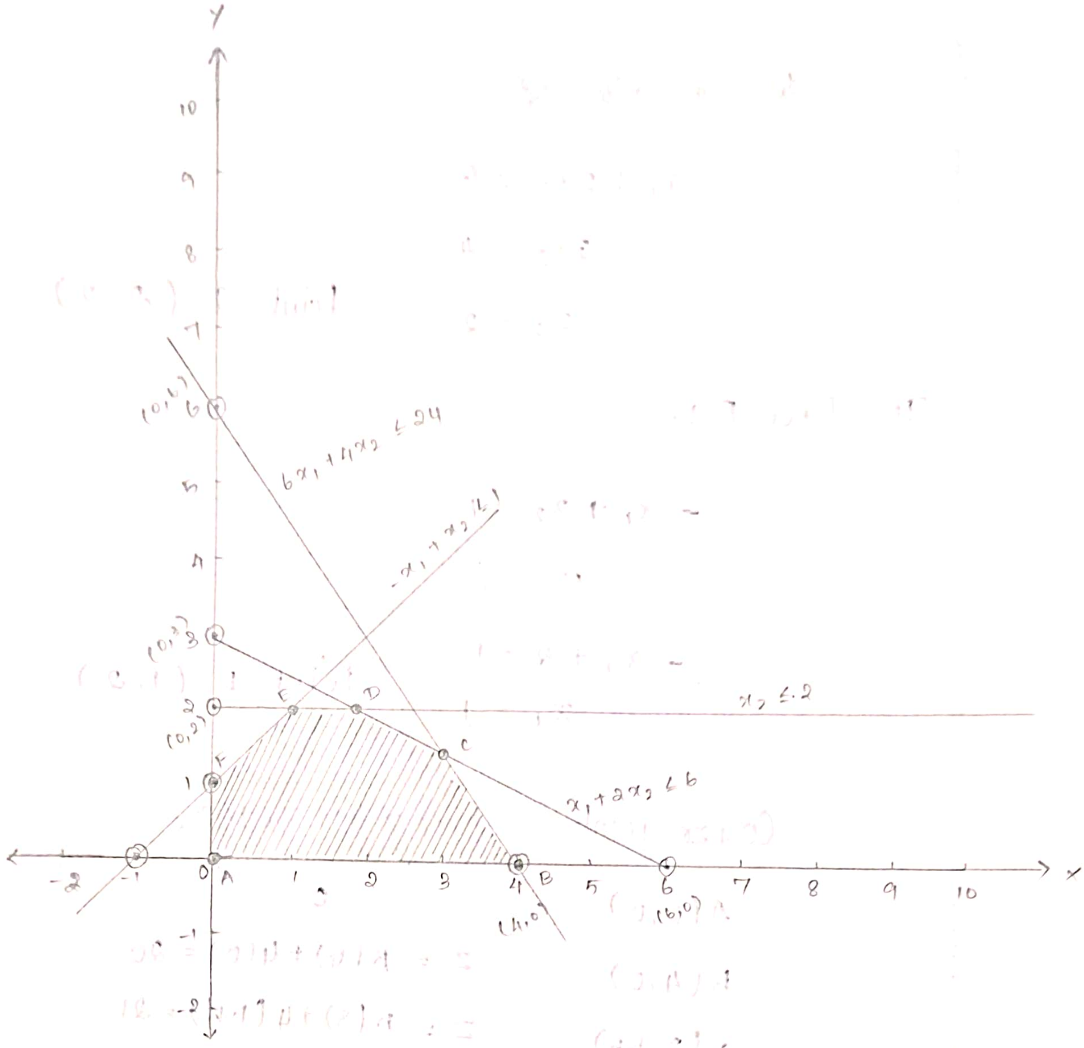
$x_2 = 1$ | $x_1 = -1$

$\boxed{(0, 1)}$ | $\boxed{(-1, 0)}$

$\Rightarrow x_2 = 2.$ —— (4)

Put $x_1 = 0$

$\boxed{(0, 2)}$



<u>To Find C :-</u>

$$① \rightarrow 6x_1 + 4x_2 = 24$$

$$② \times 2 \longrightarrow 2x_1 + 4x_2 = 12.$$

$$4x_1 = 12$$

$$x_1 = 3$$

Sub x_1 in eqn ①

$$6(3) + 4(x_2) = 24$$

$$4x_2 = 6$$

$$x_2 = 3/2$$

Point C $(3, 1.5)$

$$x_2 = 1.5$$

<u>To Find D :-</u>

$$② \rightarrow x_1 + 2x_2 = 6$$

$$④ \times 2 \rightarrow 0x_1 + 2x_2 = 4$$

$$x_1 = 2.$$

Sub in eqn (2)

$$x_1 + 2x_2 = 6$$
$$2x_2 = 4$$
$$x_2 = 2.$$

Point D (2, 2)

To Find E :-

$$-x_1 + x_2 = 1$$
$$x_2 = 2$$
$$-x_1 + 2 = 1$$
$$x_1 = 1$$

Point E (1, 2)

Corner point	Max $z = 5x_1 + 4x_2$
A (0, 0)	0
B (4, 0)	$z = 5(4) + 4(0) = 20$
C (3, 1.5)	$z = 5(3) + 4(1.5) = 21$
D (2, 2)	$z = 5(2) + 4(2) = 18$
E (1, 2)	$z = 5(1) + 4(2) = 13$
F (0, 1)	$z = 5(0) + 4(1) = 4$

∴ The maximum value of z obtained at $C = (3, 1.5)$

∴ The optimal solution is $x_1 = 3$, $x_2 = 1.5$

∴ $z = 21$.

(6) Minimize $z = 4x_1 + 3x_2$

Subject to constrains, $200x_1 + 100x_2 \geq 4000$

$$x_1 + 2x_2 \geq 50$$

$$40x_1 + 40x_2 \geq 1400$$

and $x_1, x_2 \geq 0$

Given that,

$$Z = 4x_1 + 3x_2$$

$\Rightarrow$ $200x_1 + 100x_2 \geq 4000 \div 100$

$2x_1 + x_2 \geq 40$ —— ①

$\Rightarrow$ $x_1 + 2x_2 \geq 50$ —— ②

$\Rightarrow$ $40x_1 + 40x_2 \geq 1400 \div 10$

$4x_1 + 4x_2 \geq 140 \div 4$

$x_1 + x_2 \geq 35$ —— ③

① $\longrightarrow$ $2x_1 + x_2 = 40$

Put $x_1 = 0$ | Put $x_2 = 0$

$x_2 = 40$ | $2x_1 = 40$

$x_1 = 20$

$(0, 40)$ | $(20, 0)$

② $\longrightarrow$ $x_1 + 2x_2 = 50$

Put $x_1 = 0$ | Put $x_2 = 0$

$2x_2 = 50$ | $x_1 = 50$

$x_2 = 25$ | $(50, 0)$

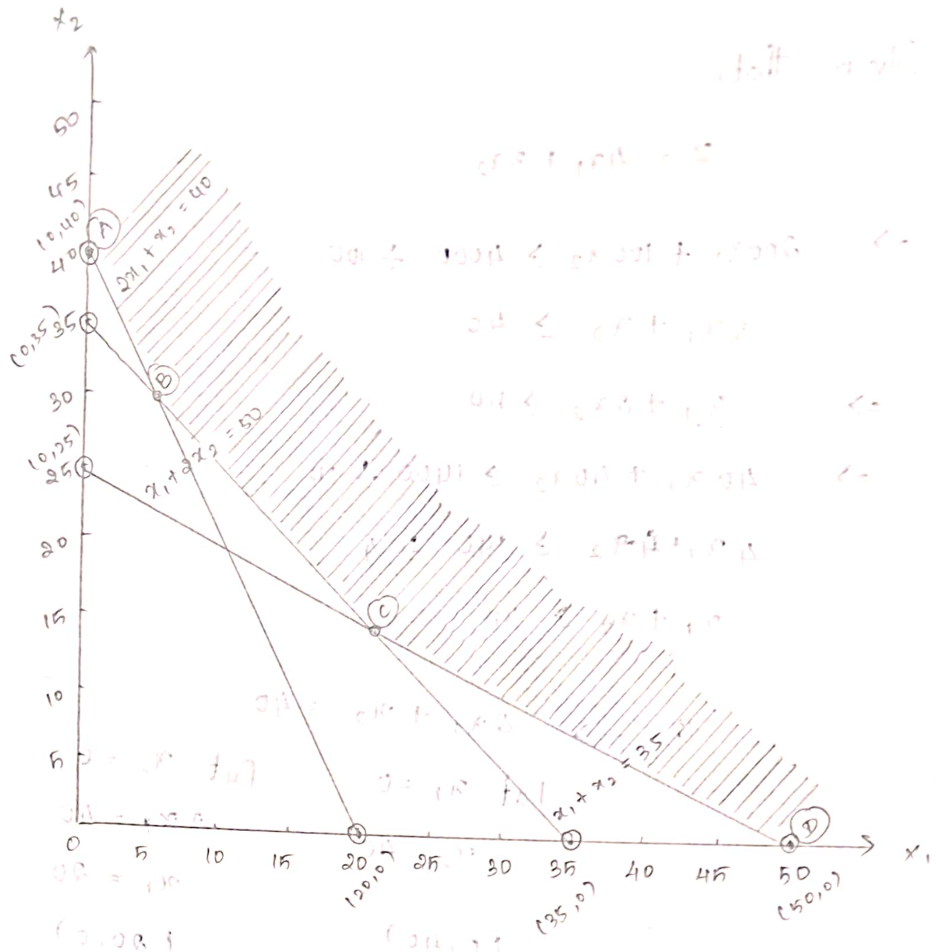
$(0, 25)$

③ $\longrightarrow$ $x_1 + x_2 = 35$

Put $x_1 = 0$ | Put $x_2 = 0$

$x_2 = 35$ | $x_1 = 35$

$(0, 35)$ | $(35, 0)$



To Find B :.

$$2x_1 + x_2 = 40$$
$$x_1 + x_2 = 35$$
$$x_1 = 5$$

$$x_1 + x_2 = 35$$
$$5 + x_2 = 35$$ Point B (5, 30)
$$x_2 = 030$$

To Find C :.

$$x_1 + 2x_2 = 50$$
$$x_1 + x_2 = 35$$
$$x_2 = 15$$

$$x_1 + x_2 = 35$$
$$x_1 + 15 = 35$$ Point C (20, 15)
$$x_1 = 20$$

Corner points	Min $z = 4x_1 + 3x_2$.
$A(0,40)$	$z = 4(0)+3(40) = 120$
$B(5,30)$	$z = 4(5)+3(30) = 110$
$C(20,15)$	$z = 4(20)+3(15) = 125$
$D(50,0)$	$z = 4(50)+3(0) = 200$

$\therefore$ The minimum value of z obtained at $B(5,30)$

$\therefore$ The optimal solution is $x_1 = 5$, $x_2 = 30$

$\therefore z = 110$.

Homework

(7) Max $z = 4x_1 + x_2$

Subject to constraints.

$x_1 + x_2 \leq 50$

$3x_1 + x_2 \leq 90$

$x_1, x_2 \geq 0$

Given that,

Max $z = 4x_1 + x_2$.

$x_1 + x_2 = 50$ —— (1).

Put $x_1 = 0$ | Put $x_2 = 0$

$x_2 = 50$ | $x_1 = 50$

$(0,50)$ | $(50,0)$

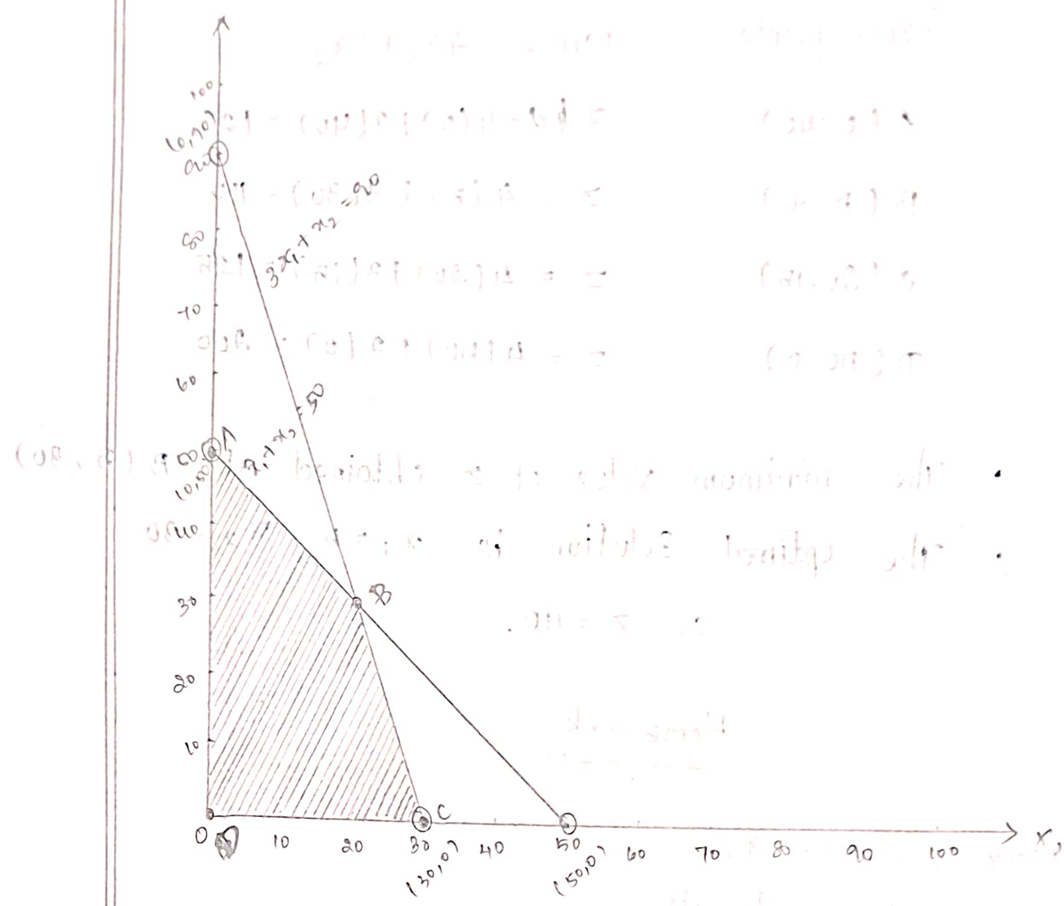
$3x_1 + x_2 = 90$ —— (2)

Put $x_1 = 0$ | Put $x_2 = 0$

$x_2 = 90$ | $3x_1 = 90$

| $x_1 = 30$

$(0,90)$ | $(30,0)$



To Find B :-

$$① \rightarrow 3x_1 + x_2 = 90$$
$$② \rightarrow x_1 + x_2 = 50$$
$$2x_1 = 40$$
$$x_1 = 20$$

Put x_1 in eqn ②

$$20 + x_2 = 50 \qquad \text{Point B } (20, 30)$$
$$x_2 = 30$$

Corner points	Max $z = 4x_1 + x_2$.
A (0, 50)	$z = 4(0) + 50 = 50$
B (20, 30)	$z = 4(20) + 30 = 110$
C (30, 0)	$z = 4(30) + 0 = 120$
D (0, 0)	$z = 4(0) + 0 = 0$

$\therefore$ The max value of z obtained at $C = (30, 0)$

$\therefore$ The optimal solution is $x_1 = 30$, $x_2 = 0$

$\therefore$ $z = 120$.

(8) Min $z = -x_1 + 2x_2$.

Subject to Constraints, $-x_1 + 3x_2 \leq 10$

$x_1 + x_2 \leq 6$

$x_1 - x_2 \leq 2$

$x_1, x_2 \geq 0$

Given that,

Min $z = -x_1 + 2x_2$.

$\rightarrow$ $-x_1 + 3x_2 = 10$ —— (1)

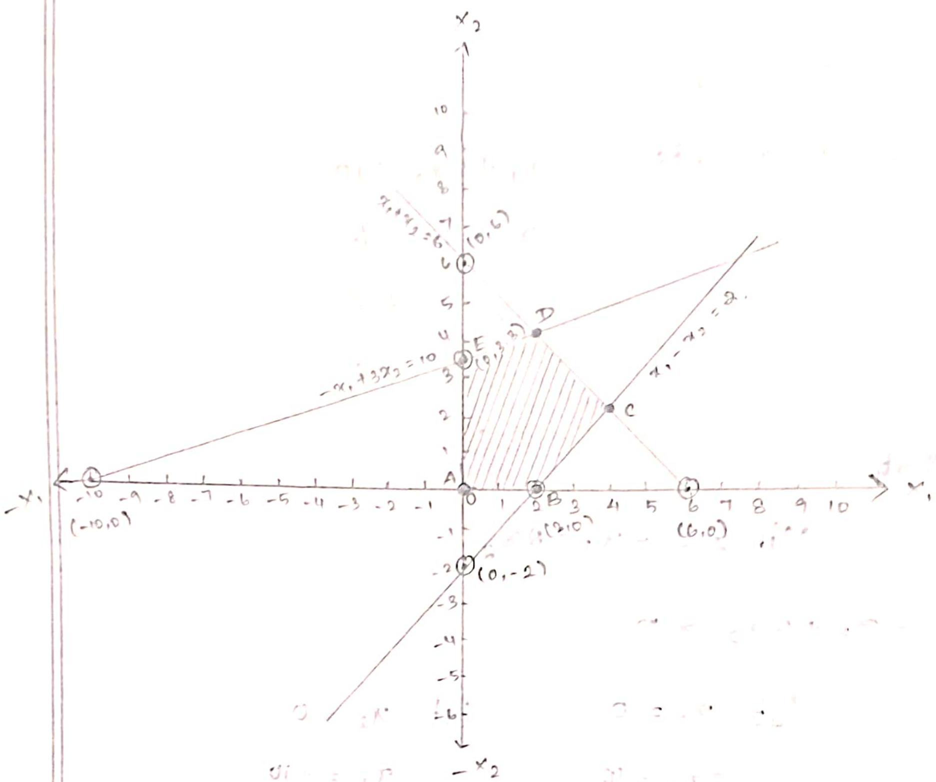
Put $x_1 = 0$	Put $x_2 = 0$
$3x_2 = 10$	$x_1 = -10$
$x_2 = 3.3$	$(-10, 0)$
$(0, 3.3)$	

$\rightarrow$ $x_1 + x_2 = 6$ —— (2)

Put $x_1 = 0$	Put $x_2 = 0$
$x_2 = 6$	$x_1 = 6$
$(0, 6)$	$(6, 0)$

$\rightarrow$ $x_1 - x_2 = 2$ —— (3)

Put $x_1 = 0$	Put $x_2 = 0$
$x_2 = -2$	$x_1 = 2$
$(0, -2)$	$(2, 0)$



To Find C :-

$$\textcircled{2} \Longrightarrow x_1 + \cancel{x_2} = 6$$
$$\textcircled{3} \Longrightarrow x_1 - \cancel{x_2} = 2$$

$$2x_1 = 8$$
$$x_1 = 4$$

x_1 in eqn ③

$$4 - x_2 = 2.$$
$$-x_2 = -2$$
$$x_2 = 2$$

Point C $(4, 2)$

To Find D :-

$$\textcircled{1} \rightarrow -\cancel{x_1} + 3x_2 = 10$$
$$\textcircled{2} \Rightarrow \cancel{x_1} + x_2 = 6$$

$$4x_2 = 16$$
$$x_2 = 4$$

Put x_2 value in eqn (2)

$x_1 + 4 = 6$

$x_1 = 6 - 4$ Point D $(2, 4)$

$x_1 = 2$

Corner points	Min $z = -x_1 + 2x_2$
$A(0,0)$	$z = -(0) + 2(0) = 0$
$B(2,0)$	$z = -2 + 2(0) = -2$
$c(4,2)$	$z = -4 + 2(2) = 0$
$D(2,4)$	$z = -2 + 2(4) = 6$
$E(0, 3.3)$	$z = -(0) + 2(3.3) = 6.6$

∴ The minimum value of z obtained at $B(2,0)$

∴ The optimal solution is $x_1 = 2$, $x_2 = 0$.

∴ $z = -2$.

SIMPLEX METHOD

Problem : 01

Using simplex method solve the LPP

Max $z = x_1 + x_2 + 3x_3$

Subject to constraints : $3x_1 + 2x_2 + x_3 \le 3$

$2x_1 + x_2 + 2x_3 \le 2$

$x_1, x_2, x_3 \ge 0$

Soln :-

By introducing the slack variables s_1, s_2. we get,

$\text{Max } z = x_1 + x_2 + 3x_3 + 0s_1 + 0s_2$

Subject to constraints,

$$3x_1 + 2x_2 + x_3 + s_1 + 0s_2 = 3$$
$$2x_1 + x_2 + 2x_3 + 0s_1 + s_2 = 2$$
$$x_1, x_2, x_3, s_1, s_2 \geq 0.$$

Initial Simplex table :.

C		C_j	1	1	3	0	0	
Cost C_B	Basis S_B	Solution X_B	x_1	x_2	x_3	s_1	s_2	Ratio
0	s_1	3	3	2	1	1	0	$3/1 = 3$
0	s_2	2	2	1	2	0	1	$2/2 = 1$ ←
	Z_j	0	0	0	0	0	0	
	$Z_j - C_j$		-1	-1	-3	0	0	

↑

Introducing variable is x_3

leaving variable is s_2

Iteration table :

C		C_j	1	1	3	0	0
C_B	S_B	X_B	x_1	x_2	x_3	s_1	s_2
0	s_1	2	2	1.5	0	1	-0.5
3	x_3	1	1	0.5	1	0	0.5
	Z_j	3	3	1.5	3	0	1.5
$Z_j - C_j$			2	0.5	0	0	1.5

Since all $Z_j - C_j \geq 0$.

The current solution is optimal :-

$x_1 = 0, \; x_2 = 0, \; x_3 = 1$

$\text{Max } z = 3.$

Problem 102

Using Simplex method Solve the LPP.

$$\text{Max } z = 3x_1 + 2x_2 + 5x_3$$

Subject to

$$x_1 + 2x_2 + x_3 \le 430$$

$$3x_1 + 2x_3 \le 460$$

$$x_1 + 4x_2 \le 420$$

$$x_1, x_2, x_3 \ge 0$$

Soln :-

By introducing the slack Variables, S_1, S_2, and S_3.

We get,

$$\text{Max } z = 3x_1 + 2x_2 + 5x_3 + 0S_1 + 0S_2 + 0S_3.$$

Subject to

$$x_1 + 2x_2 + x_3 + 1S_1 + 0S_2 + 0S_3 = 430$$

$$3x_1 + 0x_2 + 2x_3 + 0S_1 + 1S_2 + 0S_3 = 460$$

$$x_1 + 4x_2 + 0x_3 + 0S_1 + 0S_2 + 1S_3 = 420$$

$$x_1, x_2, x_3, S_1, S_2, S_3 \ge 0$$

Initial Simplex table :-

C		C_j	3	2	5	0	0	0	Ratio
C_B	S_B	X_B	x_1	x_2	x_3	S_1	S_2	S_3	
0	S_1	430	1	2	1	1	0	0	$430/1 = 430$
0	S_2	460	3	0	2	0	1	0	$\frac{460}{2} = 230$ ←
0	S_3	420	1	4	0	0	0	1	$\frac{420}{0} = 0$
	~~Z_j~~	0	0	0	0	0	0	0	
$Z_j - C_j$			-3	-2	-5	0	0	0	
					↑				

Introducing Variable is x_3,

leaving Variable is S_2.

Iteration table : 01

C		C_j	3	2	5	0	0	0	Ratio
C_B	S_B	X_B	x_1	x_2	x_3	S_1	S_2	S_3	
0	S_1	200	-0.5	2	0	1	-0.5	0	$\frac{200}{2} = 100 \leftarrow$
5	x_3	230	1.5	0	1	0	0.5	0	$\frac{230}{0} = 0$
0	S_3	420	1	4	0	0	0	1	$\frac{420}{4} = 105$
Z_j		0+1150+0 1150	7.5	0	5	0	2.5	0	
$Z_j - C_j$			4.5	-2 $\uparrow$	0	0	2.5	0	

$x_3 = \frac{S_2}{2} = 230 \quad 1.5 \quad 0 \quad 1 \quad 0 \quad 0.5 \quad 0$

Old element :-

$S_1 = 430 \quad 1 \quad 2 \quad 1 \quad 1 \quad 0 \quad 0$

(−)

New element :-

$$x_3 = \frac{230 \quad 1.5 \quad 0 \quad 1 \quad 0 \quad 0.5 \quad 0}{200 \quad -0.5 \quad 2 \quad 0 \quad 1 \quad -0.5 \quad 0}$$

old element :-

$S_1 = 420 \quad 1 \quad 4 \quad 0 \quad 0 \quad 0 \quad 1$

(−)

New element :-

$$x_3 = \frac{0[230 \quad 1.5 \quad 0 \quad 1 \quad 0 \quad 0.5 \quad 0]}{420 \quad 1 \quad 4 \quad 0 \quad 0 \quad 0 \quad 1}$$

Introducing Variable is x_2
leaving variable is S_1

Iteration table : 02

C		C_j	3	2	5	0	0	0
C_B	S_B	X_B	x_1	x_2	x_3	S_1	S_2	S_3
2	x_2	100	0.25	1	0	0.5	-0.25	0
5	x_3	230	1.5	0	1	0	0.5	0
0	S_3	20	0	0	0	-2	1	1
	Z_j	1350	8	2	5	1	2	0
$Z_j - C_j$			5	0	0	1	2	0

$x_2 = S_1/2 = 100 \quad 0.25 \quad 1 \quad 0 \quad 0.5 \quad -0.25 \quad 0$

$S_3 - 4x_2 \Rightarrow 420 \quad 1 \quad 4 \quad 0 \quad 0 \quad 0 \quad 1$

$-4\,[\,100 \quad 0.25 \quad 1 \quad 0 \quad 0.5 \quad -0.25 \quad 0\,]$

$20 \quad 0 \quad 0 \quad 0 \quad -2 \quad 1 \quad 1$

Since all $Z_j - C_j \geq 0$. The current Solution is optimal.

$x_1 = 0$, $x_2 = 100$, $x_3 = 230$ and Max $Z = 1350$.

Home work

Using Simplex method Solve the LPP

$$\text{Max } Z = 2x_1 + x_2$$

Subject to Constraints,

$$x_1 + 2x_2 \leq 10$$
$$x_1 + x_2 \leq 6$$
$$x_1 - x_2 \leq 2$$
$$x_1 - 2x_2 \leq 1$$
$$x_1, x_2 \geq 0$$

By introducing the slack variable S_1, S_2, S_3, S_4 we get,

$$\text{Max } z = 2x_1 + x_2 + 0S_1 + 0S_2 + 0S_3 + 0S_4.$$

Subject to constraints :-

$$x_1 + 2x_2 + S_1 + 0S_2 + 0S_3 + 0S_4 = 10$$

$$x_1 + x_2 + 0S_1 + S_2 + 0S_3 + 0S_4 = 6$$

$$x_1 - x_2 + 0S_1 + 0S_2 + S_3 + 0S_4 = 2.$$

$$x_1 - 2x_2 + 0S_1 + 0S_2 + 0S_3 + S_4 = 1$$

$$x_1, x_2, x_3, x_4, S_1, S_2, S_3, S_4 \geq 0.$$

Initial Simplex table :-

C		Cj	2	1	0	0	0	0	Ratio
C_B	S_B	X_B	x_1	x_2	S_1	S_2	S_3	S_4	
0	S_1	10	1	2	1	0	0	0	$10/1 = 10$
0	S_2	6	1	1	0	1	0	0	$6/1 = 6$
0	S_3	2	1	-1	0	0	1	0	$2/1 = 2$
0	S_4	1	1	-2	0	0	0	1	$1/1 = 1$ ←
Z_j		0	0	0	0	0	0	0	
$Z_j - C_j$			-2 ↑	-1	0	0	0	0	

Iteration table : 01

Introducing : x_1 , leaving : S_4

C		Cj	2	1	0	0	0	0	Ratio
C_B	S_B	X_B	x_1	x_2	S_1	S_2	S_3	S_4	
0	S_1	9	0	4	1	0	0	-1	2.25
0	S_2	5	0	3	0	1	0	-1	1.67
0	S_3	1	0	1	0	0	1	-1	1 ←
2	x_1	1	1	-2	0	0	0	1	-0.5
Z_j		2	2	-4	0	0	0	2	
$Z_j - C_j$			0	-5 ↑	0	0	0	2	

$x_2 = \frac{S_3}{1}$ => 1 0 1 0 0 1 -1

S_1 => 9 0 4 1 0 0 -1

-4×2 => $-4[7$ 0 7 0 0 1 $-1]$

5 0 0 1 0 -4 3

S_2 => 5 0 3 0 1 0 -1

x_2 => $-3[1$ 0 1 0 0 1 $-1]$

2 0 0 0 1 -3 2

x_1 => $\cancel{0}$ 1 1 -2 0 0 0 1

2×2 => $2[1$ 0 1 0 0 1 $-1]$

2 1 0 0 0 2 -1

Iteration table : 02.

Introducing variable is x_2

leaving variable is S_3 .

C		Cj	2	1	0	0	0	0	Ratio
C_B	S_B	X_B	x_1	x_2	S_1	S_2	S_3	S_4	
0	S_1	5	0	0	1	0	-4	3	1.67
0	S_2	2	0	0	0	1	-3	2	1 ←
0	x_2	1	0	1	0	0	1	-1	-1
0	x_1	3	1	0	0	0	2	-1	-3
Zj		7	2	1	0	0	5	-3	
Zj - Cj			0	0	0	0	5	-3 ↑	

Iteration table : 03

Introducing Variable is S_4

leaving variable is S_2.

C		C_j	2	1	0	0	0	0
C_B	S_B	X_B	x_1	x_2	S_1	S_2	S_3	S_4
0	S_1	2	0	0	1	-1.5	0.5	0
0	S_4	1	0	0	0	0.5	-1.5	1
1	x_2	2	0	1	0	0.5	-0.5	0
2	x_1	4	1	0	0	0.5	0.5	0
Z_j		10	2	1	0	1.5	0.5	0
$Z_j - C_j$			0	0	0	1.5	0.5	0

$S_4 = \frac{S_2}{2} =$ 1 0 0 0 0.5 -1.5 1

$S_1 \Rightarrow$ 5 0 0 1 0 -4 3

$S_4 \Rightarrow -3[$ 1 0 0 0 0.5 -1.5 1 $]$

2 0 0 1 -1.5 0.5 0

$x_2 \Rightarrow$ 1 0 1 0 0 1 -1

$S_4 \Rightarrow +1[$ 1 0 0 0 0.5 -1.5 1 $]$

2 0 1 0 0.5 -0.5 0

$x_1 \Rightarrow$ 3 1 0 0 0 2 -1

$S_4 \Rightarrow 1[$ 1 0 0 0 0.5 -1.5 1 $]$

4 1 0 0 0.5 0.5 0

Since all $z_j - c_j \geq 0$,

$\therefore$ The current solution is optimal.

$x_1 = 4$, $x_2 = 2$ & Max $Z = 10$.

$\text{Min } z = x_1 - 3x_2 + 2x_3$

Subject to Constraints,

$$3x_1 - x_2 + 2x_3 \le 7$$
$$-2x_1 + 4x_2 \le 12$$
$$-4x_1 + 3x_2 + 8x_3 \le 10$$
$$x_1, x_2, x_3 \ge 0$$

$$\text{Max } z^* = -\text{Min } z.$$

By introducing slack variables s_1, s_2, s_3 The standard form of LPP is,

$$\text{Max } z^* = -x_1 + 3x_2 - 2x_3 + 0s_1 + 0s_2 + 0s_3$$

Subject to Constraints :

$$3x_1 - x_2 + 2x_3 + s_1 + 0s_2 + 0s_3 = 7$$
$$-2x_1 + 4x_2 + 0x_3 + 0s_1 + s_2 + 0s_3 = 12$$
$$-4x_1 + 3x_2 + 8x_3 + 0s_1 + 0s_2 + s_3 = 10$$
$$x_1, x_2, x_3, s_1, s_2, s_3 \ge 0$$

Initial Table :-

C		C_j	-1	3	-2	0	0	0	
C_B	S_B	X_B	x_1	x_2	x_3	s_1	s_2	s_3	Ratio.
0	s_1	7	3	-1	2	1	0	0	$7/-1 = -7$
0	s_2	12	-2	4	0	0	1	0	$12/4 = 3 \leftarrow$
0	s_3	10	-4	3	8	0	0	1	$10/3 = 3.33$
	z_j	0	0	0	0	0	0	0	
	$z_j - c_j$		1	-3 $\uparrow$	2	0	0	0	

Iteration table : 01

Introducing variable is x_2.

leaving variable is s_2.

C		C_j	-1	3	-2	0	0	0	Ratio
C_B	S_B	X_B	x_1	x_2	x_3	s_1	s_2	s_3	
0	s_1	10	2.5	0	2	1	0.25	0	$\frac{10}{2.5} = 4 \leftarrow$
3	x_2	3	-0.5	1	0	0	0.25	0	$\frac{3}{-0.5} = -6$
0	s_3	1	-2.5	0	8	0	-0.75	0	$\frac{1}{-2.5} = -0.4$
Z_j		9	-1.5	3	0	0	0.75	0	
$Z_j - C_j$			-0.5 ↑	0	2	0	0.75	0	

$\frac{x_2}{4} \Rightarrow$ 3 -0.5 1 0 0 0.25 0

$s_1 \rightarrow$ 7 3 -1 2 1 0 0 — 3 - 2/4

$x_2 \rightarrow$ 3 0.5 1 0 0 0.25 0 — 10 - 2/4

10 2.5 0 2 1 0.25 0 — 10/4

5/2

$s_3 \rightarrow$ 10 -4 3 8 0 0 1 — 3 - 1/2

$3x_2 \rightarrow$ 9 -1.5 3 0 0 0.75 0 — 6 - 1/2

(−)

1 -2.5 0 8 0 -0.75 1 — 5/2 .

Iteration table : 02

Introducing variable is x_1

leaving variable is s_1

C		C_j	-1	3	-2	0	0	0	Ratio
C_B	S_B	X_B	x_1	x_2	x_3	s_1	s_2	s_3	
-1	x_1	4	1	0	0.8	0.4	0.1	0	
3	x_2	5	0	1	0.4	0.2	0.3	0	
0	s_3	11	0	0	10	1	-0.5	0	
Z_j		11	-1	3	0.4	0.2	0.8	0	
$Z_j - C_j$			0	0	2.4	0.2	0.8	0	

$x_3 \rightarrow$	3	-0.5	1	0	0	0.25	0
0.5×0.5 (+)	[4	1	0	0.8	0.4	0.1	0]
	5	0	1	0.4	0.2	0.3	0

$S_3 \rightarrow$	1	-2.5	0	8	0	-0.75	0
$2.5 \times x_2 \rightarrow$	4	1	0	0.8	0.4	0.1	0
	11	0	0	10	1	-0.5	0

Since all $Z_j - C_j \geq 0$,

$\therefore$ The current solution is optimal

$x_1 = 4, \; x_2 = 5, \; x_3 = 0$

Min $Z = -11$ (or) Max $Z^* = 11$

Using simplex method solve the LPP

$$\text{Max } z = 5x_1 + 7x_2.$$

Subject to constraints,

$$x_1 + x_2 \leq 4$$
$$3x_1 + 8x_2 \leq 24$$
$$10x_1 + 7x_2 \leq 25$$
$$x_1, x_2 \geq 0$$

Introducing slack variable S_1, S_2, S_3

The standard form of LPP is :

$$\text{Max } z = 5x_1 + 7x_2 + 0S_1 + 0S_2 + 0S_3$$

Subject to constraints,

$$x_1 + x_2 + S_1 + 0S_2 + 0S_3 = 4$$
$$3x_1 + 8x_2 + 0S_1 + S_2 + 0S_3 = 24$$
$$10x_1 + 7x_2 + 0S_1 + 0S_2 + S_3 = 25$$
$$x_1, x_2, S_1, S_2, S_3 \geq 0$$

Initial table :-

C		C_j	5	7	0	0	0	Ratio
C_B	S_B	X_B	x_1	x_2	S_1	S_2	S_3	
0	S_1	4	1	1	1	0	0	4
0	S_2	24	3	8	0	1	0	3 ←
0	S_3	25	10	7	0	0	1	3.5
Z_j		0	0	0	0	0	0	
$Z_j - C_j$			-5	-7 ↑	0	0	0	

Iteration table : 01

Introducing variable is x_2
leaving variable is S_2.

C		C_j	5	7	0	0	0	Ratio
C_B	S_B	X_B	x_1	x_2	S_1	S_2	S_3	
0	S_1	1	5/8	0	1	-1/8	0	8/5 = 1.6
7	x_2	3	3/8	1	0	1/8	0	8
0	S_3	4	59/8	0	0	-7/8	1	0.54 ←
Z_j		21	21/8	7	0	7/8	0	
$Z_j - C_j$			-19/8 ↑	0	0	7/8	0	

$S_1 \Rightarrow$ 4 1 1 1 0 0

$x_2 \Rightarrow$ 3 3/8 1 0 1/8 0

(−)

1 5/8 0 1 -1/8 0

$S_3 \Rightarrow$ 25 10 7 0 0 1

$7x_2 \Rightarrow 7\,[$ 3 3/8 1 0 1/8 0 $]$

(−)

4 59/8 0 0 -7/8 1

Iteration table : 02

Introducing Variable is x_1
leaving Variable is S_3

C		C_j	5	7	0	0	0	Ratio
C_B	S_B	X_B	x_1	x_2	S_1	S_2	S_3	
0	S_1	0.661	0	0	1	-0.051	-0.085	
7	x_2	2.797	0	1	0	0.1696	-0.051	
5	x_1	0.5424	1	0	0	-0.119	0.136	
Z_j		22.291	5	7	0	0.5922	0.323	
$Z_j - C_j$			0	0	0	0.59	0.323	

$x_2 \Rightarrow$ 3 $\quad 3/8 \quad 1 \quad 0 \quad 1/8 \quad 0$

$3/8\, x_1 \Rightarrow \frac{3}{8}\left[\frac{32}{59} \quad 1 \quad 0 \quad 0 \quad \frac{-7}{59} \quad \frac{8}{59}\right]$

(−)

2.797 $\quad 0 \quad 1 \quad 0 \quad 0.1696 \quad -0.051$

$S1 - 0.625\, x_1 = 0.661 \quad 0 \quad 0 \quad 1 \quad -0.051 \quad -0.085$

Since the all $z_j - c_j \geq 0$

$\therefore$ The current optimal Solution is :

$x_1 = 2.797, \quad x_2 = 0.5424$

Max $z = 22.291$

Two Phase Method

This is an alternative of Big-M method. Using this method, we obtain the solution in two phases given as follows:

In phase : 01

All the artificial variables are eliminated from the basis.

In phase : 02

We use the solution from phase-I as the initial basic feasible solution and then use the simplex method to obtain the optimal solution.

(A) For phase - 01

Step: 01

Convert the given LPP in the standard form.

Step: 02

Add the neccessary artificial variables to the constraints as done in Big-M method to obtain an initial basic feasible solution.

Step: 03

Formulate an artificial objective function z^* such that,

$z^* = -A_1 - A_2 \cdots - A_n$ (= -(sum of the artificial variables)

by assigning -1 cost to each artificial variable A_i and zero cost to all other variables.

Step: 04

Maximize z^* subject to the constraints of the original problem using the simplex method. Now, we have the following cases.

Case : I $\Rightarrow$ If $\max z^* < 0$ and at least one artificial variable appears in the optimal basis at a positive level, then given LPP will not have any feasible solution and then we will not move to phase - II.

Case : II $\Rightarrow$ If $\max z^* = 0$ and no artificial variables appears in the optimal basis then BFS is not obtained and in order to obtain optimal BFS, we move to phase - II.

Case : III $\Rightarrow$ If $\max z^* = 0$ and at least one artificial variable appears in the optimal basis at zero level, then a feasible solution of the auxilliary LPP is also a feasible solution of the given LPP by setting all artificial variables to zero. Finally to obtain the basis feasible solution, remove all the artificial variable from the basis matrix.

(B) For phase - 02

Step : 01

Take the basic feasible solution, which are found at the end of phase - 1 as the (?) BFS for the given LPP.

Step : 02

Apply simplex method to find the optimal basic feasible solution.

Problem : 01

Use two - phase simplex method to solve

Maximize $z = 5x_1 + 3x_2$

Subject to constraint,

$2x_1 + x_2 \le 1$

$x_1 + 4x_2 \ge 6$

$x_1, x_2 \ge 0$

we convert the given problem into a standard form by adding slack, surplus and artificial variables, we form the auxillary LPP by assigning the cost -1 to the artificial variable and 0 to the other variables.

Phase - I :

$$\text{Max } z^* = 0x_1 + 0x_2 + 0s_1 + 0s_2 - 1A_1$$

Subject to constraints,

$$2x_1 + x_2 + s_1 = 1$$

$$x_1 + 4x_2 - s_2 + A_1 = 6$$

$$x_1, x_2, s_1, s_2, A_1 \geq 0$$

Initial table :-

C		C_j	0	0	0	0	-1	Ratio
C_B	S_B	X_B	x_1	x_2	S_1	S_2	A_1	
0	S_1	1	2	1	1	0	0	1 ←
-1	A_1	6	1	4	0	-1	1	1.5
Z_j		-6	-1	-4	0	1	-1	
$Z_j - C_j$			-1	-4 ↑	0	1	0	

Iteration table : 01

Entering variable = x_2

leaving variable = S_1

Pivot element = 1

C		C_j	0	0	0	0	-1
C_B	S_B	X_B	x_1	x_2	S_1	S_2	A_1
0	x_2	1	2	1	1	0	0
-1	A_1	2	-7	0	-4	-1	1
Z_j		-2	7	0	4	1	-1
$Z_j - C_j$			7	0	4	1	0

$A_1 \Rightarrow$	6	1	4	0	-1	1
$4 \times 2 \Rightarrow$	4	8	4	4	0	0
(−)						
	2	-7	0	-4	-1	1

Since all $z_j - c_j \geq 0$, an optimum feasible solution to the auxillary LPP is obtained, But as Max $z^* < 0$ and an artificial variable A_1 is in the basis at positive level the original LPP does not possess any feasible solution.

Problem : 02

$$\text{Max } z = 3x_1 - x_2.$$

Subject to constraints,

$$2x_1 + x_2 \geq 2$$
$$x_1 + 3x_2 \leq 2$$
$$x_2 \leq 4$$
$$x_1, x_2 \geq 0$$

we convert the given problem into a standard form by adding slack, surplus and artificial variables. we form the auxillary LPP by assigning the cost -1 to the artificial variable and 0 to all the other variables.

Phase - I :

$$\text{Max } z^* = 0x_1 + 0x_2 + 0S_1 + 0S_2 + 0S_3 - 1A_1.$$

Subject to constraints,

$$2x_1 + x_2 - S_1 + A_1 = 2$$
$$x_1 + 3x_2 + S_2 = 2$$
$$0x_1 + x_2 + S_3 = 4$$
$$x_1, x_2, S_1, S_2, S_3, A_1 \geq 0$$

Initial table :-

C		C_j	0	0	0	0	0	-1	Ratio
C_B	S_B	X_B	x_1	x_2	S_1	S_2	S_3	A_1	
-1	A_1	2	2	1	-1	0	0	1	1 ←
0	S_2	2	1	3	0	1	0	0	2
0	S_3	4	0	1	0	0	1	0	0
Z_j		-2	-2	-1	1	0	0	-1	
$Z_j - C_j$			-2 ↑	-1	1	0	0	0	

Iteration table :01

Introducing : x_1 , leaving : A_1

C		C_j	0	0	0	0	0
C_B	S_B	X_B	x_1	x_2	S_1	S_2	S_3
0	x_1	1	1	0.5	-0.5	0	0
0	S_2	1	0	2.5	0.5	1	0
0	S_3	4	0	1	0	0	1
Z_j		0	0	0	0	0	0
$Z_j - C_j$			0	0	0	0	0

$S_2 \Rightarrow$ 2 1 3 0 1 0 0

$x_1 \Rightarrow$ 1 1 0.5 -0.5 0 0 0.5

(−)

1 0 2.5 0.5 1 0 -0.5

$\therefore$ Max $z^* = 0$

Phase : II

Iteration table : 01

C		C_j	3	-1	0	0	0	Ratio
C_B	S_B	X_B	x_1	x_2	S_1	S_2	S_3	
3	x_1	1	1	1/2	-1/2	0	0	-
0	S_2	1	0	5/2	1/2	1	0	2 ←
0	S_3	4	0	1	0	0	1	-
Z_j		3	3	3/2	-3/2	0	0	
$Z_j - C_j$			0	1/2	-3/2 ↑	0	0	

Iteration table : 02

Entering variable is S_1

leaving variable is S_2.

Pivot element = 1/2

C		C_j	3	-1	0	0	0
C_B	S_B	X_B	x_1	x_2	S_1	S_2	S_3
3	x_1	2	1	3	0	1	0
0	S_1	2	0	5	1	2	0
0	S_3	4	0	1	0	0	1
Z_j		6	3	9	0	3	0
$Z_j - C_j$			0	10	0	3	0

x_1 ⇒	1	1	1/2	-1/2	0	0
1/2 S_1 ⇒ (+)	1	0	5/2	1/2	1	0
	2	1	3	0	1	0

Since all the $Z_j - C_j \geq 0$

∴ The solution is optimum

$x_1 = 2, \; x_2 = 0$

Max $Z = 6$

Problem : 03

Use two phase simplex method solve the LPP.

Minimize $z = 12x_1 + 18x_2 + 15x_3$

Subject to Constraints,

$$4x_1 + 8x_2 + 6x_3 \geq 64$$

$$3x_1 + 6x_2 + 12x_3 \geq 96$$

$$x_1, x_2, x_3 \geq 0$$

Phase - I :

$$\text{Max } z^* = 0x_1 + 0x_2 + 0x_3 + 0S_1 + 0S_2 - A_1 - A_2$$

Subject to Constraints,

$$4x_1 + 8x_2 + 6x_3 - S_1 + A_1 = 64$$

$$3x_1 + 6x_2 + 12x_3 - S_2 + A_2 = 96$$

$$x_1, x_2, x_3, S_1, S_2, A_1, A_2 \geq 0$$

Initial table :-

C		Cj	0	0	0	0	0	-1	-1	
C_B	S_B	X_B	x_1	x_2	x_3	S_1	S_2	A_1	A_2	Ratio
-1	A_1	64	4	8	6	-1	0	1	0	10.66..
-1	A_2	96	3	6	12	0	-1	0	1	8 ←
Z_j		-160	-7	-14	-18	1	1	-1	-1	
$Z_j - C_j$			-7	-14	-18 ↑	1	1	0	0	

Iteration table : 01

Entering variable : x_3

leaving variable : S_2.

C		Cj	0	0	0	0	0	-1	
C_B	S_B	X_B	x_1	x_2	x_3	S_1	S_2	A_1	Ratio
-1	A_1	16	5/2	5	0	-1	1/2	0	3.2 ←
0	x_3	8	1/4	1/2	1	0	-1/12	0	16
Z_j		-16	-5/2	-5	0	1	-1/2	0	
$Z_j - C_j$			-5/2	-5 ↑	0	1	-1/2	1	

$A_1 \Rightarrow$	64	4	8	6	-1	0	1
$6 \times x_3 \Rightarrow$ (−)	48	3/2	3	6	0	-1/2	0
	16	5/2	5	0	-1	1/2	0

Iteration table : 02.

Entering variable : x_2

leaving variable : A_1

C		C_j	0	0	0	0	0
C_B	S_B	X_B	x_1	x_2	x_3	S_1	S_2.
0	x_2	3.2	1/2	1	0	-1/5	1/10
0	x_3	6.4	0	0	1	1/4	-1/12
Z_j		0	0	0	0	0	0
$Z_j - C_j$			0	0	0	0	0

$x_3 \Rightarrow$	8	1/4	1/2	1	0	-1/12
$1/2\, x_2 \Rightarrow$ (−)	$\frac{3.2}{2}$	1/4	1/2	0	$-1/5 \times 2$	$1/10 \times 2$
	6.4	0	0	1	1/10	-2/15

$$\text{Max } z^* = 0$$

Phase - II :

$$\text{Max } z^* = -12x_1 - 18x_2 - 15x_3 + 0S_1 + 0S_2.$$

Iteration table : 01

C		C_j	-12	-18	-15	0	0
C_B	S_B	X_B	x_1	x_2	x_3	S_1	S_2.
-18	x_2	3.2	1/2	1	0	-1/5	1/10
-15	x_3	6.4	0	0	1	1/10	-2/15
Z_j		-153.6	-9	-18	-15	21/10	1/5
$Z_j - C_j$			3	0	0	21/10	1/5

Since, All $z_j - c_j \geq 0$

$\therefore$ The solution is optimum.

$x_2 = 3.2$; $x_3 = 6.4$

Max $z^* = -153.6$ (or) Min $z = 153.6$

DUALITY

A generalised format of the linear programming Problem is represented here.

Maximize (or) minimize $z = c_1 x_1 + c_2 x_2 + \dots + c_n x_n$

Subject to constraints,

$$a_{11} x_1 + a_{12} x_2 + \dots + a_{1n} x_n \leq, = (\text{or}) \geq b_1$$
$$a_{21} x_1 + a_{22} x_2 + \dots + a_{2n} x_n \leq, = (\text{or}) \geq b_2$$
$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$
$$a_{i1} x_1 + a_{i2} x_2 + \dots + a_{in} x_n \leq, = (\text{or}) \geq b_i$$
$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$
$$a_{m1} x_1 + a_{m2} x_2 + \dots + a_{mn} x_n \leq, = (\text{or}) \geq b_m$$

where, $x_1, x_2, x_3 \dots x_n \geq 0$.

Let this problem be called as a primal linear programming Problem. If the constraints in the primal problem are too many then the time taken to solve the problem is expected to be higher. Under such situation, the primal linear programming Problem can be converted into its dual linear programming Problem which requires relatively lesser time to solve. Then the Solution of the primal problem can be obtained from the optimal table of its deal problem by following certain rules.

Formulation of Dual problem :-

The primal problem is again reproduced below :-

Maximize or minimize $z = c_1 x_1 + c_2 x_2 + \dots + c_n x_n$

Subject to constraints,

$$a_{11} x_1 + a_{12} x_2 + \dots + a_{1n} x_n \leq b_1 \leftarrow y_1$$
$$a_{21} x_1 + a_{22} x_2 + \dots + a_{2n} x_n \leq b_2 \leftarrow y_2$$
$$\vdots$$
$$a_{i1} x_1 + a_{i2} x_2 + \dots + a_{in} x_n \leq b_i \leftarrow y_i$$
$$\vdots$$
$$a_{m1} x_1 + a_{m2} x_2 + \dots + a_{mn} x_n \leq b_m \leftarrow y_m$$

where,

$$x_1, x_2, x_3 \dots x_n \geq 0.$$

In the above model, the variable y_i is called as the dual variable associated with the constraint i.

Objective function :-

The number of variables in the dual problem is equal to the number of constraints in the primal problem. The objective function of the dual problem is constructed by adding the multiplies of the right-hand side ~~constraints~~ constants of the constraints of the primal problem with the respective dual variables.

Constraints :-

The number of constraints in the dual problem is equal to the number of variables in the primal problem. Each dual constraint corresponding to each primal variable. The left-hand side of the dual constraint corresponding to the j^{th} primal variable is the sum of the multiple of the left-hand side constraint coefficients of the j^{th} primal variable with the corresponding dual variables. The right hand side constant of the dual constraint corresponding to the j^{th} primal variable is the objective function coefficient of the j^{th} primal variable.

Some more guideline for forming the dual problem are Presented.

Guidelines for Dual Formation.

Type of Problem	objective function	Constraints type	Nature of Variables.
Primal Dual	Maximize minimize	$\leq$ $\geq$	Restricted in sign Restricted in sign
Primal Dual	minimize maximize	$\geq$ $\leq$	Restricted in Sign. Restricted in sign
Primal Dual	maximize minimize	$\nleq$ $\geq$	Restricted in Sign unrestricted in sign
Primal Dual	minimize maximize	$\ngeq$ $\leq$	Restricted in sign unrestricted in sign
Primal Dual	maximize minimize	$\leq$ $\ngeq$	Unrestricted in sign restricted in sign.
Primal Dual	minimize maximize	$\geq$ $\nleq$	Unrestricted in sign restricted in sign.

Steps for a standard primal Form :-

Step : 01

change the objective function to maximization form.

Step : 02

If the Constraints have an inequality sign " $\geq$ " then multiply both sides by -1 and convert the inequality sign to " $\leq$ ".

Step : 03

If the Constraint has an '=' sign then replace it by two Constraints ? the inequalities going in opposite directions.

For eg :- $x_1 + 2x_2 = 4$ is written as,

$$x_1 + 2x_2 \le 4$$

$$x_1 + 2x_2 \ge 4 \text{ (using step 2)}$$

$$\longrightarrow -x_1 - 2x_2 \le -4$$

Step : 04

Every unrestricted variable is replaced by the difference of two non-negative variables.

Step : 05

we get the standard primal form of the given LPP in which,

* All constraints have '$\le$' sign, where the objective function is of maximization form.
* All constraints have '$\ge$' sign, where the objective function is of minimization form.

Dual Simplex Algorithm :-

The ? procedure for dual simplex method is listed below.

Step : 01

Convert the problem to maximization form, if it is initially in the minimization form.

Step : 02

Convert '$\ge$' type constraints if any to '$\le$' type, by multiplying both sides by -1.

Step : 03

Express the problem in standard form by introducing slack variables. obtain the initial basic solution. display this solution in the simplex table.

Step : 04

Test the nature of $z_j - c_j$ (optimal condition).

Case - I => If all $z_j - c_j \geq 0$ and all $x_{Bi} \geq 0$, then the current solution is an optimum feasible solution.

Case - II => If all $z_j - c_j \geq 0$ and atleast one $x_{Bi} < 0$, then the current solution is not an optimum basic feasible solution. In this case go to the next step.

Case - III => If any $z_j - c_j < 0$, then the method fails.

Step : 05

In this step, we find the learning variable, which is the basic variable corresponding to the most negative value of x_{Bi}. Let x_k be the learning variable, (i.e)., $x_{Bk} = \min\{x_{Bi}, x_{Bi} < 0\}$. To find out the variable entering the basis, we compute the ratio between $z_j - c_j$ row and the key row (i.e)., Compute $\max\{z_j - c_j / c_{ik}, a_{ik} < 0\}$. (Consider the ratios with negative Dr alone). The entering variable is the one having the maximum ratio. If there is no such ratio with negative Dr, then the Problem does not have a feasible solution.

Step : 06

Convert the leading element to unity and all the other elements of key column to zero, to get an improved solution.

Step : 07

Repeat step (4) and (5) until either an optimum basic feasible solution is attained or an indication of no feasible solution is obtained.

(1) write the dual of the primal problem given below :-

$$\text{Max } z = 3x_1 + x_2 + 4x_3 + x_4 + 9x_5$$

Subject to constraints;

$$4x_1 - 5x_2 - 9x_3 + x_4 - 2x_5 \leq 6$$
$$2x_1 + 3x_2 + 4x_3 - 5x_4 + x_5 \leq 9$$
$$x_1 + x_2 - 5x_3 - 7x_4 + 11x_5 \leq 10$$

this is the primal problem.

The dual of the given primal is,

$$\text{Min } z = 6x_1 + 9x_2 + 10x_3$$

Subject to Constraint,

$$4x_1 + 2x_2 + x_3 \geq 3$$
$$-5x_1 + 3x_2 + x_3 \geq 1$$
$$-9x_1 + 4x_2 - 5x_3 \geq 4$$
$$x_1 - 5x_2 - 7x_3 \geq 1$$
$$-2x_1 + x_2 + 11x_3 \geq 9$$
$$x_1, x_2, x_3 \geq 0$$

(2) write the dual of primal problem given below.

$$\text{Min } z = 3x_1 + 3x_2 + 8x_3$$

Subject to Constraints,

$$8x_1 + 2x_2 + x_3 \geq 3$$
$$3x_1 + 6x_2 + 4x_3 \geq 4$$
$$4x_1 + x_2 + 5x_3 \geq 1$$
$$x_1 + 5x_2 + 2x_3 \geq 7$$
$$x_1, x_2, x_3 \geq 0$$

The dual of the given primal is

$$\text{Max } z = 3x_1 + 4x_2 + x_3 + 7x_4.$$

Subject to Constraint,

$$8x_1 + 3x_2 + 4x_3 + x_4 \leq 3$$
$$2x_1 + 6x_2 + x_3 + 5x_4 \leq 3$$
$$x_1 + 4x_2 + 5x_3 + 2x_4 \leq 8$$
$$x_1, x_2, x_3 \leq 0$$

3) write the dual of the given problem.

$$\text{Min } z = 3x_1 - 2x_2 + 4x_3$$

Subject to constraint,

$$3x_1 + 5x_2 + 4x_3 \geq 7$$
$$6x_1 + x_2 + 3x_3 \geq 4$$
$$7x_1 - 2x_2 - x_3 \leq 10$$
$$x_1 - 2x_2 + 5x_3 \geq 3$$
$$4x_1 + 7x_2 - 2x_3 \geq 2.$$
$$x_1, x_2, x_3 \geq 0$$

The given problem can be written as,

$$\text{Min } z = 3x_1 - 2x_2 + 4x_3$$

Subject to constraint,

$$3x_1 + 5x_2 + 4x_3 \geq 7$$
$$6x_1 + x_2 + 3x_3 \geq 4$$
$$-7x_1 + 2x_2 + x_3 \geq -10$$
$$x_1 - 2x_2 + 5x_3 \geq 3$$
$$4x_1 + 7x_2 - 2x_3 \geq 2$$

$$\text{Max } z = 7x_1 + 4x_2 - 10x_3 + 3x_4 + 2x_5$$

Subject to

$$3x_1 + 6x_2 - 7x_3 + x_4 + 4x_5 \leq 3$$
$$5x_1 + x_2 + 2x_3 - 2x_4 + 7x_5 \leq -2$$
$$4x_1 + 3x_2 + x_3 - 5x_4 - 2x_5 \leq 4$$
$$x_1, x_2, x_3 \geq 0$$

(4) write the dual of the primal problem given below:.

$$\text{Max } z = 2x_1 + 3x_2 + x_3$$

Subject to

$$4x_1 + 3x_2 + x_3 = 6$$
$$x_1 + 2x_2 + 5x_3 = 4$$
$$x_1, x_2, x_3 \geq 0$$

The given problem can be written as

$$\text{Max} = 2x_1 + 3x_2 + x_3$$

Subject to

$$4x_1 + 3x_2 + x_3 \le 6$$
$$4x_1 + 3x_2 + x_3 \ge 6$$
$$x_1 + 2x_2 + 5x_3 \le 4$$
$$x_1 + 2x_2 + 5x_3 \ge 4$$
$$x_1, x_2, x_3 \ge 0$$

Again the given problem can be written as

$$\text{Max } z = 2x_1 + 3x_2 + x_3$$

Subject to

$$4x_1 + 3x_2 + x_3 \le 6$$
$$-4x_1 - 3x_2 - x_3 \le -6$$
$$x_1 + 2x_2 + 5x_3 \le 4$$
$$-x_1 - 2x_2 - 5x_3 \le -4$$

The dual of the given primal is:

$$\text{Min } z = 6x_1 - 6x_2 + 4x_3 - 4x_4$$

Subject to,

$$4x_1 - 4x_2 + x_3 - x_4 \ge 2$$
$$3x_1 - 3x_2 + 2x_3 - 2x_4 \ge 3$$
$$x_1 - x_2 + 5x_3 - 5x_4 \ge 1$$

x_1, x_2 are unrestricted.

DUAL SIMPLEX METHOD

) Use dual simplex method to solve the following LPP.

$$\text{Max } z = -3x_1 - x_2$$

Subject to

$$x_1 + x_2 \ge 1$$
$$2x_1 + 3x_2 \ge 2$$
$$x_1, x_2 \ge 0.$$

$$\text{Max } z = -3x_1 - x_2$$

Subject to

$$-x_1 - x_2 \le -1$$

$$-2x_1 - 3x_2 \le -2$$

$$x_1, x_2 \ge 0.$$

$$\text{Max } z = -3x_1 - x_2 + 0s_1 + 0s_2.$$

Subject to

$$-x_1 - x_2 + s_1 = -1$$

$$-2x_1 - 3x_2 + s_2 = -2$$

$$x_1, x_2, s_1, s_2 \ge 0.$$

~~Iteration~~ Initial table :-

C		C_j	-3	-1	0	0
C_B	S_B	X_B	x_1	x_2	s_1	s_2
0	s_1	-1	-1	-1	1	0
0	s_2	-2	-2	-3	0	1
	z_j	0	0	0	0	0
$z_j - c_j$		3	3	1	0	0
Ratio		~~$\lvert 3/-2 \rvert$ = 1.5~~	$\lvert 3/-2 \rvert$ = 1.5	$\lvert 1/-3 \rvert$ = 0.3	-	-

Iteration table : 01

leaving variable = s_2

introducing variable = x_2

key element = -3

$s_1 \Rightarrow$	-1	-1	-1	1	0
$x_2 \Rightarrow$	$+2/3$	$+2/3$	$+1$	0	$-1/3$
	$-1/3$	$-1/3$	0	1	$+1/3$

Iteration table : 01

C		C_j	-3	-1	0	0
C_B	S_B	X_B	x_1	x_2	s_1	s_2
0	s_1	$-1/3$	$-1/3$	0	1	$-1/3$
-1	x_2	$+2/3$	$+2/3$	+1	0	$+1/3$
z_j		$-2/3$	$-2/3$	-1	0	$+1/3$
$z_j - c_j$			$7/3$	0	0	$+1/3$
Ratio			$\|-7\| = 7$	0	0	$\|-1\| = 1$

$$\text{Ratio} = \frac{z_j - c_j}{s_1}$$

Iteration table : 02

Introducing variable = s_2

leaving variable = s_1

Key element = $-1/3$

C		C_j	-3	-1	0	0
C_B	S_B	X_B	x_1	x_2	s_1	s_2.
0	s_2	1	1	0	-3	1
-1	x_2	1	1	1	-1	0
z_j		-1	-1	-1	1	0
$z_j - c_j$			2	0	1	0
Ratio			-	-	-	-

$s_2 \times 1/3 \Rightarrow$ $1/3$ $1/3$ 0 -1 $1/3$

$x_2 \Rightarrow$ $2/3$ $2/3$ +1 0 $-1/3$

1 1 +1 -1 0

Since all $z_j - c_j \geq 0$ and $x_{B_i} \geq 0$, then the current solution is an optimum feasible solution

$\therefore$ $x_1 = 0$, $x_2 = 1$, $s_1 = 0$, $s_2 = 1$

$\therefore$ Max $z = -1$

Home work :

Dual Simplex method to solve the following LPP :

$$\text{Min } z = 3x_1 + 2x_2 + x_3$$

Subject to

$$3x_1 + x_2 + x_3 \geq 3$$

$$-3x_1 + 3x_2 + x_3 \geq 6$$

$$x_1 + x_2 + x_3 \leq 3$$

$$x_1, x_2, x_3 \geq 0.$$

By introducing slack Variable S_1, S_2, S_3

$$\text{Max } z = -3x_1 - 2x_2 - x_3 + 0S_1 + 0S_2 + 0S_3$$

Subject to

$$-3x_1 - x_2 - x_3 + S_1 = -3$$

$$3x_1 - 3x_2 - x_3 + S_2 = -6$$

$$x_1 + x_2 + x_3 + S_3 = 3$$

Initial Table :-

C		C_j	-3	-2	-1	0	0	0
C_B	S_B	X_B	x_1	x_2	x_3	S_1	S_2	S_3
0	S_1	-3	-3	-1	-1	1	0	0
0	S_2	-6	3	-3	-1	0	1	0
0	S_3	3	1	1	1	0	0	1
Z_j		0	0	0	0	0	0	0
$Z_j - C_j$			3	2	1	0	0	0
Ratio			$3/3 = 1$	$\lvert 2/-3 \rvert = 0.66$	$\lvert -1/-1 \rvert = 1$	0	0	0

First Iteration :-

C.		C_j	-3	-2	-1	0	0	0	
C_B	S_B	X_B	x_1	x_2	x_3	S_1	S_2	S_3	Ratio
0	S_1	-1	-4	0	-0.67	1	-0.33	0	$\lvert 5/-4 \rvert = 1.25$
-2	x_2	2	-1	1	0.33	0	-0.33	0	$\left\lvert \frac{0.34}{-0.67} \right\rvert = 0.51$
0	S_3	1	2	0	0.67	0	0.33	1	$\left\lvert \frac{0.66}{-0.33} \right\rvert = 2$
Z_j		-4	2	-2	-0.66	0	0.66	0	
$Z_j - C_j$			5	0	0.34	0	0.66	0	

$S_2/_{-3} = x_2 = 2 \quad -1 \quad 1 \quad 0.33 \quad 0 \quad -0.33 \quad 0$

$S_1 + x_2 = -1 \quad -4 \quad 0 \quad -0.67 \quad 1 \quad -0.33 \quad 0$

$S_3 - x_2 = 1 \quad 2 \quad 0 \quad 0.67 \quad 0 \quad 0.33 \quad 1$

Introducing : x_2, leaving : S_2.

Iteration : 02

Introducing : x_3, leaving : S_1

C		C_j	-3	-2	-1	0	0	0
C_B	S_B	X_B	x_1	x_2	x_3	S_1	S_2	S_3.
-1	x_3	1.5	5.97	0	1	-1.5	0.5	0
-2	x_2	1.51	-2.97	1	0	0.5	-0.5	0
0	S_3	0.005	-1.999	0	0	1.005	0.005	1
Z_j		-4.52	-0.03	-2	-1	0.5	0.5	0
$Z_j - C_j$			2.97	0	0	0.5	0.5	0

$\frac{S_1}{-0.67} = x_3 = 1.5 \quad 5.97 \quad 0 \quad 1 \quad -1.5 \quad 0.5 \quad 0$

$x_2 - 0.33x_3 = 1.51 \quad -2.97 \quad 1 \quad 0 \quad 0.5 \quad -0.5 \quad 0$

$S_3 - 0.67x_3 = 0.005 \quad -1.999 \quad 0 \quad 0 \quad 1.005 \quad 0.005 \quad 1$

Since all $Z_j - C_j \geq 0$ and $X_{B_i} \geq 0$ then the current solution is an optimum feasible solution.

$x_1 = 0, \quad x_2 = 1.51, \quad x_3 = 1.5$

Max $z = -4.52$.

Transportation problem and Assignment problem

Transportation problem is a special kind of linear Programming problem (LPP) in which goods are transported from a set of sources to a set of destinations subject to the supply and demand of the sources and destination respectively. Such that the total cost of transportation is minized. It is also sometimes called as Hitchcock problem.

Types of Transportation problem :-

Balanced :-

when both supplies and demands are equal then the Problem is said to be a balanced transportation problem.

Unbalanced :-

when the supply and demand are not equal then it is said to be an unbalanced transportation problem. In this type of Problem, either a dummy row or a dummy column is added according to the requirement to make it a balanced problem. Then it can be solved similar to the balanced problem.

Methods to solve :-

To find the initial basic feasible solution there are three methods.

Nortwest Corner Cell method.
least Cost Cell method (or) matrix minimum method
Vogel's Approximation method (VAM)

Basic structure of transportation problem :-

Destination

		D_1	D_2	D_3	D_4	Supply (s_i)
Source	O_1	C_{11}	C_{12}	C_{13}	C_{14}	S_1
	O_2	C_{21}	C_{22}	C_{23}	C_{24}	S_2
	O_3	C_{31}	C_{32}	C_{33}	C_{34}	S_3
	O_4	C_{41}	C_{42}	C_{43}	C_{44}	S_4
Demand (d_i):		d_1	d_2	d_3	d_4	

In the above table D_1, D_2, D_3 and D_4 are the destinations where the products / goods are to be delivered from different Sources S_1, S_2, S_3 and S_4. S_i is the Supply from the Source O_i, d_j is the demand of the destination D_j, C_{ij} is the cost when the product is delivered from Source S_i to destination D_j.

Definitions :-

Transportation problem :-

The objective of transportation problem is to determine the amount to be transported from each origin to each destinations such that the total transportation cost is minimized.

Feasible Solution :-

A feasible solution to a transportation problem is a set of non-negative values

x_{ij} ($i = 1, 2 \ldots, m$, $j = 1, 2 \ldots n$)

that satisfies the Constraints.

Basic Feasible Solution :-

A feasible solution is called a basic feasible solution if it contains not more than $m+n-1$ allocations, where m is the number of rows and n is the no. of columns in a transportation problem.

Optimal Solution :-

optimal Solution is a feasible Solution (not necessarily basic) which optimizes (minimize) the total transportation cost.

Difference between Transportation and Assignment problems :-

Transportation problem	Assignment problem.
Number of Sources and destinations need not be equal. Hence the cost matrix is not necessarily a square matrix.	Since assignment is done on a one to one basis, the number of Sources and destinations are equal. Hence, the Cost matrix must be a square matrix.
x_{ij}, the quantity to be transported from i^{th} Origin to j^{th} destination can take any possible positive value, and it Satisfies the rim requirements.	x_{ij}, the j^{th} job is to be assignment to the i^{th} person and can take either the value 1 or zero.
The capacity and the requirement value is equal to a_i and b_j for the i^{th} Source and j^{th} destination. $(i = 1, 2, \ldots m ; j = 1, 2 \ldots, m)$	The Capacity and the requirement value is exactly one (i.e)., for each Source of each destination, the Capacity and the requirement value is exactly one.
The problem is unbalanced if the total Supply and total demand are not equal.	The problem is unbalanced if the cost matrix is not a Square matrix.

North-west Corner :-

In this method, we apply the following steps :-

Step : 01

Start with the cell (1, 1) at the upper left (north-west) corner of the matrix and allocate it as much as possible amount equal to the minimum of the supply - demand values, (i.e)., We allocate x_{11} to the cell (1, 1) where

$$x_{11} = \min \{a, b\}$$

where a_1 is the supply amount for the first row and b_1 is the demand for the first column.

Step : 02

(i) If $a_1 > b_1$, then move to the cell (1, 2) and allocate x_{12} where $x_{12} = \min \{a_1 - x_{11}, b_2\}$.

(ii) If $a_1 < b_1$, then move to the call (2, 11) and allocate it as x_{21} where $x_{21} = \min (a_2 - b_1 - x_{11})$.

Step : 03

Continue this process step by step till an allocation is made in the south east corner of the cell (i.e)., until all available amount is exhausted.

108 TRANSPORTATION PROBLEM

North West Corner Method :.

Problem : 01

Solve the transportation problem using NWCR method.

				Supply
Source.	2	7	5	200
	3	4	2	300
	5	4	7	500
Demand	200	400	400	

Step :01

Check whether Demand and Supply is equal.

				Supply
Source	2	7	5	200
	3	4	2	300
	5	4	7	500
Demand	200	400	400	1000

Step:02 :-

				Supply
Source	$\boxed{200}$ 2	7	5	~~200~~ 0
	3	$\boxed{300}$ 4	2	~~300~~ 0
	5	$\boxed{100}$ 4	$\boxed{400}$ 7	~~500~~ ~~400~~ 0
Demand	~~200~~ 0	~~400~~ 100	400 0	

Step:03

$$\text{Cost} = (2\times200) + (4\times300) + (4\times100) + (7\times400)$$

$$= 400 + 1200 + 400 + 2800$$

$$= ₹4800.$$

Homework :-

	D_1	D_2	D_3	D_4	Supply
S_1	6	1	9	3	70
S_2	11	5	2	8	55
S_3	10	12	4	7	70
Demand	85	35	50	45	

Step :01

Check whether Demand and Supply is equal

	D_1	D_2	D_3	D_4	Supply
S_1	6	1	9	3	70
S_2	11	5	2	8	55
S_3	10	12	4	7	70
S_4	0	0	0	0	20
Demand	85	35	50	45	

Step :02

	D_1	D_2	D_3	D_4	Supply
S_1	$\boxed{70}$ 6	1	9	3	0 ~~70~~
S_2	$\boxed{15}$ 11	$\boxed{35}$ 5	$\boxed{5}$ 2	8	0 ~~5~~ ~~40~~ ~~55~~
S_3	10	12	$\boxed{45}$ 4	$\boxed{25}$ 7	0 ~~25~~ ~~70~~
S_4	0	0	0	$\boxed{20}$ 0	0 ~~20~~
Demand	0 ~~15~~ ~~85~~	0 ~~35~~	0 ~~45~~ ~~50~~	0 ~~25~~ ~~45~~	

Step : 03

$$\text{Cost} = (6\times 70) + (11\times 15) + (5\times 35) + (2\times 5) + (4\times 45) + (7\times 25) + (0\times 20)$$

$$= 420 + 165 + 175 + 10 + 180 + 175 + 0$$

$$= ₹1125.$$

Problem :- 02.

	D_1	D_2	D_3	Supply
S_1	2	7	4	5
S_2	3	3	1	8
Source S_3	5	4	7	7
S_4	1	6	2	14
Demand	4	9	18	

Step : 01

Check whether Demand and Supply is equal

	D_1	D_2	D_3	D_4	Supply
S_1	2	7	4	0	5
S_2	3	3	1	0	8
Source S_3	5	4	7	0	7
S_4	1	6	2	0	14
Demand	4	9	18	3	34

Step : 02

	D_1	D_2	D_3	Supply
S_1	$\boxed{5}$ 2	7	4	~~5~~ 0
S_2	$\boxed{2}$ 3	$\boxed{6}$ 3	1	~~8~~ ~~6~~ 0
S_3	5	$\boxed{3}$ 4	$\boxed{4}$ 7	~~7~~ ~~4~~ 0
S_4	1	6	$\boxed{14}$ 2	~~14~~ 0
Demand	~~7~~ ~~2~~ 0	~~9~~ ~~3~~ 0	~~18~~ ~~14~~ 0	

Step : 03

$$\text{Cost} = (5 \times 2) + (2 \times 3) + (6 \times 3) + (3 \times 4) + (4 \times 7) + (14 \times 2)$$

$$= 10 + 6 + 18 + 12 + 28 + 28$$

$$= ₹\ 102.$$

Least cost Method (or) Matrix minima method :-

Transportation problem Involving ~~and~~ 3 ~~and~~ sources and 4 destination. The cell entries represent the cost of transportation per unit.

Problem : 01

obtain the initial basic feasible Solution using least cost method.

		1	2	3	4	Supply
	I	3	1	7	4	300
Source	II	2	6	5	9	400
	III	8	3	3	2	500
Demand		250	350	400	200	

Step : 01

Check whether Demand and Supply in equal

Supply = 300 + 400 + 500

= 1200

Demand = 250 + 350 + 400 + 200

= 1200

∴ Balanced.

Step : 02.

Destination

Source	1	2	3	4	Supply
I	3	$\boxed{300}$ 1	7	4	~~300~~ 0
II	2	6	5	9	400
III	8	3	3	2	500
Demand	250	~~350~~ 50	400	200	

Step : 03

	1	2	3	4	Supply
II	$\boxed{250}$ 2	6	5	9	~~400~~ 150
III	8	3	3	2	500
Demand	~~250~~ 0	50	400	200	

Step : 04

	2	3	4	Supply
II	6	5	9	150
III	3	3	$\boxed{200}$ 2	~~500~~ 300
Demand	50	400	~~200~~ 0	

Step : 05

	2	3	Supply
II	6	5	150
III	50 3	3	~~300~~ 250
Demand	~~50~~ 0	400	

Step : 06

	3	Supply
II	5	150
III	250 3	~~250~~ 0
Demand	~~400~~ 150	

Step : 07

	3	Supply
II	150 5	~~150~~ 0
Demand	~~150~~ 0	

Step : 08

Cost = (300 × 1) + (250 × 2) + (200 × 2) + (50 × 3) + (250 × 3) + (150 × 5)

= 300 + 500 + 400 + 150 + 750 + 750

= ₹ 2850 .

Homework !-

	D_1	D_2	D_3	D_4	Supply
S_1	6	4	1	5	14
S_2	8	9	2	1	16
S_3	4	3	6	2	5
Demand	6	10	15	4	

Step : 01

Check whether Demand and Supply is equal

Supply = 14 + 16 + 5

= 35

Demand = 6 + 10 + 15 + 4

= 35

Step : 02

	D_1	D_2	D_3	D_4	Supply
S_1	6	4	1	5	14
S_2	8	9	2	$\boxed{4}$ 1	~~16~~ 12
S_3	4	3	6	2	5
Demand	6	10	15	~~4~~ 0	

Step : 03

	D_1	D_2	D_3	Supply
S_1	6	4	$\boxed{14}$ 1	~~14~~ 0
S_2	8	9	2	12
S_3	4	3	6	5
Demand	6	10	~~15~~ 1	

Step : 04

	D_1	D_2	D_3	Supply
S_2	8	9	$\boxed{1}$ 2	~~12~~ 11
S_3	4	3	6	5
Demand	6	10	~~1~~ 0	

Step : 05

	D_1	D_2	Supply
S_2	8	9	11
S_3	4	$\boxed{5}$ 3	5 → 0
Demand	6	16	

Step : 06

	D_1	D_2	Supply
S_2	$\boxed{6}$ 8	9	11 → 5
Demand	6 → 0	5	

Step : 07

	D_2	Supply.
S_2	$\boxed{5}$ 9	5 → 0
Demand	5 → 0	

Result :-

$$\text{Cost} = (4 \times 1) + (14 \times 1) + (1 \times 2) + (5 \times 3) + (6 \times 8) + (5 \times 9)$$
$$= 4 + 14 + 2 + 15 + 48 + 45$$
$$= ₹\,128.$$

Least Cost method (or) Matrix minima method !.

Transportation problem involving 3 sources and 4 destination. The cell entries represent the cost of transportation per unit.

In lowest cost entry method, we use the following steps :-

Step : 01

Identify the cell with lowest cost. Let it be (i, j). Then allocate x_{ij} to the cell (i, j) such that,

$$x_{ij} = \min\{a_i, b_j\}$$

Step:02

If $x_{ij} = a_i$, then remove the i^{th} row from the table and then demand b_j is reduced to $(b_j - a_i)$. Then go to step-3.

If $x_{ij} = b_j$, then remove the j^{th} column from transportation table and the supply a_i is reduced to $a_i - b_j$, then go to step-3.

If $x_{ij} \neq b_j$, then remove either i^{th} row or j^{th} column but not both.

Step:03

Repeat the above steps with reduced transportation table thus obtained in step-2 until all the available amount is exhausted.

09/0

Vogel's Approximation Method :-

To find the IBFS by vogel's approximation method, we use the following steps.

Step:01

Find the smallest and next to smallest costs for each row of the transportation table and then find the difference between them for each row. write these difference (penalties) alongside the transportation table against the respective rows, similar ~~exercste~~ exercise will be done on case of columns.

Step:02

Select the maximum penalty among the rows and columns penalties and if there is a tie, choose any one arbitrarily.

Step : 03

Allocate the maximum possible amount to the cell with lowest cost in that particular row or column.

Let the largest penalty correspond to i^{th} row and let C_{ij} be the smallest cost in the i^{th} row. Allocate the amount

$$x_{ij} = \min\{a_i , b_j\} \text{ in the cell } (i,j)$$

and then cross out i^{th} row and j^{th} columns and obtain reduced matrix.

Step : 04

Now compute the row and column penalties for the reduced table and repeat step 2 and 3.

VOGEL'S APPROXIMATION METHOD :

Problem : 01

Consider the following transportation problem involving three sources and four destination. The cell entries represent the cost of transportation per unit.

		Destination				Supply
		1	2	3	4	
Sources	I	3	1	7	4	300
	II	2	6	5	9	400
	III	8	3	3	2	500
Demand		250	350	400	200	

obtain the initial basic feasible solution using VAM.

Solution :-

Step : 01

	1	2	3	4	Supply	
I	3	1	7	4	300	3-1=2
II	$\boxed{250}$ 2	6	5	9	~~400~~ 150	5-2=3
III	8	3	3	2	500	3-2=1
Demand	~~250~~ 0	350	400	200		
	3-2=1	3-1=2	5-3=2	4-2=2		

Step : 02

	2	3	4	Supply	
I	$\boxed{300}$ 1	7	4	~~300~~ 0	4-1=3
II	6	5	9	150	6-5=1
III	3	3	2	500	3-2=1
Demand	~~350~~ 50	400	200		
	3-1=2	5-3=2	4-2=2		

Step : 03

	2	3	4	Supply	
II	6	5	9	150	6-5=1
III	3	3	$\boxed{200}$ 2	~~500~~ 300	3-2=1
Demand	50	400	~~200~~ 0		
	6-3=3	5-3=2	9-2=7		

Step : 04

	2	3	Supply	
II	6	5	150	6-5=1
III	$\boxed{50}$ 3	3	~~300~~ 250	3-3=0
Demand	~~50~~ 0	400		
	6-3=3	5-3=2		

Step : 05

	3	Supply
II	5	150
III	250 3	~~250~~ 0
Demand	~~400~~ 150	

Step : 06

	3	Supply
III	150 5	~~150~~ 0
Demand	~~150~~ 0	

Cost $= (2 \times 250) + (1 \times 300) + (2 \times 200) + (3 \times 50) + (3 \times 250) + (5 \times 150)$

$= 500 + 300 + 400 + 150 + 750 + 750$

$=$ ₹ 2850.

Problem : 02

Find the initial basic feasible solution for the following transportation problem by VAM.

Destination

Origin		D_1	D_2	D_3	D_4	Supply
	O_1	11	13	17	14	250
	O_2	16	18	14	10	300
	O_3	21	24	13	10	400
	Demand	200	225	275	250	950

Solution :

	D_1	D_2	D_3	D_4	Supply	
O_1	$\boxed{200}$ 11	13	17	14	~~250~~ 50	13 - 11 = 2
O_2	16	18	14	10	300	14 - 10 = 4
O_3	21	24	13	10	400	13 - 10 = 3
Demand	~~200~~ 0	225	275	250	950	
	16 - 11 = 5	18 - 13 = 5	14 - 13 = 1	10 - 10 = 0		

Step : 02

	D_2	D_3	D_4	Supply	
O_1	$\boxed{50}$ 13	17	14	~~50~~ 0	14 - 13 = 1
O_2	18	14	10	300	14 - 10 = 4
O_3	24	13	10	400	13 - 10 = 3
Demand	~~225~~ 175	275	250		
	18 - 13 = 5	14 - 13 = 1	10 - 10 = 0		

Step : 03

	D_2	D_3	D_4	Supply	
O_2	$\boxed{175}$ 18	14	10	~~300~~ 125	14 - 10 = 4
O_3	24	13	10	400	13 - 10 = 3
Demand	~~175~~ 0	275	250		
	24 - 18 = 6	14 - 13 = 1	10 - 10 = 0		

Step : 04

	D_3	D_4	Supply	
O_2	14	$\boxed{125}$ 10	~~125~~ 0	14 - 10 = 4
O_3	13	10	400	13 - 10 = 3
Demand	275	~~250~~ 125		
	14 - 13 = 1	10 - 10 = 0		

Step : 05

	D_3	D_4	Supply
O_3	13	(125) 10	~~400~~ 275
Demand	275	~~125~~ 0	

13 - 10 = 3

Step : 06

	D_3	Supply
O_3	(275) 13	~~275~~ 0
Demand	~~275~~ 0	

$$\text{Cost} = (11 \times 200) + (13 \times 50) + (18 \times 175) + (10 \times 125) + (10 \times 125) + (13 \times 275).$$

$$= 2200 + 650 + 3150 + 1250 + 1250 + 3575$$

$$= ₹\ 12075.$$

ASSIGNMENT PROBLEM

Definitions :

Suppose there are n jobs to be performed and n persons are available for doing that jobs. Assume that each person can do each job at a time, though with varying degree of efficiency. Let C_{ij} be the cost if the i^{th} person is assigned to the j^{th} job. The problem is to find an assignment (which job should be assigned to which person, on a one to one basis). So that the total cost of performing all the jobs is minimum, problem of this kind are known as assignment problems.

An assigment problem can be stated in the form of $n \times n$ cost matrix $[C_{ij}]$ of real numbers as given in the following table.

$$\begin{array}{c c} & \text{Jobs} \\ & \begin{array}{cccccc} 1 & 2 & 3 \cdots & j & \cdots & n \end{array} \\ \text{Persons} \begin{array}{c} 1 \\ 2 \\ 3 \\ \vdots \\ i \\ \vdots \\ n \end{array} & \begin{bmatrix} c_{11} & c_{12} & c_{13} \cdots & c_{1j} & \cdots & c_{1n} \\ c_{21} & c_{22} & c_{23} \cdots & c_{2j} & \cdots & c_{2n} \\ c_{31} & c_{32} & c_{33} \cdots & c_{3j} & \cdots & c_{3n} \\ \vdots & & & & & \\ c_{i1} & c_{i2} & c_{i3} \cdots & c_{ij} & \cdots & c_{in} \\ \vdots & & & & & \\ c_{n1} & c_{n2} & c_{n3} \cdots & c_{nj} & \cdots & c_{nn} \end{bmatrix} \end{array}$$

Mathematical Formulation of an Assignment problem:-

Mathematically, an assignment problem can be state as,

$$\text{Minimize } z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}, \quad \text{where, } i = 1, 2 \ldots n \text{ and } j = 1, 2 \ldots n$$

Subject to the restrictions,

$$x_{ij} = \begin{cases} 1, & \text{if the } i^{th} \text{ person is assigned } j^{th} \text{ job} \\ 0, & \text{if not.} \end{cases}$$

$\sum_{j=1}^{n} x_{ij} = 1$ (one job is done by the i^{th} person).

and $\sum_{j=1}^{n} x_{ij} = 1$ (only one person should be assigned the j^{th} job).

where, x_{ij} denotes that the j^{th} job is to be assigned to the i^{th} person.

Hungarian Method Procedure

Solution of an assignment problem can be arrived at, by using the Hungarian method. The steps involved in this method are as follows.

Step : 01

Prepare a cost matrix. If the cost matrix is not a square matrix then add a dummy row (column) with zero cost element.

Step : 02

Subtract the minimum element in each row from all the elements of the respective rows.

Step : 03

Further, modify the resulting matrix by subtracting the minimum element of each column from all the elements of the respective columns. Thus, obtain the modified matrix.

Step : 04

Then, draw the minimum number of horizontal and vertical lines to cover all zeros in the resulting matrix. Let the minimum number of lines be N. Now there are two possible cases.

Case I : If $N = n$, where n is the order of matrix, then an optimal assignment can be made. So make the assignment to get the required solution.

Case II : If $N < n$, then proceed to step 5.

Step : 05

Determine the smallest uncovered element in the matrix (element not covered by N lines). Subtract this minimum element from all uncovered elements and add the same element at the intersection of horizontal and vertical lines. Thus, the second modified matrix is obtained.

Step : 06

Repeat steps 3 and 4 until we get the case (i) of step 4

Step : 07

(To make zero assignment) Examine the rows successively until a row-wise exactly single zero is found. Circle (o) this zero to make the assignment. Then mark a cross (x) over all zeros if lying in the column of the circled zero, showing that they cannot be considered for future assignment. Continue in this manner until all the zeros have been examined. Repeat the same procedure for columns also.

Step : 08

Repeat step 6 successively until one of the following situation arises

(i) If no unmarked zero is left, then the process ends or

(ii) If there lie more than one unmarked zero in any column or row, circle one of the unmarked zeros arbitrarily and mark a cross in the cells of remaining zeros in its row or column. Repeat the process until no unmarked zero is left in the matrix

Step : 09

Thus, exactly one marked circled zero in each row and each column of the matrix is obtained. The assignment corresponding to these marked circled zeros will give the optimal assignment.

Assigment problems :-

Hungarian Method :-

Problem : 01

There are 5 Jobs to be assigned on each 5 machines and the associated cost matrix is as follows.

Machines

Jobs		1	2	3	4	5
	A	10	3	3	2	8
	B	9	7	8	2	7
	C	7	5	6	2	4
	D	3	5	8	2	4
	E	9	10	9	6	10

Solve using hungarian method.

Solution :-

Row Reduction :-

10	3	3	2	8	2
9	7	8	2	7	2
7	5	6	2	4	2
3	5	8	2	4	2
9	10	9	6	10	6

8	1	1	0	6
7	5	6	0	5
5	3	4	0	2
1	3	6	0	2
3	4	3	0	4

1 1 1 0 2

Step : 02

Column Reduction :-

7	0	0	0	4
6	4	5	0	3
4	2	3	0	0
0	2	5	0	0
2	3	2	0	2

Step : 03

Row Scanning :-

7	0	0	0	4
6	4	5	$\boxed{0}$	3
4	2	3	0	$\boxed{0}$
$\boxed{0}$	2	5	0	0
2	3	2	0	2

Step : 04

Column Scanning :-

7	$\boxed{0}$	0	0	4
6	4	5	$\boxed{0}$	3
4	2	3	0	$\boxed{0}$
$\boxed{0}$	2	5	0	0
2	3	2	0	2

No. of Selected cells ≠ No. of machin

Step : 05

9	0	0	2	6
6	2	3	0	3
4	0	1	0	0
0	0	3	0	0
2	1	0	0	2

Take smallest numbers from unselected cells and add it is the interested elements. Also subtract it from remaining unselected cells.

Step : 06

Perform Row and Column Scanning :-

9	$\boxed{0}$	0	2	6
6	2	3	$\boxed{0}$	3
4	0	1	0	$\boxed{0}$
$\boxed{0}$	0	3	0	0
2	1	$\boxed{0}$	0	2

Step : 07

	M_1	M_2	M_3	M_4	M_5
J_1	10	$\boxed{3}$	3	2	8
J_2	9	7	8	$\boxed{2}$	7
J_3	7	5	6	2	$\boxed{4}$
J_4	$\boxed{3}$	5	8	2	4
J_5	9	10	$\boxed{9}$	6	10

~~Ans~~ Answer :-

Job	Machine	Assignment.
J_1	M_2	3 hours
J_2	M_4	2 hours
J_3	M_5	4 hours
J_4	M_1	3 hours
J_5	M_3	9 hours
		21 hours.

Homework Sum ...

Problem : 01

Job

		1	2	3	4
	A	20	25	22	28
Person	B	15	18	23	17
	C	19	17	21	24

Solution :.

Row Reduction

20	25	22	28	20
15	18	23	17	15
19	17	21	24	17
0	0	0	0	0

Column Reduction.

0	5	2	8
0	3	8	2
2	0	4	7
0	0	0	0
0	0	0	0

Row & Column Scanning :-

$\boxed{0}$	5	2	8
0	3	8	2
2	$\boxed{0}$	4	7
0	0	$\boxed{0}$	0

3 ≠ 4

0	5	$\boxed{0}$	6
$\boxed{0}$	3	6	0
2	$\boxed{0}$	2	5
2	2	0	$\boxed{0}$

20	25	$\boxed{22}$	28
$\boxed{15}$	18	23	17
19	$\boxed{17}$	21	24
0	0	0	$\boxed{0}$

Answer :-

Job	persons	Assignment
1	A	22
2	B	15
3	C	17
4	D	0
		54

Problem : 02

Ans : 40

	A	B	C	D	E
1	13	8	16	18	19
2	9	15	24	9	12
3	12	9	4	4	4
4	6	12	10	8	13
5	15	17	18	10	20

Solution :-

Row Reduction

	A	B	C	D	E	
1	13	8	16	18	19	8
2	9	15	24	9	12	9
3	12	9	4	4	4	4
4	6	12	10	8	13	6
5	15	17	18	10	20	10

Column Reduction.

1	5	0	8	10	11
2	0	6	15	0	3
3	8	5	0	0	0
4	0	6	4	2	7
5	5	7	8	0	10

Row Scanning

	A	B	C	D	E
1	5	$\boxed{0}$	8	10	11
2	0	6	15	0	3
3	8	5	$\boxed{0}$	0	0
4	$\boxed{0}$	6	4	2	7
5	5	7	8	$\boxed{0}$	10

Column Scanning

	A	B	C	D	E
1	5	$\boxed{0}$	5	10	8
2	0	6	12	0	$\boxed{0}$
3	11	8	$\boxed{0}$	3	0
4	$\boxed{0}$	6	1	2	1
5	5	7	5	$\boxed{0}$	7

	A	B	C	D	E
1	13	$\boxed{8}$	16	18	19
2	9	15	24	9	$\boxed{12}$
3	12	9	$\boxed{4}$	4	4
4	$\boxed{6}$	12	10	8	13
5	15	17	18	$\boxed{10}$	20

RESULT :-

Job	Machine	Assignment.
J_1	M_2	8
J_2	M_5	12
J_3	M_3	4
J_4	M_1	6
J_5	M_4	10
		40

Sequencing Problems

Definition :-

Suppose there are n jobs (1, 2, ... n), each of which has to be processed one at a time at m machines (A, B, C ...). The order of processing each job through each machine is given. The problem is to find a sequence among $(n!)^m$ number of all possible sequence for processing the jobs, so that the total elapsed time for all the jobs will be minimum.

Terminology and Notations :-

The following are the terminologies and notations used in this chapter.

Number of machines :

It means the service facilities through which a job must pass before it is completed.

Processing order :

It refers to the order in which various machines are required for completing the job.

Processing time :

It means the time required by each job on each machine.

Idle time on a machine :-

This is the time for which a machine remains idle during the total elapsed time. The notation x_{ij} is used to denote the idle time of a machine j between the end of the (i-1) th job and the start of the i^{th} job

No passing rule :

It means, passing is not allowed (i.e.,) maintaining the same order of jobs over each machine. If each of N-jobs is to be Processed through 2 machines M_1 and M_2 in the order M_1, M_2 then this rule will mean that each job will go to machine M_1 first and then to M_2. If a job is finished on M_1, it goes directly to machine M_2, if it is free, otherwise it starts a waiting line or joins the end of the waiting line, if one already exists. Jobs that form a waiting line are processing on machine M_2 when it become free.

TYPE : 01

Problems with n jobs through two machines.

The algorithms which is used to optimize the total elapsed time for processing n jobs through two machine is Called 'Johnson's algorithm' and has the following table.

Machine / Job	1	2	3	....	n
A	A_1	A_2	A_3	....	A_n
B	B_1	B_2	B_3	....	B_n

The problem is to sequence the jobs so as to minimize the total elapsed time. The Solution procedure adopted by Johnson is given below.

Step : 01

Select the least processing time occuring in the list $A_1, A_2 \ldots A_n$ and $B_1, B_2 \ldots B_n$. Let this minimum processing time occur for a job k.

Step : 02

If the shortest processing is for machine A, process the k^{th} job last and place it at the end of the sequence.

Step : 03

when there is a tie in selecting the minimum processing time, then there may be three solutions.

(i) If the equal minimum values occur only for machine A, Select the job with larger processing time in B to be Placed first in the job sequence.

(ii) If the equal minimum values occur only for machine B, Select the job with larger processing time in A to be Placed last in the job sequence.

(iii) If there are equal minimum values, one for each machine, then place the job in machine A first and the one in machine B last.

Step : 04

Delete the jobs already sequenced. If all the jobs have been sequenced, go to the next step. otherwise, repeat step 1 to 3.

Step : 05

In this step, determine the overall or total elapsed time and also the idle time on machine A and B as follows.

Total elapsed time = The time between starting the first job in the optimal sequence on machine B.

Idle time on A = (Time when the last job in the optimal sequence on machine B) − (Time when the last Job in the optimal Sequence is Completed on machine A)

Idle time on B = when the first job in the optimal Sequence starts on machine B + $\sum_{k=2}^{n}$ (time k^{th} job starts on machine − time $(k-1)^{th}$ job finished on machine B)

TYPE : 02

Processing n jobs through three machines A, B, C.

Consider n jobs (1, 2 . . . n) processing on three machine A, B, C in the order ABC. The optimal sequence can be obtained by converting the problem into a two-machine problem. From this, we get the optimum sequence using Johnson's algorithm.

The following steps are used to convert the given problem into a two-machine problem.

Step : 01

Find the minimum processing time for the jobs on the first and last machines and the maximum processing time for the second machine.

(i.e)., find $\underset{i}{\text{Min}}(A_i, C_i)$ $i = 1, 2, \ldots n$ and $\underset{i}{\text{Max}}(B_i)$

Step : 02

Check the following inequality

$$\underset{i}{\text{Min}}\, A_i \geq \underset{i}{\text{Mix}}\, B_i \quad (\text{or}) \quad \underset{i}{\text{Min}}\, C_i \geq \underset{i}{\text{Mix}}\, B_i$$

Step : 03

If at least one of the inequalities in step 2 are satified, this method cannot be applied.

Step : 04

If at least one of the inequalities in step 2 is satisfied, we define two machines G and H, such that the processing time on G and H are given by,

$$G_i = A_i + B_i \quad ; \quad i = 1, 2, \ldots n$$

$$H_i = B_i + C_i \quad ; \quad i = 1, 2 \ldots n$$

Step : 05

For the converted machines G and H, we obtain the optimum sequence using two-machine algorithm.

SEQUENING PROBLEM

Two machines problems :-

Problem : 01

There are 5 jobs each of which must go through the two machines A and B in the order AB processing times are given below.

Jobs	1	2	3	4	5
machine A	5	1	9	3	10
machine B	2	6	7	8	4

Determine a sequence for the 5 jobs the will minimize the total elapsed time ?

Solution :-

Step : 01

Given the table

minimum value

Jobs	1	2	3	4	5	
machine A	5	1	9	3	10	left
machine B	2	6	7	8	4	Right.

Step : 02

2	4	3	5	1

Step :03

Jobs	A	B	Machines A		Machines B		Idle time	
			In	out	In	out	A	B
				In + A		In for B +		
2	1	6	0	1	1	7	0	1
4	3	8	1	4	7	15	0	0
3	9	7	4	13	15	22	0	0
5	10	4	13	23	23	27	0	1
1	5	2	23	28	28	30	2	1

Total elapsed time = 30 hrs

Idle time on machine A = (30 - 28) = 2 hrs

Idle time on machine B = 3 hrs.

Homework :-

Jobs	1	2	3	4	5	6
machine A	5	9	4	7	8	6
machine B	7	4	8	3	9	5

Solution :-

Step :01

Given the table.

Jobs	1	2	3	4	5	6	
machine A	5	9	4	7	8	6	left
machine B	7	4	8	3	9	5	Right

Step :02.

3	1	5	6	2	4

Step : 03

Jobs	A	B	machines A		machine B		Idle time	
			In	out	In	out	A	B
3	4	8	0	4	4	12	0	4
1	5	7	4	9	12	19	0	0
5	8	9	9	17	19	28	0	0
6	6	5	17	23	28	33	0	0
2	9	4	23	32	33	37	0	0
4	7	3	32	39	39	42	3	2

Total elapsed time = 42 hrs

Idle time on machine A = (42 - 39) = 3 hrs

Idle time on machine B = (39 - 37) = 2 hrs.

6

08/24