BHARATHIDASAN UNIVERSITY

Tiruchirappalli- 620024

Tamil Nadu, India

Programme: M.Sc., Physics

Course Title : Mathematical Physics

Course Code : 22PH101

Dr. M. Senthilvelan

Professor

Department of Nonlinear Dynamics

UNIT - II

Orthogonal Polynomials

Frobenius' Theorem

If $x = x_0$ is a regular singular point of the differential equation, then there exists **at least one solution of the form**

$$y = (x - x_0)^r \sum_{n=0}^{\infty} c_n (x - x_0)^n \quad = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r}$$

where the number r is a constant to be determined. The series will converge at least on some interval $0 < x - x_0 < R$.

Example 1: Solve : $3xy'' + y' - y = 0$ using Frobenius series method

Recall Frobenius theorem

$$y = (x - x_0)^r \sum_{n=0}^{\infty} c_n (x - x_0)^n$$

$$y = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r}$$

Indicial roots to be determined

Regular singular point

Unknown coefficients to be determined

<u>To begin:</u> Find out Regular singular points of the given DE

$$3xy'' + y' - y = 0 \Longrightarrow y'' + \underbrace{\frac{1}{3x}}_{P(x)} y' - \underbrace{\frac{1}{3x}}_{Q(x)} y = 0$$

$x_0 = 0$ is a Singular Point

Whether the SP is a regular SP or irregular SP ?

$$\boxed{(x-x_o)P(x)} = x\left(\frac{1}{3x}\right) = \frac{1}{3}$$ (Not infinity) Analytic at $x = 0$

$$\boxed{(x-x_o)^2\,Q(x)} = x^2\left(\frac{1}{3x}\right) = \frac{x}{3}\bigg|_{x=0}$$ (Not infinity) Analytic at $x = 0$

Analytic $\Rightarrow$ $\boxed{x = 0 \text{ is a Regular Singular Point}}$

$\boxed{\text{Frobenius Theorem can be applied}}$

Given equation $3xy'' + y' - y = 0$

Assume $$y = \sum_{n=0}^{\infty} c_n (x-x_0)^{n+r} = \sum_{n=0}^{\infty} c_n x^{n+r}$$

Frobenius Series

$$y = \sum_{n=0}^{\infty} c_n x^{n+r}$$

On differentiating,

$$y' = \sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1};$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-2}$$

$3xy'' + y' - y = 0 \rightarrow$

$$3x \sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-2} + \sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1} - \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

$$3 \sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-1} + \sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1} - \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

n = variable
r = constant

x^r can be taken outside the summation

Frobenius Series

$$3\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n-1}\cdot x^r+\sum_{n=0}^{\infty}(n+r)c_n x^{n-1}\cdot x^r-\sum_{n=0}^{\infty}c_n x^n\cdot x^r=0$$

No connection with 'n'
It can be taken outside

$$3x^r\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n-1}+x^r\sum_{n=0}^{\infty}(n+r)c_n x^{n-1}-x^r\sum_{n=0}^{\infty}c_n x^n=0$$

$$x^r\left\{\underbrace{3\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n-1}+\sum_{n=0}^{\infty}(n+r)c_n x^{n-1}-\sum_{n=0}^{\infty}c_n x^n}\right\}=0$$

A . B = 0

$A=0, B\neq 0 \rightarrow x^r=0, B\neq 0$

Contradictory We assumed $x^r\neq 0$

$A\neq 0, B=0 \rightarrow x^r\neq 0, B=0$

Valid choice

Valid Choice

$$3\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n-1}+\sum_{n=0}^{\infty}(n+r)c_n x^{n-1}-\sum_{n=0}^{\infty}c_n x^{n}=0$$

The first two terms looks similar.

$$\sum_{n=0}^{\infty}[3(n+r)(n+r-1)+(n+r)]c_n x^{n-1}-\sum_{n=0}^{\infty}c_n x^{n}=0$$

$$\sum_{n=0}^{\infty}(n+r)[3(n+r-1)+1]c_n x^{n-1}-\sum_{n=0}^{\infty}c_n x^{n}=0$$

$$\sum_{n=0}^{\infty}(n+r)[3n+3r-2]c_n x^{n-1}-\sum_{n=0}^{\infty}c_n x^{n}=0$$

Determination of Indicial Roots (r)

- The lowest power of x yields the value of r
- Let us find the lowest power of x
- *Expanding*

$$\sum_{n=0}^{\infty}(n+r)[3n+3r-2]c_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n = 0$$

Lowest power of x → Comes out from $n = 0$

$$r(3r-2)c_0 x^{-1} + \boxed{(1+r)(3+3r-2)c_1 x^0 \; + (2+r)(6+3r-2))c_2 x^1 + \;...}$$

$$\boxed{-[c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + ...] = 0}$$

$$\underbrace{r(3r-2)c_0 x^{-1}} + \sum_{n=1}^{\infty}(n+r)[3n+3r-2]c_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n = 0$$

Choose 'r' such that this term vanishes

$$\rightarrow r(3r-2) = 0 \rightarrow r = 0\,; r = \frac{3}{2}$$

Indicial roots

Indicial Equation

For second order ODEs one gets 2 indicial values

Simplifying the summation

$$r(3r-2)c_0x^{-1} + \sum_{n=1}^{\infty}(n+r)[3n+3r-2]c_nx^{n-1} - \sum_{n=0}^{\infty}c_nx^n = 0$$

Let us rewrite the second and third terms as a single term

$$\sum_{n=1}^{\infty}(n+r)[3n+3r-2]c_nx^{n-1}$$

$$= [(1+r)[1+3r]]c_1x^0 + [(2+r)[4+3r]]c_2x + (\ )c_3x^2 + (\ )c_4x^3 + \ldots\ldots$$

$$\sum_{n=0}^{\infty}c_nx^n = c_0x^0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + c_6x^6 + \ldots$$

Every term that present in the first sum also present in the second sum

We rewrite the first sum so that the index *n* starts from **0 instead of 1**

Simplifying the summation

$$\sum_{n=1}^{\infty}(n+r)[3n+3r-2]c_n x^{n-1}$$

Let us define $n-1=l \rightarrow n=l+1$

When $n=1 \rightarrow l=0$

$n=\infty \rightarrow l=\infty$

Dummy index

$$\sum_{n=1}^{\infty}(n+r)[3n+3r-2]c_n x^{n-1} \rightarrow \sum_{l=0}^{\infty}(l+1+r)[3l+3r+1]c_{l+1} x^{l}$$

Dummy index

$$\sum_{n=0}^{\infty} c_n x^n = \sum_{l=0}^{\infty} c_l x^l$$

$$\sum_{n=1}^{\infty}(n+r)[3n+3r-2]c_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{l=0}^{\infty}(l+1+r)[3l+3r+1]c_{l+1}\, x^{l} - \sum_{l=0}^{\infty} c_l x^l = 0$$

Starts and ends at same value

$$\sum_{l=0}^{\infty}[(l+1+r)[3l+3r+1]c_{l+1} - c_l]x^l = 0$$

Values of unknown constants

Recurrence Relation

$$c_{l+1} = \frac{c_l}{(l+r+1)(3l+3r+1)} \qquad , l = 0,1,2,3,4,\ldots$$

Case 1:

$$r = 0 \qquad c_{l+1} = \frac{c_l}{(l+1)(3l+1)}, \quad l = 0,1,2,3,4,\ldots$$

Case 2:

$$r = \frac{2}{3} \qquad c_{l+1} = \frac{c_l}{(3l+5)(l+1)}, \quad l = 0,1,2,3,4,\ldots$$

Case 1: $c_{l+1} = \dfrac{c_l}{(l+1)(3l+1)}, \quad l = 0,1,2,3,4,\ldots$

$\underline{l = 0}$ $\qquad c_1 = \dfrac{c_0}{1}$

$\underline{l = 1}$ $\qquad c_2 = \dfrac{c_0}{8}$

$\underline{l = 2}$ $\qquad c_3 = \dfrac{c_0}{(3)(7)(8)}$

$\underline{l = 3}$ $\qquad c_4 = \dfrac{c_0}{(3)(4)(7)(8)(10)}$

Case 2: $c_{l+1} = \dfrac{c_l}{(3l+5)(l+1)}$

$\underline{l = 0}$ $\qquad c_1 = \dfrac{c_0}{5}$

$\underline{l = 1}$ $\qquad c_2 = \dfrac{c_0}{(16)(5)}$

$\underline{l = 2}$ $\qquad c_3 = \dfrac{c_0}{(33)(16)(5)}$

Two independent solutions

Case 1: $r = 0$ (first indicial root)

$$y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+r} = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots\ldots.$$

$$= c_0 + c_0 x + \frac{c_0}{8} x^2 + \frac{c_0}{3(7)(8)} x^3 + \frac{c_0}{3(4)(7)(8)(10)} c_0 x^4 + \frac{c_0}{3(4)(5)(7)(8)(10)(13)} c_0 x^5 + \ldots$$

$$\boxed{y_1(x) = c_0 \left[1 + x + \frac{x^2}{8} + \frac{x^3}{3(7)(8)} + \frac{x^4}{3(4)(7)(8)(10)} + \frac{x^5}{3(4)(5)(7)(8)(10)(13)} \ldots\right]}$$

Case 2: $r = \frac{2}{3}$ (second indicial root)

$$y_2(x) = \sum_{n=0}^{\infty} c_n x^{n+r} = \sum_{n=0}^{\infty} c_n x^{n+2/3}$$

$$y_2(x) = x^{2/3} \sum_{n=0}^{\infty} c_n x^n = x^{2/3} [c_0 x^0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \ldots..]$$

$$\boxed{y_2(x) = c_0 x^{2/3} \left[1 + \frac{1}{5} x + \frac{1}{(16)5} x^2 + \frac{1}{(33)(16)5} x^3 + \ldots..\right]}$$

General Solution

$$\left.\begin{array}{l} r = 0 \rightarrow y = y_1(x) \\ \\ r = \frac{2}{3} \rightarrow y = y_2(x) \end{array}\right\} \quad y_{Gs} = Ay_1(x) + By_2(x)$$

$$\boxed{\begin{gathered} y_{Gs} = A\left[1 + x + \frac{x^2}{8} + \frac{x^3}{3(7)(8)} + \frac{x^4}{3(4)(7)(8)(10)} + \frac{x^5}{3(4)(5)(7)(8)(10)(13)} \cdots\right] \\ +Bx^{2/3}\left[1 + \frac{1}{5}x + \frac{1}{(16)5}x^2 + \frac{1}{(33)(16)5}x^3 + \ldots\ldots\right] \end{gathered}}$$

Note:

- $\boldsymbol{r_1 - r_2 \neq}$ **Integer.**
- **In this case, we get two independent solutions.**

Frobenius' Series: Case 1

$$m_1 \neq m_2$$

$$m_1 - m_2 \neq \text{not an integer}$$

Frobenius Method

Type I : (m_1, m_2 = distinct, $m_1 - m_2$ = not an integer)

(Roots of Indicial equations <u>are unequal</u> and <u>not differing</u> by an integer)

Example 2: Solve : $9x(1-x)y'' - 12y' + 4y = 0$

Step 1 : Write the given DE in the form $y'' + P(x)y' + Q(x)y = 0$ and identify $P(x)$ & $Q(x)$

$$y'' - \frac{4}{3x(1-x)}y' + \frac{4}{9x(1-x)}y = 0$$

$$P(x) = \frac{4}{3x(1-x)} \quad ; \quad Q(x) = \frac{4}{9x(1-x)}$$

Step 2: Identify Singular points and its nature.

$$\left.\begin{array}{l} P(x) \text{ goes to infinity at } x = 0, x = 1 \\ Q(x) \text{ goes to infinity at } x = 0, x = 1 \end{array}\right\} \text{Singular points}$$

<u>**Nature of Singular points**</u>

Case 1 : $x_o = 0$

$$(x - x_o)P(x) = (x-0)\frac{4}{3x(1-x)} = \frac{4}{3(1-x)} = \frac{4}{3(1)} = \text{Constant}$$

$$(x-x_o)^2\,Q(x) \;=(x-x_o)^2\frac{4}{9x(1-x)}$$

$$=\frac{4\,x^2}{9x(1-x)} = \left.\frac{4x}{9(1-x)}\right|_{x=0} = 0$$

Both $(x-x_o)P(x)$ and $(x-x_o)^2\,Q(x)$ are analytic at $x_o = 0$

$$\boxed{\boldsymbol{x_o = 0} \textbf{ is a Regular Singular Point}}$$

Case 2 : $x_o = 1$

$$(x-x_o)P(x) = (x-1)\frac{4}{3x(1-x)} \qquad = \left.\frac{4}{3x}\right|_{x=1} = \text{Constant}$$

$$(x-x_o)^2\,Q(x) \;= (x-1)^2\frac{4}{3x(1-x)}$$

$$= \left.\frac{4(1-x)}{3x}\right|_{x=1} = 0$$

Both $(x-x_o)P(x)$ and $(x-x_o)^2\,Q(x)$ are analytic at $x_o = 1$

$$\boxed{\boldsymbol{x_o = 1} \textbf{ is a Regular Singular Point}}$$

Frobenius Series

Step 2: Let us construct a series solution.

$$y = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+m}$$

Indicial roots to be determined

Unknown coefficients to be determined

Singular Point at which we are expanding the solution

In the present case $x_o = 0 \quad \therefore \; y = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+m} = \sum_{n=0}^{\infty} c_n x^{n+m}$

$$y' = \sum_{n=0}^{\infty} (n+m) c_n x^{n+m-1} \quad ; \qquad y'' = \sum_{n=0}^{\infty} (n+m)(n+m-1) c_n x^{n+m-2}$$

The given equation : $9x(1-x)y'' - 12y^1 + 4y = 0$

$$9xy'' - 9x^2y'' - 12y'' + 4y = 0$$

Recall : $9xy'' - 9x^2y'' - 12y' + 4y = 0$

Substituting the expressions for derivatives,

On solving powers, $x^{n+m-2+1} = x^{n+m-1}$

On solving powers, $x^{n+m-2+2} = x^{n+m}$

$$9x\sum_{n=0}^{\infty}(n+m)(n+m-1)c_nx^{n+m-2} - 9x^2\sum_{n=0}^{\infty}(n+m)(n+m-1)c_nx^{n+m-2}$$

$$-12\sum_{n=0}^{\infty}(n+m)c_nx^{n+m-1} + 4\sum_{n=0}^{\infty}c_nx^{n+m} = 0$$

$$-9\sum_{n=0}^{\infty}(n+m)(n+m-1)c_nx^{n+m} + 9\sum_{n=0}^{\infty}(n+m)(n+m-1)c_nx^{n+m-1}$$

$$-12\sum_{n=0}^{\infty}(n+m)c_nx^{n+m-1} + 4\sum_{n=0}^{\infty}c_nx^{n+m} = 0$$

Multiply all terms by $(-)$

$$+9\sum_{n=0}^{\infty}(n+m)(n+m-1)c_nx^{n+m} - 9\sum_{n=0}^{\infty}(n+m)(n+m-1)c_nx^{n+m-1}$$

$$+12\sum_{n=0}^{\infty}(n+m)c_nx^{n+m-1} - 4\sum_{n=0}^{\infty}c_nx^{n+m} = 0$$

$$9\sum_{n=0}^{\infty}(n+m)(n+m-1)c_n x^{n+m} - 9\sum_{n=0}^{\infty}(n+m)(n+m-1)c_n x^{n+m-1}$$

$$+12\sum_{n=0}^{\infty}(n+m)c_n x^{n+m-1} - 4\sum_{n=0}^{\infty}c_n x^{n+m} = 0$$

$$9\sum_{n=0}^{\infty}(n+m)(n+m-1)c_n x^{n+m} - \sum_{n=0}^{\infty}\underbrace{[9(n+m)(n+m-1)-12(n+m)]}c_n x^{n+m-1}$$

$$-4\sum_{n=0}^{\infty}c_n x^{n+m} = 0$$

On taking $(n+m)$ commonly we get $(n+m)[9n+9m-9-12]$

$$\sum_{n=0}^{\infty}[9(n+m)(n+m-1)-4]c_n x^{n+m} - \sum_{n=0}^{\infty}(n+m)[9n+9m-21]c_n x^{n+m-1} = 0$$

Summation involves only n

Independent Parameter

We can take x^m outside the summation

Let us pull out the term x^m outside the summation

$$x^m \left\{ \sum_{n=0}^{\infty} [9(n+m)(n+m-1)-4]c_n x^n - \sum_{n=0}^{\infty} (n+m)[9n+9m-21]c_n x^{n-1} \right\} = 0$$

Product of two terms =0

or

Either $x^m = 0$

Not sensible

Terms inside curl bracket $= 0$

Go for it

$$\sum_{n=0}^{\infty} [9(n+m)(n+m-1)-4]c_n x^n - \sum_{n=0}^{\infty} (n+m)[9n+9m-21]c_n x^{n-1} = 0$$

$$\sum_{n=0}^{\infty} (n+m)[9n+9m-21]c_n x^{n-1} - \sum_{n=0}^{\infty} [9(n+m)(n+m-1)-4]c_n x^n = 0$$

$$\sum_{n=0}^{\infty}(n+m)[9n+9m-21]c_n x^{n-1} - \sum_{n=0}^{\infty}[9(n+m)(n+m-1)-4]c_n x^n = 0$$

By substituting $n = 0$ we get the first term

Lowest power of x

No change
Summation starts from n=0

$$m(9m-21)c_0 x^{-1} + \sum_{n=1}^{\infty}(n+m)[9n+9m-21]c_n x^{n-1} - \sum_{n=0}^{\infty}[9(n+m)(n+m-1)-4]c_n x^n = 0$$

The coefficients fix the indicial roots

These two terms can be written as a single term

Since we separated the first term n=0
Summation starts from n=1

$$\boxed{A}\qquad\qquad\qquad\qquad\boxed{B}$$

$$m(9m-21)c_0\,x^{-1}+\underbrace{\sum_{n=1}^{\infty}(n+m)[9n+9m-21]c_nx^{n-1}-\sum_{n=0}^{\infty}[9(n+m)(n+m-1)-4]c_nx^n}_{\text{Let us make it as a single summation}}=0$$

$$\boxed{A}\ \sum_{n=1}^{\infty}(n+m)[9n+9m-21]c_nx^{n-1}=(1+m)[9m-12]c_1x^0+(2+m)[9m-3]c_2x^1$$
$$\underbrace{\qquad\qquad+(3+m)[9m+6]c_3x^2+\ldots\ldots}_{}$$

Powers Match with each other

$$\boxed{B}\ \sum_{n=0}^{\infty}[9(n+m)(n+m-1)-4]c_nx^n=\overbrace{[9m(m-1)-4]c_0x^0+[9(1+m)m-4]c_1x^1}^{}$$
$$+[9(2+m)[1+m]-4]c_2\,x^2+\ldots\ldots$$

Let us relabel $\boxed{A}$ Define $\mathrm{n}-1=l$ and express in terms of l

$$\mathrm{n}-1=l\ \rightarrow\ \boxed{n=1;l=0}\ \ \boxed{n=\infty;l=\infty}$$

$$\boxed{A}\ \sum_{l=0}^{\infty}(l+m+1)[9l+9m-12]c_{l+1}x^l$$

$$\boxed{B}\ \sum_{I=0}^{\infty}[9(l+m)(l+m-1)-4]c_lx^l$$

Note we **relabelled with *l*** instead of ***n***

$$m(9m-21)c_0\, x^{-1} + \sum_{l=0}^{\infty} (l+m+1)[9l+9m-12]c_{l+1}x^{l} - \sum_{l=0}^{\infty}[9(l+m)(l+m-1)-4]c_l x^{l} = 0$$

$$m(9m-21)c_0\, x^{-1} + \sum_{l=0}^{\infty} \{(l+m+1)[9l+9m-12\,]c_{l+1} \; - [9(l+m)(l+m-1)-4]c_l\}x^{l} = 0$$

$$(\;\;)x^{-1} + (\;\;)x^{0} + (\;\;)x^{1} + (\;\;)x^{2} + (\;\;)x^{2} + (\;\;)x^{2} + \ldots. = 0$$

Indicial equation

Yields the value of m

$\boxed{\text{Unknown coefficients } c_0, c_1, c_2, c_3, .. \text{ can be determined from these}}$

As usual the LHS can be solved by equating the coefficients of various powers of x to zero

$c_0 \neq 0$

$\underline{\boldsymbol{x^{-1}}}$ $\qquad \underbrace{m(9m-21)c_0}_{\text{3 terms}} = 0$

Indicial equation

we can choose $m = 0$ or $9m - 21 = 0$

$m = 0$ $\qquad m = \frac{21}{9} = \frac{7}{3}$

Indicial roots

$\underline{\boldsymbol{x^{l}}}$ $\qquad (l+m+1)[9l+9m-12\,]c_{l+1} - [9(l+m)(l+m-1)-4]c_l$

$$\boxed{c_{l+1} = \frac{9(l+m)(l+m-1)-4}{(l+m+1)[9l+9m-12\,]}c_l}$$

l= 0,1,2,3,4, ...

Recursion relation connects c_l with c_{l+1}

Meaning of Recursion Relations

- If we know c_l, we can get c_{l+1}
- If we know c_{l+1}, we can get c_{l+2}
- If we know c_{l+2}, we can get c_{l+3}

It recursively gives
The values of the
coefficients

Recurrence Relation

$$\boxed{c_{l+1} = \frac{9(l+m)(l+m-1)-4}{(l+m+1)[9l+9m-12\,]}c_l}$$

$\underbrace{l = 0,1,2,3,4,\ldots\ldots}_{\text{Allowed values of } l}$

- We have two m's, $m = 0$ and $m = \frac{7}{3}$
- First go for the case $m = 0$.
- Substitute $m = 0$ in the above equation and determine the values of $c_0, c_1, c_2, c_3, ..$
- Then for the case $m = \frac{7}{3}$.
- Substitute $m = \frac{7}{3}$ in the above equation and determine the values of $c_0, c_1, c_2, c_3, ..$

Case: 1 $m = 0$ $$c_{l+1} = \frac{9l(l-1)-4}{(l+1)[9l-12\,]}c_l$$ l= 0,1,2,3,4, ...

<u>$l = 0$</u> $c_1 = \frac{-4}{-12}c_0 \quad = \frac{1}{3}c_0$

$l = 1$ $c_2 = \frac{-4}{2(-3)}c_1 \quad = \frac{2}{3}c_1 = \frac{2}{9}c_0$

$l = 2$ $c_3 = \frac{9\text{x}2\text{x}1(-4)}{3(6)}c_2 = \frac{14}{18}c_2 = \frac{7}{9}c_2 \quad = \frac{7}{9}.\frac{2}{9}c_0 \quad = \frac{14}{81}c_0$

$l = 3$ $c_4 =$ ***H.W***

$l = 4$ $c_5 =$ ***H.W***

$$y = \sum_{n=0}^{\infty} c_n x^{n+m} \; = \sum c_n x^n \; = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + c_6 x^6 + \ldots$$

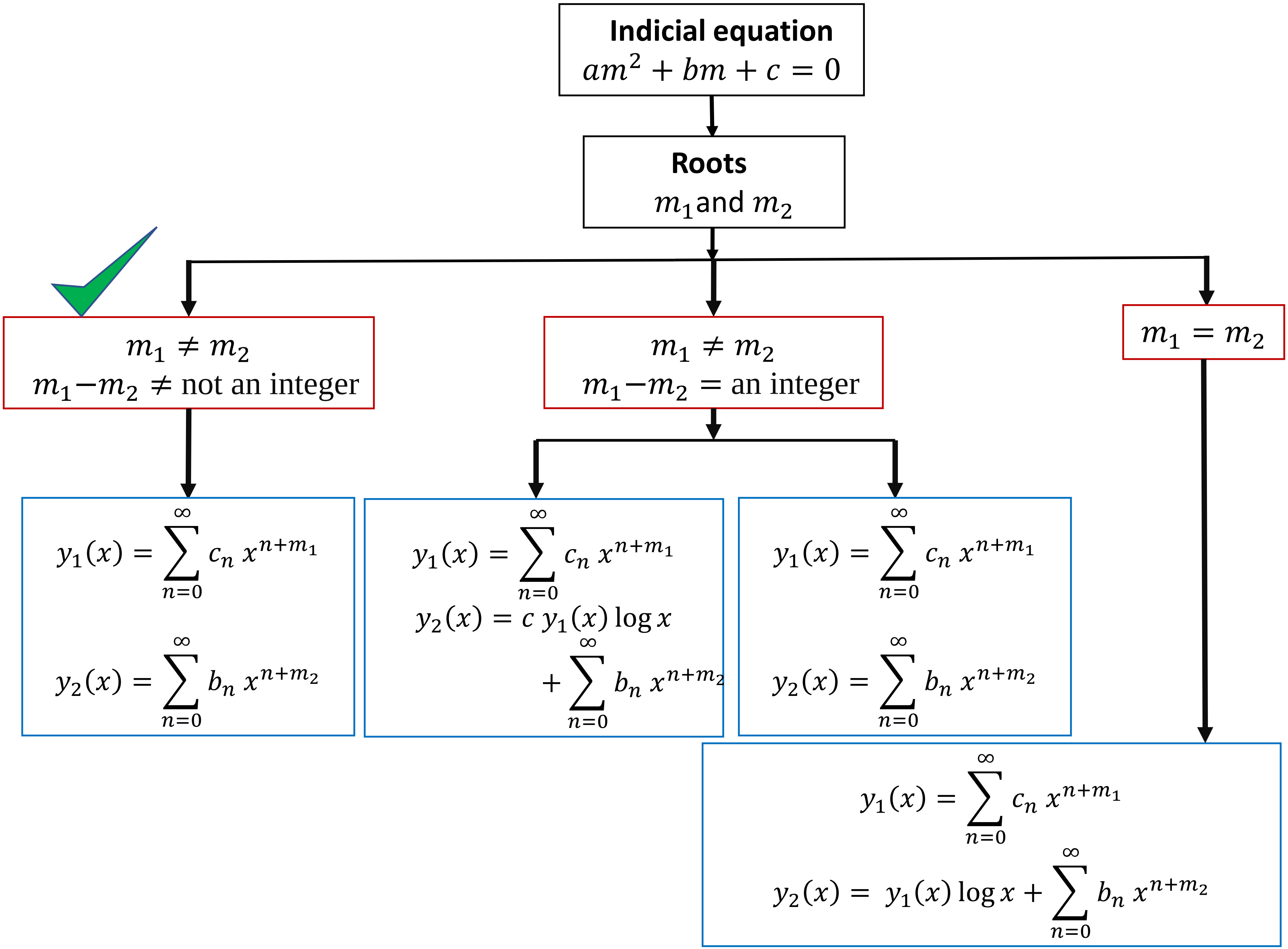
$$= c_0 + \frac{c_1}{3}x + \frac{2}{9}c_0 x^2 + \frac{14}{81}c_0 x^3 + (\;?\;)c_0 x^4 + (\;?\;)c_0 x^5 + \ldots$$

$$\boxed{y = c_0\left[1 + \frac{x}{3} + \frac{2}{9}x^2 + \frac{14}{81}x^3 + (\;?\;)x^4 + (\;?\;)x^5 + \ldots\right]}$$

Recurrence Relation $$\boxed{c_{l+1} = \frac{9(l+m)(l+m-1)-4}{(l+m+1)[9l+9m-12\,]}c_l}$$ l= 0,1,2,3,4, ...

Home Work:

- Substitute $m = \frac{7}{3}$ in the above equation and determine the values of $c_0, c_1, c_2, c_3, ..$
- Write down the second independent solution $y_2(x)$.
- Write down the general solution.



Indicial equation
$am^2 + bm + c = 0$
Roots
m_1 and m_2
$m_1 \neq m_2$
$m_1 - m_2 \neq$ not an integer
$m_1 \neq m_2$
$m_1 - m_2 =$ an integer
$m_1 = m_2$
$y_1(x) = \sum_{n=0}^{\infty} c_n \, x^{n+m_1}$
$y_2(x) = \sum_{n=0}^{\infty} b_n \, x^{n+m_2}$
$y_1(x) = \sum_{n=0}^{\infty} c_n \, x^{n+m_1}$
$y_2(x) = c \, y_1(x) \log x + \sum_{n=0}^{\infty} b_n \, x^{n+m_2}$
$y_1(x) = \sum_{n=0}^{\infty} c_n \, x^{n+m_1}$
$y_2(x) = \sum_{n=0}^{\infty} b_n \, x^{n+m_2}$
$y_1(x) = \sum_{n=0}^{\infty} c_n \, x^{n+m_1}$
$y_2(x) = y_1(x) \log x + \sum_{n=0}^{\infty} b_n \, x^{n+m_2}$

Frobenius' Series: Case 2

$$m_1 \neq m_2$$

$$m_1 - m_2 = \text{an integer}$$

Frobenius' Theorem

If $x = x_0$ is a regular singular point of the differential equation, then there exists **at least one solution of the form**

$$y = (x - x_0)^r \sum_{n=0}^{\infty} c_n (x - x_0)^n = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r}$$

Where the number r is a constant to be determined. The series will converge at least on some interval $0 < x - x_0 < R$.

Method of constructing the solution

Singular Points

Solve in series: $x(1-x)y'' - 3xy' - y = 0$ near x=0

(i) Comparing with $y'' + P(x)y' + Q(x)y = 0$, we infer

$$P(x) = \frac{-3x}{x(1-x)} = \frac{-3}{(1-x)} \quad ; \quad Q(x) = \frac{-1}{x(1-x)}$$

(ii) $x = 0\ and\ x = 1$ are singular points

(iii) A $\quad (x - x_o)P(x) = x\left(\frac{-3}{(1-x)}\right)\Bigg|_{x=0} = 0$

$$(x - x_o)^2\, Q(x) = x\left(\frac{-1}{1-x}\right)\Bigg|_{x=0} = \frac{-x}{(1-x)} = 0$$

$P(x)$ & Q(x) don't go infinity at x=*0*

$\boxed{x_o = 0 \text{ is a Regular Singular Point}}$

Singular Points

(iii) B $\quad (x - x_o)P(x) = (x-1)\left(\dfrac{-3}{(1-x)}\right) = 3$

$$(x - x_o)^2\, Q(x) = (x-1)^2\left(\dfrac{-1}{x(1-x)}\right) = \left.\dfrac{x-1}{x}\right|_{x=1} = 0$$

$P(x)$ & Q(x) don't go infinity at $x = 1$

$\boxed{x_o = 1 \text{ is a Regular Singular Point}}$

Question asked: At $x = 0$ we have to develop a series solution.

Frobenius' Series

$$y(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+m} = \sum_{n=0}^{\infty} c_n x^{n+m}$$

$$y'(x) = \sum_{n=0}^{\infty} (n+m) c_n x^{n+m-1} \qquad ; \; y''(x) = \sum_{n=0}^{\infty} (n+m)(n+m-1) c_n x^{n+m-2}$$

Given Equation: $xy'' - x^2 y'' - 3xy' - y = 0$

$$\sum_{n=0}^{\infty} (n+m)(n+m-1) c_n x^{n+m-1} - \sum_{n=0}^{\infty} (n+m)(n+m-1) c_n x^{n+m}$$

$$-3 \sum_{n=0}^{\infty} (n+m) c_n x^{n+m} - \sum_{n=0}^{\infty} c_n x^{n+m} = 0$$

Same exponent *(n +m)*
We can group these 3 terms into 1 term.

All three summations start and end at same numbers

$$\sum_{n=0}^{\infty}(n+m)(n+m-1)c_n x^{n+m-1} - \sum_{n=0}^{\infty}\underbrace{[(n+m)(n+m-1)+3(n+m)}+1]c_n x^{n+m} = 0$$

$$(n+m)(n+m-1+3) = (n+m)(n+m+2)$$

$$x^m\underbrace{\left\{\sum_{n=0}^{\infty}(n+m)(n+m-1)c_n x^{n-1} - \sum_{n=0}^{\infty}[(n+m)(n+m+2)\ +1]c_n x^n\right\}}_{=0} = 0$$

$\neq 0$

$=0$

$$m(m-1)c_0 x^{-1} + \sum_{n=1}^{\infty}(n+m)(n+m-1)c_n x^{n-1} - \sum_{n=0}^{\infty}[\overset{(n+m+1)^2}{\cancel{(n+m)(n+m+2)+1}}]c_n x^n = 0$$

Lowest power of x

$n-1 = l \rightarrow n = l+1$

$n = 1 \rightarrow l = 0$

$n = \infty \rightarrow l = \infty$

Dummy index

Can be replaced by l

$$m(m-1)c_0x^{-1}+\sum_{l=0}^{\infty}(l+m+1)(l+m)c_{l+1}\,x^l-\sum_{l=0}^{\infty}[(l+m+1)^2]c_lx^l=0$$

$$m(m-1)c_0x^{-1}+\sum_{l=0}^{\infty}[(l+m+1)(l+m)c_{l+1}-(l+m+1)^2c_l]x^l=0$$

Indicial equation

Lowest power of *x*

Cross Check $(\quad)x^{-1}+(\quad)x^0+(\quad)x\;+(\quad)x^2\;+(\quad)x^2\;+\ldots.$

$m(m-1)c_0=0$ Indicial equation

$m_1=1, m_2=0$ Indicial roots

Lesson: Equating the coefficients of different powers of x to zero we can determine the coefficient values

The coefficients of all other powers of x can be identified from

$$(l+m+1)(l+m)c_{l+1} - (l+m+1)^2 c_l = 0, \quad l = 0,1,2,3,4,\ldots$$

Allowed values of l

$$(l+m)c_{l+1} - (l+m+1)c_l = 0, \quad l = 0,1,2,3,4,\ldots$$

$$\boxed{c_{l+1} = \frac{(l+m+1)}{(l+m)} c_l, \qquad l = 0,1,2,3,4,\ldots.}$$

Recursion relation.

For $l = 0$, we can determine c_1 in terms of c_0

For $l = 1$, we can determine c_2 in terms of c_1

For $l = 2$, we can determine c_3 in terms of c_2

Coefficients can be determined recursively

We have,

$$\boxed{c_{l+1} = \frac{(l+m+1)}{(l+m)} c_l \qquad , l = 0,1,2,3,4,\ldots.}$$

$l = 0 \qquad c_1 = \frac{(m+1)}{m} c_0$

$l = 1 \qquad c_2 = \frac{(m+2)}{(m+1)} c_1 = \frac{(m+2)\cancel{(m+1)}}{m\cancel{(m+1)}} c_0 = \frac{(m+2)}{m} c_0$

$l = 2 \qquad c_3 = \frac{(m+3)}{(m+2)} c_2 = \frac{(m+3)\cancel{(m+2)}}{m\cancel{(m+2)}} c_0 = \frac{(m+3)}{m} c_0$

$l = 3 \qquad c_4 = \frac{(m+4)}{(m+3)} c_3 = \frac{(m+4)\cancel{(m+3)}}{m\cancel{(m+3)}} c_0 = \frac{(m+4)}{m} c_0$

$\vdots \qquad \vdots$

We have,

$$y(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+m} = \sum_{n=0}^{\infty} c_n x^{n+m}$$

$$= x^m \left[\mathrm{c}_0 + \frac{(m+1)}{m} c_0 x + \frac{(m+2)}{m} c_0 x^2 + \frac{(m+3)}{m} c_0 x^3 + \frac{(m+4)}{m} c_0 x^4 + \ldots . \right]$$

$$= x^m c_0 \left[1 + \frac{(m+1)}{m} x + \frac{(m+2)}{m} x^2 + \frac{(m+3)}{m} x^3 + \frac{(m+4)}{m} x^4 + \ldots . \right]$$

Substituting $\boxed{m = 0 \text{ provide } y_1(x) \text{ and } m = 1 \text{ provide } y_2(x)}$

Case $m = 0$:

- Substituting $m = 0$ in the solution we get $y(x) = \infty$ (not admissible solution)
- We can easily overcome this obstacle

We have,

$$y(x) = c_0 x^m \left[1 + \frac{(m+1)}{m}x + \frac{(m+2)}{m}x^2 + \frac{(m+3)}{m}x^3 + \frac{(m+4)}{m}x^4 + \dots\right]$$

$$= \frac{c_0}{m} x^m [m + (m+1)x + (m+2)x^2 + (m+3)x^3 + (m+4)x^4 + \dots]$$

$\boxed{d = \text{ another constant}}$ $\boxed{d = \frac{c_0}{m}}$

$$y_1(x) = d\, x^m [m + (m+1)x + (m+2)x^2 + (m+3)x^3 + (m+4)x^4 + \dots]$$

At $m = 0$:

$$y_1(x) = d[x + 2x^2 + 3x^3 + 4x^4 + \dots]$$

At $m = 1$:

$$y_2(x) = dx[1 + 2x + 3x^2 + 4x^3 + 5x^4 + \dots]$$

$$y_2(x) = d[x + 2x^2 + 3x^3 + 4x^4 + 5x^5 + \dots]$$

$y_2(x)$ is **NOT** AN INDEPENDENT SOLUTION

Note: We have obtained only one solution **(Read Frobenius Theorem Carefully)**

How to Obtain Second Solution?

Frobenius' Theorem

If $x = x_0$ is a regular singular point of the differential equation, then there exists **at least one solution of the form**

$$y = (x - x_0)^r \sum_{n=0}^{\infty} c_n (x - x_0)^n = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r}$$

Where the number r is a constant to be determined. The series will converge at least on some interval $0 < x - x_0 < R$.

$$m_1 \neq m_2$$
$$m_1 - m_2 = \text{an integer}$$

How to determine the second solution

Method of Reduction of Order

$$y_2(x) = y_1(x) \int \frac{1}{y_1^2(x)} \, e^{-\int p\,dx} \, dx$$

Method of partial differentiation

$$y_2(x) = \frac{\partial}{\partial m} \{(m - m_2) y(x, m)\}|_{m=m_2}$$

Method of substitution

$$y_2(x) = c \, y_1(x) \log x + \sum b_n x^{n+m_2}$$

$$y_1(x) = \sum_{n=0}^{\infty} c_n \, x^{n+m_1}$$

$$y_2(x) = c \, y_1(x) \log x + \sum_{n=0}^{\infty} b_n \, x^{n+m_2}$$

A Note on Differentiation

Suppose we have $y = x^m$ then $\dfrac{dy}{dx} = m\,x^{m-1}$

Suppose we have $y = m^x$ then $\dfrac{dy}{dx} =??$

Answer :

$$y = m^x \Longrightarrow log\ y = log\ m^x \quad = x \log m$$

Differentiating on both sides

$$\frac{1}{y}\frac{dy}{dx} = \log m + x.0 \qquad = \log m$$

$$\frac{dy}{dx} = y \log m \boxed{= m^x \log m}$$

Method of Partial Differentiation

Obtaining: Second independent solution

$$y_2(x) = \frac{\partial}{\partial m}\{(m - m_2)y(x,m)\}\Bigg|_{m=0}$$

$\boxed{m_2 \ - \text{Lowest indicial root}}$

In our case, $m_1 = 1, \qquad m_2 = 0$

$$y_2(x) = \frac{\partial}{\partial m}\{(m)y(x,m)\}$$

$$y_2(x) = \frac{\partial}{\partial m}\left\{m\frac{c_0}{m}\,x^m[m + (m+1)x + (m+2)x^2 + (m+3)x^2 + (m+4)x^4 + \ldots.]\right\}_{m=0}$$

$$= \frac{\partial}{\partial m}\{c_0\,x^m\underbrace{[m + (m+1)x + (m+2)x^2 + (m+3)x^2 + (m+4)x^4 + \ldots.]}\}_{m=0}$$

$$= c_0\,[x^m \log x[m + (m+1)x + (m+2)x^2 + (m+3)x^2 + \ldots.]\,]_{m=0}$$

$$+c_0x^m[1 + x + x^2 + x^3 + x^4 + \ldots.]_{m=0}$$

$$y_2(x) = c_0 \log x\,[x + 2x^2 + 3x^3 + 4x^4 + \ldots\ldots] + c_0\,[1 + x + x^2 + x^3 + x^4 + \ldots\ldots]$$

$$y_2(x) = y_1(x) \log x + \sum_{n=0}^{\infty} x^n$$

$$y_{GS} = Ay_1(x) + By_2(x)$$

$$\boxed{y_{GS} = A[x + 2x^2 + 3x^3 + 4x^4 + \ldots\ldots] + B \log x\,[x + 2x^2 + 3x^3 + 4x^4 + \ldots\ldots] + \mathrm{B} \sum_{n=0}^{\infty} x^n}$$

$m_1 \neq m_2$

$m_1 - m_2 =$ an integer

How to determine the second solution

Method of Reduction of Order

$$y_2(x) = y_1(x) \int \frac{1}{y_1^2(x)} \, e^{-\int p dx} \, dx$$

Method of partial differentiation

$$y_2(x) = \frac{\partial}{\partial m} \{(m - m_2) y(x, m)\}|_{m=m_2}$$

Method of substitution

$$y_2(x) = c \, y_1(x) \log x + \sum b_n x^{n+m_2}$$

$$y_1(x) = \sum_{n=0}^{\infty} c_n \, x^{n+m_1}$$

$$y_2(x) = c \, y_1(x) \log x + \sum_{n=0}^{\infty} b_n \, x^{n+m_2}$$

Method of Reduction of Order

Recall

$$x(1-x)y'' - 3xy' - y = 0$$

$$y_2(x) = y_1(x)\int \frac{e^{-\int P\,dx}}{y_1^2(x)}\,dx$$

Step 1: $$\int P\,dx = \int \frac{3}{x-1}dx = 3\log(x-1) = \log(x-1)^3$$

Step 2: $$e^{-\int P\,dx} = e^{-\log(x-1)^3} = \frac{1}{(x-1)^3}$$

Step 3: $$\frac{1}{y_1^2(x)} = \frac{1}{\left(\frac{x}{(1-x)^2}\right)^2} = \frac{(1-x)^4}{x^2}$$

Step 4: $$\int \frac{e^{-\int P\,dx}}{y_1^2(x)}\,dx = \int \frac{1}{(x-1)^3}.\frac{(1-x)^4}{x^2}\,dx = \int \frac{(x-1)^4}{(x-1)^3x^2}\,dx$$

$$= \int \frac{(x-1)}{x^2}\,dx = -\int \frac{(1-x)}{x^2}\,dx = -\int \frac{dx}{x^2} + \int \frac{dx}{x} = \frac{1}{x} + \log x$$

Step 5: $$y_1(x)\int \frac{e^{-\int P\,dx}}{y_1^2(x)}dx = \frac{x}{(1-x)^2}\,\frac{(1+x\log x)}{x}$$

$$= \frac{1+x\log x}{(1-x)^2} = \frac{x\log x}{(1-x)^2} + \frac{1}{(1-x)^2}$$

$$y_{GS} = Ay_1(x) + By_2(x) = A\frac{x}{(1-x)^2} + B\frac{1}{(1-x)^2} + B\frac{\boldsymbol{x}\log x}{(\mathbf{1}-\boldsymbol{x})^{\mathbf{2}}}$$

$$= Ay_1(x) + By_1(x)\log x + B[1 + 2x + 3x^2 + 4x^3 + \ldots\ldots]$$

$$y_{GS} = Ay_1(x) + By_1(x)\log x + B\sum_{n=0}^{\infty}(n+1)x^n$$

Same result as obtained through first method

$$m_1 \neq m_2$$
$$m_1 - m_2 = \text{an integer}$$

How to determine the second solution

Method of Reduction of Order

$$y_2(x) = y_1(x) \int \frac{1}{y_1^2(x)}\, e^{-\int p\,dx}\, dx$$

Method of partial differentiation

$$y_2(x) = \frac{\partial}{\partial m}\{(m - m_2)y(x,m)\}|_{m=m_2}$$

Method of substitution

$$y_2(x) = c\, y_1(x) \log x + \sum b_n x^{n+m_2}$$

$$y_1(x) = \sum_{n=0}^{\infty} c_n\, x^{n+m_1}$$

$$y_2(x) = c\, y_1(x) \log x + \sum_{n=0}^{\infty} b_n\, x^{n+m_2}$$

Method of Substitution

$$y_2(x) = ky_1(x)\log x + \sum_{n=0}^{\infty} a_n x^{n-m_2}$$

$$\boxed{\begin{aligned} m &= m_1 - m_2 > 0 \\ &= 1 > 0 \end{aligned}}$$

$$y_2(x) = ky_1(x)\log x + \sum_{n=0}^{\infty} a_n x^{n}$$

$$y_2'(x) = ky_1'(x)\log x + \frac{ky_1}{x} + \sum_{n=0}^{\infty} a_n n x^{n-1}$$

$$y_2''(x) = ky''_1(x)\log x + \frac{2ky_1'}{x} - \frac{ky_1}{x^2} + \sum_{n=0}^{\infty} a_n n(n-1)x^{n-2}$$

Substituting in the given equation $x(1-x)y'' - 3xy' - y = 0$

$$x(1-x)\left\{ky_1''(x)\log x + \frac{2ky_1'}{x} - \frac{ky_1}{x^2} + \sum_{n=0}^{\infty} a_n n(n-1)x^{n-2}\right\}$$

$$-3x\left\{ky_1'(x)\log x + \frac{ky_1}{x} + \sum_{n=0}^{\infty} a_n n x^{n-1}\right\} - \left\{ky_1(x)\log x + \sum_{n=0}^{\infty} a_n x^{n}\right\} = 0$$

$$k\log x\,\underbrace{\{x(1-x)y'' - 3xy' - y\}}_{=0} + 2(1-x)ky_1' - \frac{ky_1}{x} - 2ky_1$$

$$+\sum_{n=0}^{\infty} a_n n(n-1)x^{n-1} - \sum_{n=0}^{\infty} a_n (n+1)^2 x^{n} = 0$$

$$2(1-x)ky_1' - \frac{ky_1}{x} - 2ky_1 + \sum_{n=0}^{\infty} a_n n(n-1)x^{n-1} - \sum_{n=0}^{\infty} a_n (n+1)^2 x^n = 0$$

$$\therefore y_1(x) = x + 2x^2 + 3x^3 + 4x^4 + \cdots \Rightarrow y_1'(x) = 1 + 4\mathrm{x} + 9\mathrm{x}^2 + 16\mathrm{x}^3 + \ldots$$

$$2(1-x)k[1 + 4\mathrm{x} + 9\mathrm{x}^2 + 16\mathrm{x}^3 + \ldots] - \frac{k}{x}[x + 2x^2 + 3x^3 + 4x^4 + \cdots]$$
$$-2k[x + 2x^2 + 3x^3 + 4x^4 + \cdots] + \{2a_2x + 6a_3x^2 + 12a_4x^3 + \cdots\}$$
$$-\{a_0 + 4a_1x + 9a_2x^2 + 16a_3x^3 + ..\} = 0$$

$\underline{x^0}$ $\quad a_0 = k$

$\underline{x^1}$ $\quad 2k + 2a_0 - 4a_1 = 0 \Rightarrow a_2 = 2a_1 - a_0$

$\underline{x^2}$ $\quad 3\mathrm{k} + 6\mathrm{a}_3 - 9\mathrm{a}_2 = 0 \Rightarrow a_3 = 3a_1 - 2a_0$

$\underline{x^3}$ $\quad 4k + 12a_4 - 16a_3 = 0 \Rightarrow a_4 = 4a_1 - 3a_0$

Substituting back in $y_2(x) = ky_1(x)\log x + \sum_{n=0}^{\infty} a_n x^n$

$$y_2(x) = a_0 y_1(x)\log x + a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$$

$$y_2(x) = a_0 y_1(x)\log x + a_0 + a_1x + 2a_1x^2 - a_0x^2 + 3a_1x^3 - 2a_0x^3 + \ldots$$

$$= a_0\, y_1(x)\log x + a_0(1 - x^2 - 2x^3 - \cdots) + a_1(x + 2x^2 + 3x^3 + \cdots)$$

$$= y_1(x)\log x + (1 + x + x^2 + x^3 + \cdots)$$

$y_2\ (x)$=$y_1(x)\log x$ + $(1 + x + x^2 + x^3 + \cdots)$

$$y_{G.S} = Ay_1(x) + By_2(x)$$

$$= Ay_1(x) + By_1(x)\log x \ + B[1 + x + x^2 + x^3 + \ldots\ldots]$$

$$= Ay_1\ (x) + By_1(x)\log x \ + B\sum_{n=0}^{\infty} x^n$$

$m_1 \neq m_2$

$m_1 - m_2 =$ an integer

How to determine the second solution

Method of Reduction of Order

$$y_2(x) = y_1(x) \int \frac{1}{y_1^2(x)}\, e^{-\int p\,dx}\, dx$$

Method of partial differentiation

$$y_2(x) = \frac{\partial}{\partial m}\{(m - m_2)y(x,m)\}\big|_{m=m_2}$$

Method of substitution

$$y_2(x) = c\, y_1(x)\log x + \sum b_n x^{n+m_2}$$

Note :

(i) All methods are not equally suitable for a DE.

(ii) One method may find a second solution more easily than the other two methods.

$$y_1(x) = \sum_{n=0}^{\infty} c_n\, x^{n+m_1}$$

$$y_2(x) = c\, y_1(x)\log x + \sum_{n=0}^{\infty} b_n\, x^{n+m_2}$$

Summary

$x(1-x)y'' - 3xy' - y = 0$ Indicial roots $m_1 = 1, m_2 = 0$

At $m = 0$

$$y_1(x) = d[x + 2x^2 + 3x^3 + 4x^4 + \ldots.]$$ **(First Solution)**

$$y_2(x) = c_0 \log x\, [x + 2x^2 + 3x^3 + 4x^4 + \ldots..] + c_0\, [1 + x + x^2 + x^3 + x^4 + \ldots..]$$

(Second Solution)

$$\boxed{y_{GS} = A[x + 2x^2 + 3x^3 + 4x^4 + \ldots..] + B \log x\, [x + 2x^2 + 3x^3 + 4x^4 + \ldots..] + \mathrm{B} \sum_{n=0}^{\infty} x^n}$$

We have constructed the second solution $y_2(x)$ through three different series

Case 2

$$m_1 \neq m_2$$

$$m_1 - m_2 = \text{an integer}$$

Without Logarithmic term in the Second Solution

Use the method of Frobenius to obtain the series solution about x=0

$$xy'' + 2y' - xy = 0$$

Step 1: $\boxed{x_o = 0 \text{ is a Regular Singular Point}}$

Step 2: Substituting $y(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+m}$ in the equation, we obtain

$$y'(x) = \sum_{n=0}^{\infty} (n+m) c_n x^{n+m-1} \quad ; \; y''(x) = \sum_{n=0}^{\infty} (n+m)(n+m-1) c_n x^{n+m-2}$$

$$\sum_{n=0}^{\infty} (n+m)(n+m-1) c_n x^{n+m-1} + 2 \sum_{n=0}^{\infty} (n+m) c_n x^{n+m-1} - \sum_{n=0}^{\infty} c_n x^{n+m+1} = 0$$

$$\sum_{n=0}^{\infty} (n+m)(n+m+1) c_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^{n+1} = 0$$

$$m(m+1) c_0 x^{-1} + \sum_{n=1}^{\infty} (n+m)(n+m+1) c_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^{n+1} = 0$$

Indicial roots $m_1 = 0, m_2 = -1.$

$$(m+1)(m+2)c_1+\sum_{n=2}^{\infty}(n+m)(n+m+1)c_n x^{n-1}-\sum_{n=0}^{\infty}c_n x^{n+1}=0$$

$$(m+1)(m+2)c_1+\sum_{l=1}^{\infty}(l+m+1)(l+m+2)c_{l+1}x^{l}-\sum_{l=1}^{\infty}c_{l-1}x^{l}=0$$

$$\boxed{c_{l+1}=\frac{c_{l-1}}{(l+m+1)(l+m+2)} \qquad ,l=1,2,3,4,\ldots.}$$

Case 1: $m_1=0$ $\Rightarrow$ $\boxed{c_{l+1}=\frac{c_{l-1}}{(l+1)(l+2)},}$ $\boxed{c_1=0}$

$l=1$ $\Rightarrow$ $c_2=\frac{c_0}{3!}$

$l=2$ $\Rightarrow$ $c_3=0$

$l=3$ $\Rightarrow$ $c_4=\frac{c_0}{5!}$

$l=4$ $\Rightarrow$ $c_5=0$

$l=5$ $\Rightarrow$ $c_6=\frac{c_0}{7!}$

$$y(x)=\sum_{n=0}^{\infty}c_n x^{n+m} \quad =[c_0+c_2x^2+c_4x^4+c_6\,x^6+\ldots]$$

$$y_1 = c_0\left(1 + \frac{x^2}{3!} + \frac{x^4}{5!} + \frac{x^6}{7!}\dots\right) \quad = \frac{c_0}{x}\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!}\dots\right) \quad = \frac{c_0}{x}\sinh x$$

Case 2: $m_2 = -1$ $\Rightarrow$ $\boxed{c_{l+1} = \frac{c_{l-1}}{l(l+1)} \qquad l = 1,2,3,4,\dots}$

$l = 1 \Rightarrow c_2 = \frac{c_0}{2!}$

$l = 2 \Rightarrow c_3 = \frac{c_1}{3!}$

$l = 3 \Rightarrow c_4 = \frac{c_0}{4!}$

$l = 4 \Rightarrow c_5 = \frac{c_1}{5!}$

$l = 5 \Rightarrow c_6 = \frac{c_0}{6!}$

$$y_2(x) = \sum_{n=0}^{\infty} c_n x^{n-1} = \left[\frac{c_0}{x} + c_1 + c_2\mathrm{x} + c_3 x^2 + \dots\right] = \left[\frac{c_0}{x} + \mathrm{c}_1 + \frac{c_0}{2!}\mathrm{x} + \frac{c_1}{3!}x^2 + \frac{c_0}{4!}x^3 + \dots\right]$$

$$= c_0\left(\frac{1}{x} + \frac{x}{2!} + \frac{x^3}{4!} + \dots\right) + c_1\left(1 + \frac{x^2}{3!} + \frac{x^4}{5!} + \dots\right)$$

$$= \frac{c_0}{x}\cosh x + \frac{c_1}{x}\sinh x$$

Let us determine the second solution y_2 using **METHOD -1**:

$$y_2(x) = y_1(x)\int \frac{e^{-\int P\,dx}}{y_1^2(x)}\,dx$$

Recall:

$$xy'' + 2y' - xy = 0$$

$$y_1(x) = \frac{1}{x}\sinh x$$

Step 1: $\int P(x)\,\mathrm{d}x = \int \frac{2}{x}dx = 2\log x = \log x^2$

Step 2: $\mathrm{e}^{-\int P\,\mathrm{d}x} = \mathrm{e}^{-\log x^2} = \frac{1}{x^2}$

Step 3: $$\int \frac{\mathrm{e}^{-\int P\,\mathrm{d}x}}{y_1^2(x)}\mathrm{d}x = \int \frac{\frac{1}{\cancel{x^2}}}{\frac{\sinh^2 x}{\cancel{x^2}}}\mathrm{d}x = \int \frac{1}{\sinh^2 x}\mathrm{d}x = \frac{\cosh x}{\sinh x}$$

Step 4: $$y_2(x) = y_1(x)\int \frac{e^{-\int P\,dx}}{y_1^2(x)}dx = \frac{\sinh x}{x}\,\frac{\cosh x}{\sinh x} = \frac{\cosh x}{x}$$

$$y_{G.S} = Ay_1(x) + By_2(x) = A\frac{\sinh x}{x} + B\frac{\cosh x}{x}$$

Method of Substitution:

$$y_2(x) = ky_1(x)\log x + \sum_{n=0}^{\infty} a_n x^{n+m_2} \qquad y_2(x) = ky_1(x)\log x + \sum_{n=0}^{\infty} a_n x^{n-1}$$

$$y_2'(x) = ky_1'(x)\log x + \frac{ky_1}{x} + \sum_{n=0}^{\infty} a_n(n-1)x^{n-2}$$

$$y_2''(x) = ky''_1(x)\log x + \frac{2ky_1'}{x} - \frac{ky_1}{x^2} + \sum_{n=0}^{\infty} a_n(n-1)(n-2)x^{n-3}$$

Substituting in the given equation $xy'' + 2y' - xy = 0$

$$x\left\{ky_1''(x)\log x + \frac{2ky_1'}{x} - \frac{ky_1}{x^2} + \sum_{n=0}^{\infty} a_n(n-1)(n-2)x^{n-3}\right\}$$

$$+\,2\left\{ky_1'(x)\log x + \frac{ky_1}{x} + \sum_{n=0}^{\infty} a_n(n-1)x^{n-2}\right\} - \left\{xky_1(x)\log x + \sum_{n=0}^{\infty} a_n x^{n+1}\right\} = 0$$

$$k\log x\,\underbrace{\{xy'' + 2y' - xy\}}_{=\,0} + 2ky_1' + \frac{ky_1}{x} + \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\therefore y_1(x) = 1 + \frac{x^2}{6} + \frac{x^4}{120} + \cdots \quad \Rightarrow \quad y_1'(x) = \frac{x}{3} + \frac{x^3}{30} + \ldots$$

$$2k\left[\frac{x}{3} + \frac{x^3}{30} + \ldots\right] + \frac{k}{x}\left[1 + \frac{x^2}{6} + \frac{x^4}{120} + \cdots\right] + \{2a_2 + 6a_3x + 12a_4x^2 + \cdots\}$$

$$-\{a_0 + a_1x + a_2x^2 + a_3x^3 + ..\} = 0$$

$\underline{x^{-1}}$ $\quad k = 0$

$\underline{x^0}$ $\quad 2a_2 - a_0 = 0 \quad \Rightarrow \quad a_2 = \frac{a_0}{2!}$

$\underline{x^1}$ $\quad 6\mathrm{a}_3 - \mathrm{a}_1 = 0 \quad \Rightarrow \quad a_3 = \frac{a_1}{3!}$

$\underline{x^2}$ $\quad 12a_4 - a_2 = 0 \quad \Rightarrow \quad a_4 = \frac{a_0}{4!}$

Substituting back in $y_2(x) = ky_1(x)\log x + \sum_{n=0}^{\infty} a_n x^{n-1} = \frac{a_0}{x} + a_1 + a_2x + a_3x^2 + \ldots$

$$= a_0\left(\frac{1}{x} + \frac{x}{2!} + \frac{x^3}{4!} + \ldots\right) + a_1\left(1 + \frac{x^2}{3!} + \frac{x^4}{5!} + \ldots\right)$$

$$= \frac{a_0}{x}\cosh x + \frac{a_1}{x}\sinh x$$

$$y_{G.S} = Ay_1(x) + By_2(x) = C\frac{\sinh x}{x} + B\frac{\cosh x}{x}$$

Method of partial differentiation:

$$y_2(x) = \frac{\partial}{\partial m}\{(m - m_2)y(x,m)\}$$

In our case, $m_1 = 0,$ $m_2 = -1$

$$y_2(x) = \frac{\partial}{\partial m}\{(m+1)y(x,m)\}$$

Recursion relation $\longrightarrow$ $\boxed{c_{l+1} = \frac{c_{l-1}}{(l+m+1)(l+m+2)} \quad ,l = 1,2,3,4,\ldots.}$

$$c_2 = \frac{c_0}{(m+2)(m+3)} \;;\; c_3 = \frac{c_1}{(m+3)(m+4)} \;;\; c_4 = \frac{c_0}{(m+2)(m+3)(m+4)(m+5)}$$

$$y(x,m) = c_0 + c_1 x + \frac{c_0}{(m+2)(m+3)}x^2 + \frac{c_1}{(m+3)(m+4)}x^3 + \cdots$$

Substituting $y(x,m)$ in $y_2(x)$, we obtain

$$y_2(x) = \frac{\partial}{\partial m}\left\{(m+1)\;\; x^m\left[c_0 + c_1 x + \frac{c_0}{(m+2)(m+3)}x^2 + \frac{c_1}{(m+3)(m+4)}x^3 + \cdots\right]\right\}$$

$$y_2(x) = \left\{ x^m \log x (m+1) \left[c_0 + c_1 x + \frac{c_0}{(m+2)(m+3)} x^2 + \frac{c_1}{(m+3)(m+4)} x^3 + \cdots \right] \right\}_{m=-1}$$

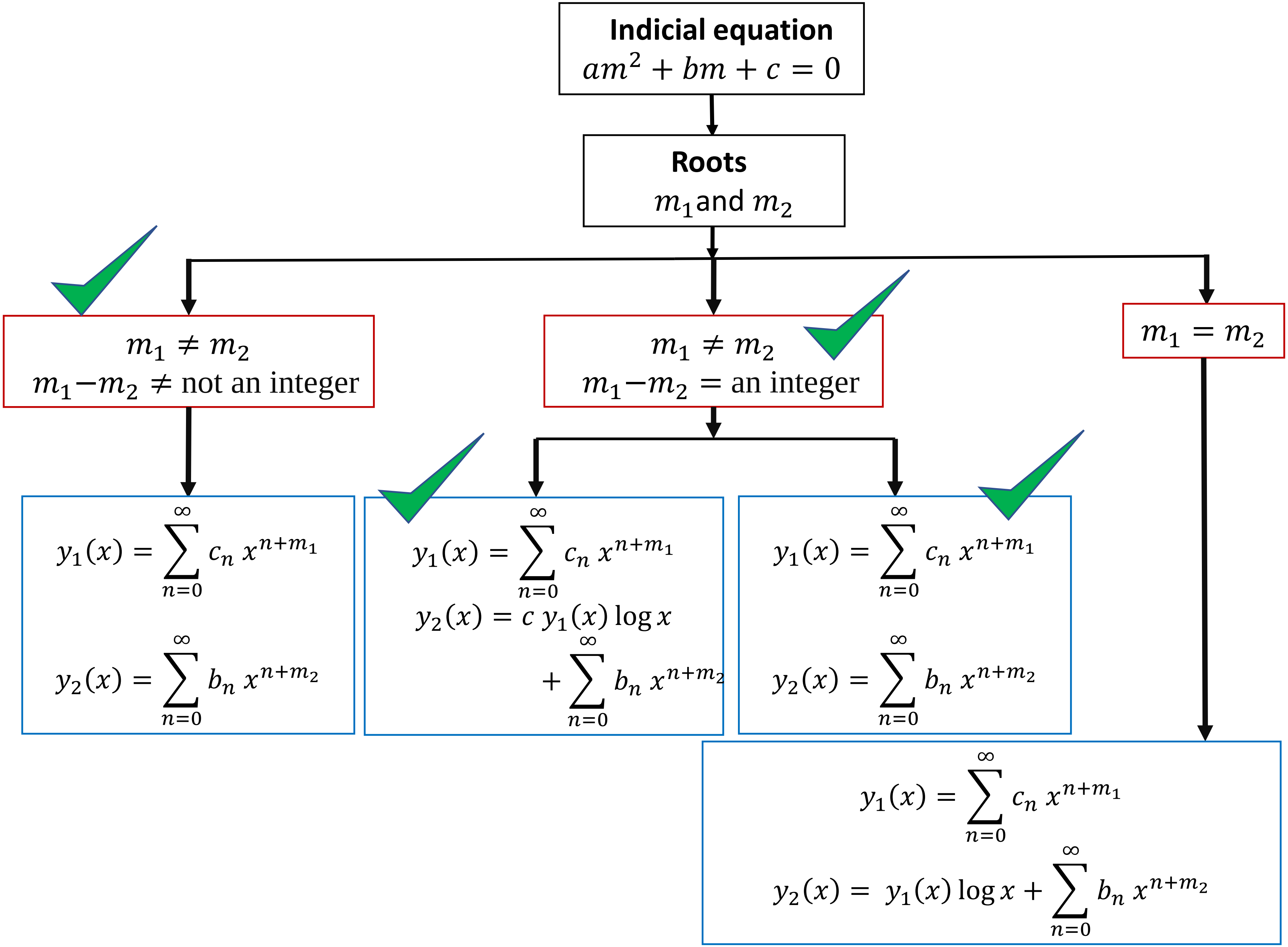
$$+c_0 x^m \left(1 - \frac{(m^2+2m-1)}{(m+2)^2(m+3)^2} x^2 - \cdots \right)_{m=-1} +c_1 x^m \left(x - \frac{(m^2+2m-5)}{(m+3)^2(m+4)^2} x^3 - \cdots \right)_{m=-1}$$

$$= \frac{c_0}{x} \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots \right) + \frac{c_1}{x} \left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} \ldots \right)$$

$$= \frac{c_0}{x} \cosh x + \frac{c_1}{x} \sinh x$$

$$y_{G.S} = A y_1(x) + B y_2(x)$$

$$= A \frac{\sinh x}{x} + B \frac{\cosh x}{x}$$



Indicial equation
$am^2 + bm + c = 0$
Roots
m_1 and m_2
$m_1 \neq m_2$
$m_1 - m_2 \neq$ not an integer
$m_1 \neq m_2$
$m_1 - m_2 =$ an integer
$m_1 = m_2$
$y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+m_1}$
$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+m_2}$
$y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+m_1}$
$y_2(x) = c\, y_1(x) \log x + \sum_{n=0}^{\infty} b_n x^{n+m_2}$
$y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+m_1}$
$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+m_2}$
$y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+m_1}$
$y_2(x) = y_1(x) \log x + \sum_{n=0}^{\infty} b_n x^{n+m_2}$

Frobenius' Series: Case 3

$$m_1 = m_2$$

Case 3: Roots of Indicial equation equal.

Solve using $xy'' + y' + xy = 0$ Frobenius series method

Solution: $y'' + \left(\frac{1}{x}\right)y' + y = 0$

Step 1: $P(x) = \frac{1}{x}$ and $Q(x) = 1$ $\Rightarrow$ $x = 0$ is a Singular point.

Step 2:

$$\left.\begin{aligned} xP(x) &= 1 \\ x^2Q(x) &= x^2 \end{aligned}\right\}$$

Both are analytic at $x = 0$

$\therefore x = 0$ is a Regular Singular point.

We can apply Frobenius series method

Step 3:

$$y'(x) = \sum_{n=0}^{\infty}(n+m)c_n x^{n+m-1}$$

$$y(x) = \sum_{n=0}^{\infty} c_n x^{n+m}$$

$$y''(x) = \sum_{n=0}^{\infty}(n+m)(n+m-1)c_n x^{n+m-2}$$

Substituting in $xy'' + y' + xy = 0$

$$x\sum_{n=0}^{\infty}(n+m)(n+m-1)c_n x^{n+m-2}+\sum_{n=0}^{\infty}(n+m)c_n x^{n+m-1}+x\sum_{n=0}^{\infty}c_n x^{n+m}=0$$

$$\sum_{n=0}^{\infty}(n+m)(n+m-1)c_n x^{n+m-1}+\sum_{n=0}^{\infty}(n+m)c_n x^{n+m-1}+\sum_{n=0}^{\infty}c_n x^{n+m+1}=0$$

Upon Simplification,

1 $$m^2c_0x^{-1}+(m+1)^2c_1+\sum_{l=1}^{\infty}[(m+l+1)^2c_{l+1}+c_{l-1}]x^l=0$$

Lowest power of x

2 Indicial roots $m^2=0 \rightarrow m=0,0$ **Equal roots**

3 $c_1=0$

4 Recurrence relation $$\boxed{c_{l+1}=-\frac{c_{l-1}}{(m+l+1)^2}, \qquad l=1,2,3,\ldots}$$

Verify all the steps (H. W)

$$c_{l+1} = -\frac{c_{l-1}}{(m+l+1)^2} \quad , \qquad l = 1,2,3,\ldots$$

$\underline{l=1}$ $$c_2 = \frac{-c_0}{(m+2)^2}$$

$\underline{l=2}$ $$c_3 = \frac{-c_1}{(m+3)^2} = 0$$

$\underline{l=3}$ $$c_4 = \frac{-c_2}{(m+4)^2} = \frac{c_0}{(m+4)^2(m+2)^2}$$

$\underline{l=4}$ $$c_5 = \frac{-c_3}{(m+5)^2} = 0$$

$\underline{l=5}$ $$c_6 = \frac{-c_0}{(m+2)^2(m+4)^2(m+6)^2}$$

$$y(x) = x^m \sum_{n=0}^{\infty} c_n x^n$$

$$y(x) = x^m \left(c_0 - \frac{c_0}{(m+2)^2} x^2 + \frac{c_0}{(m+4)^2(m+2)^2} x^4 - \frac{c_0}{(m+2)^2(m+4)^2(m+6)^2} x^6 + .. \right)$$

First solution:
$m = 0$

$$\boxed{y(x) = c_0 \left(1 - \frac{x^2}{2^2} + \frac{x^4}{2^2.4^2} - \frac{x^6}{2^2.4^2.6^2} + .. \right)}$$

Second solution

$$y_2(x) = \frac{\partial}{\partial m}\{y(x,m)\} \Bigg|_{m = m_1}$$

We know,

$$y(x,m) = c_0 x^m \underbrace{\left(1 - \frac{x^2}{(m+2)^2} + \frac{x^4}{(m+4)^2(m+2)^2} + \frac{x^6}{(m+2)^2(m+4)^2(m+6)^2} + \ldots\right)}$$

$$y(x,m) = u(x,m)\,.\,v(x,m)$$

$$\frac{\partial y}{\partial m}(x,m) = \overset{\mathbf{1}}{\frac{\partial u}{\partial m}} v + \overset{\mathbf{2}}{\frac{\partial v}{\partial m}} u$$

1 $u = x^m \Rightarrow \dfrac{\partial u}{\partial m} = x^m \log x$

2 $v = 1 - \dfrac{x^2}{(m+2)^2} + \dfrac{x^4}{(m+4)^2(m+2)^2} + \dfrac{x^6}{(m+2)^2(m+4)^2(m+6)^2} + ..$

$$v = 1 - \frac{x^2}{(m+2)^2} + \frac{x^4}{(m+4)^2(m+2)^2} - \frac{x^6}{(m+2)^2(m+4)^2(m+6)^2} + ..$$

$$\frac{\partial v}{\partial m} = \frac{2x^2}{(m+2)^3} + x^4\left[\frac{-2}{(m+2)^3(m+4)^2} - \frac{2}{(m+2)^2(m+4)^3}\right]$$

$$-x^6\left[\frac{-2}{(m+2)^3(m+4)^2(m+6)^2} - \frac{2}{(m+2)^2(m+4)^3(m+6)^2} - \frac{-2}{(m+2)^2(m+4)^2(m+6)^3}\right] + \dots$$

$$= \frac{2x^2}{(m+2)^3} - \frac{2x^4}{(m+2)^3(m+4)^3}[(m+4)+(m+2)]$$

$$+ \frac{2x^6}{(m+2)^3(m+4)^3(m+6)^3}[(m+4)(m+6)+(m+2)(m+6)+(m+2)(m+4)] + \cdots$$

$$\frac{\partial v}{\partial m} = \frac{2x^2}{(m+2)^3} - \frac{2x^4(2m+6)}{(m+2)^3(m+4)^3} + \frac{2x^6(3m^2+24m+44)}{(m+2)^2(m+4)^3(m+6)^3} + \dots..$$

$$\frac{dy}{dm}(x,m) = \frac{\partial u}{\partial m} v + \frac{\partial v}{\partial m} u$$

$$\frac{dy}{dm} = x^m \log x \left\{ 1 - \frac{x^2}{(m+2)^2} + \frac{x^4}{(m+4)^2(m+2)^2} - \frac{x^6}{(m+2)^2(m+4)^2(m+6)^2} +.. \right\}$$

$$+x^m \left\{ \frac{2x^2}{(m+2)^3} - \frac{2x^4(2m+6)}{(m+2)^3(m+4)^3} + \frac{2x^6(3m^2+24m+44)}{(m+2)^3(m+4)^3(m+6)^3} + \ldots. \right\}$$

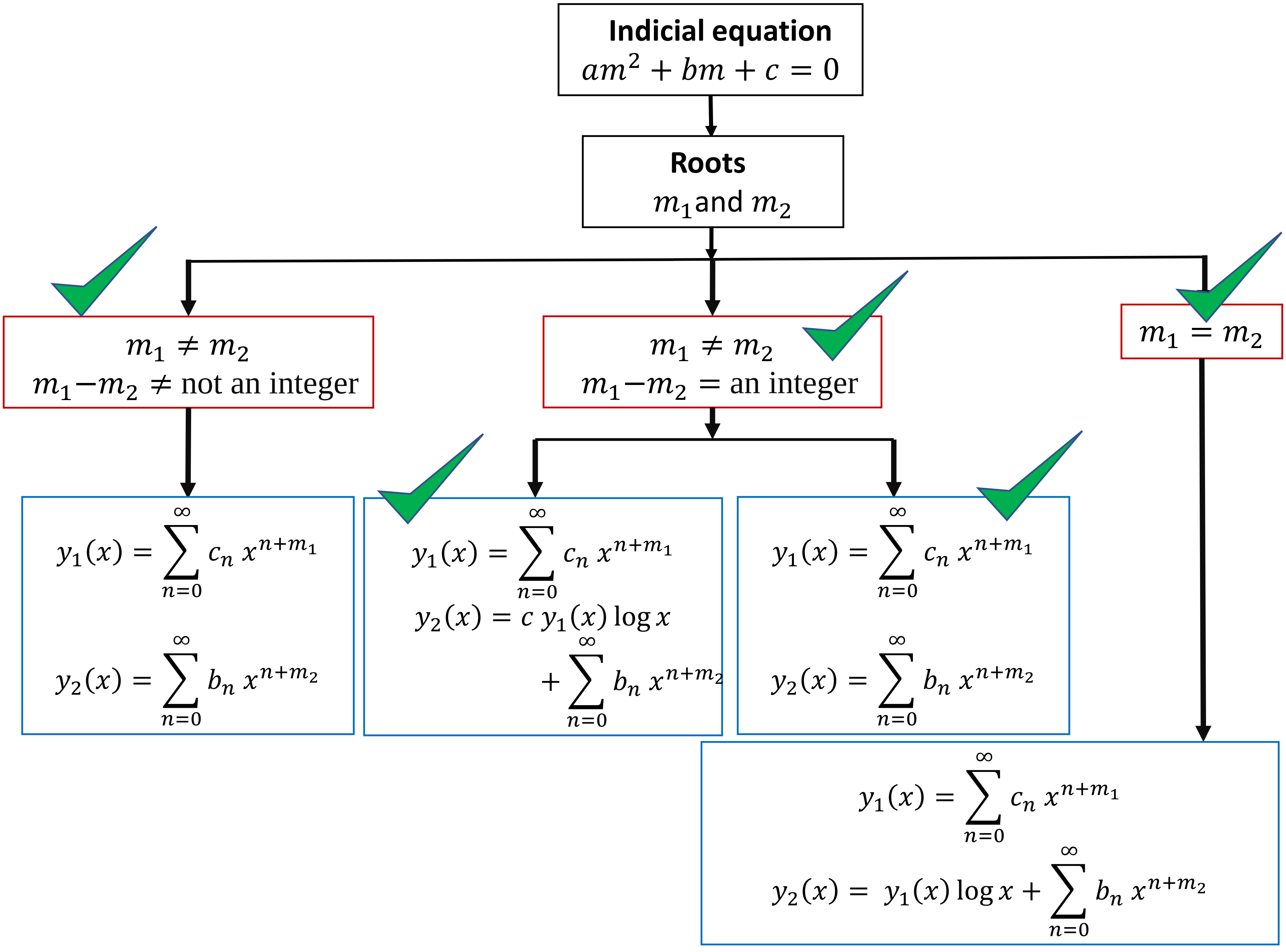
$$\left.\frac{dy}{dm}(x,m)\right|_{m=0} = \log x \left\{ 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2.4^2} - \frac{x^6}{2^2.4^2.6^2} +.. \right\} + 2x^2 \left\{ \frac{1}{2^3} - \frac{6x^2}{2^3.4^3} + \frac{44x^6}{2^3.4^3.6^3} + .. \right\}$$

$$y_2(x) = \log x \left\{ 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2.4^2} - \frac{x^6}{2^2.4^2.6^2} +.. \right\} + 2x^2 \left\{ \frac{1}{2^3} - \frac{6x^2}{2^3.4^3} + \frac{44x^6}{2^3.4^3.6^3} + .. \right\}$$

General Solution: $y(x) = Ay_1(x) + By_2(x)$

$$y_{GS} = A\left(1 - \frac{x^2}{2^2} + \frac{x^4}{2^2.4^2} - \frac{x^6}{2^2.4^2.6^2} +.. \right)$$

$$+B\left(\log x \; y_1(x) + 2x^2 \left\{ \frac{1}{2^3} - \frac{6x^2}{2^3.4^3} + \frac{44x^6}{2^3.4^3.6^3} + \ldots. \right\} \right)$$


Indicial equation
$am^2 + bm + c = 0$
Roots
m_1 and m_2
$m_1 \neq m_2$
$m_1 - m_2 \neq$ not an integer
$m_1 \neq m_2$
$m_1 - m_2 =$ an integer
$m_1 = m_2$
$y_1(x) = \sum_{n=0}^{\infty} c_n \, x^{n+m_1}$
$y_2(x) = \sum_{n=0}^{\infty} b_n \, x^{n+m_2}$
$y_1(x) = \sum_{n=0}^{\infty} c_n \, x^{n+m_1}$
$y_2(x) = c \, y_1(x) \log x + \sum_{n=0}^{\infty} b_n \, x^{n+m_2}$
$y_1(x) = \sum_{n=0}^{\infty} c_n \, x^{n+m_1}$
$y_2(x) = \sum_{n=0}^{\infty} b_n \, x^{n+m_2}$
$y_1(x) = \sum_{n=0}^{\infty} c_n \, x^{n+m_1}$
$y_2(x) = y_1(x) \log x + \sum_{n=0}^{\infty} b_n \, x^{n+m_2}$

References:

1) D. G. Zill and M. R. Cullen, Advanced Engineering Mathematics (Narosa, New Delhi, 2020).
2) E. Kreysig, Advanced Engineering Mathematics (John Wiley, New Delhi, 2011).
3) Srimanta Pal, Subodh C. Bhunia, Engineering Mathematics (Oxford University Press, 2015).
4) T. L. Chow, Mathematical Methods for Physicists: A Concise Introduction (Cambridge University Press, Cambridge, 2014).
5) K. F. Reily, M. P. Hobson and S. J. Bence, Mathematical Methods for Physics and Engineering (Cambridge University Press, Cambridge, 2006).
6) V. Balakrishnan, Mathematical Physics with Applications, Problems and Solutions (Ane Books, New Delhi, 2019).
7) B. S. Rajput, Mathematical Physics (Pragati Prakashan, Meerut, 2019).
8) G. B. Arfken, H. J. Weber and R. E. Harris, Mathematical Method for Physicists (Academic Press, Cambridge, 2011).
9) M. P. Boas, Mathematical Methods in the Physical Sciences (Wiley, New York, 2018).
10) L. Kantorovica, Mathematics for National Scientists Vols. I and II (Springer, New York, 2016).
11) James R. Schott, Matrix Analysis for Statistics (Wiley, New Jersey, 2017).
12) F. Ayres, Theory and Problems of Matrices (Schaum, New York, 1962).
13) https://nptel.ac.in/courses/115103036
14) http://www.issp.ac.ru/ebooks/books/open/Mathematical%20Methods.pdf