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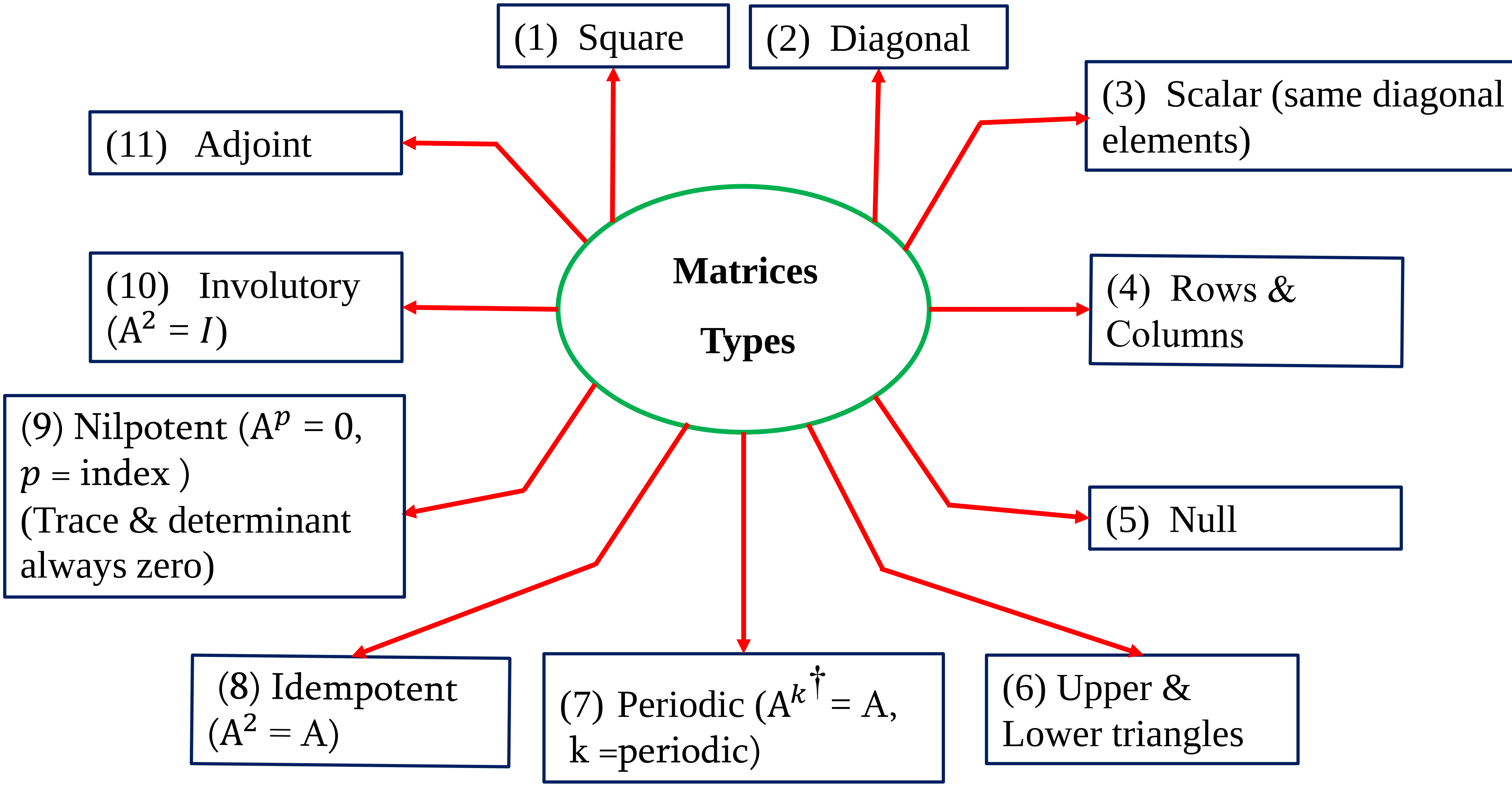
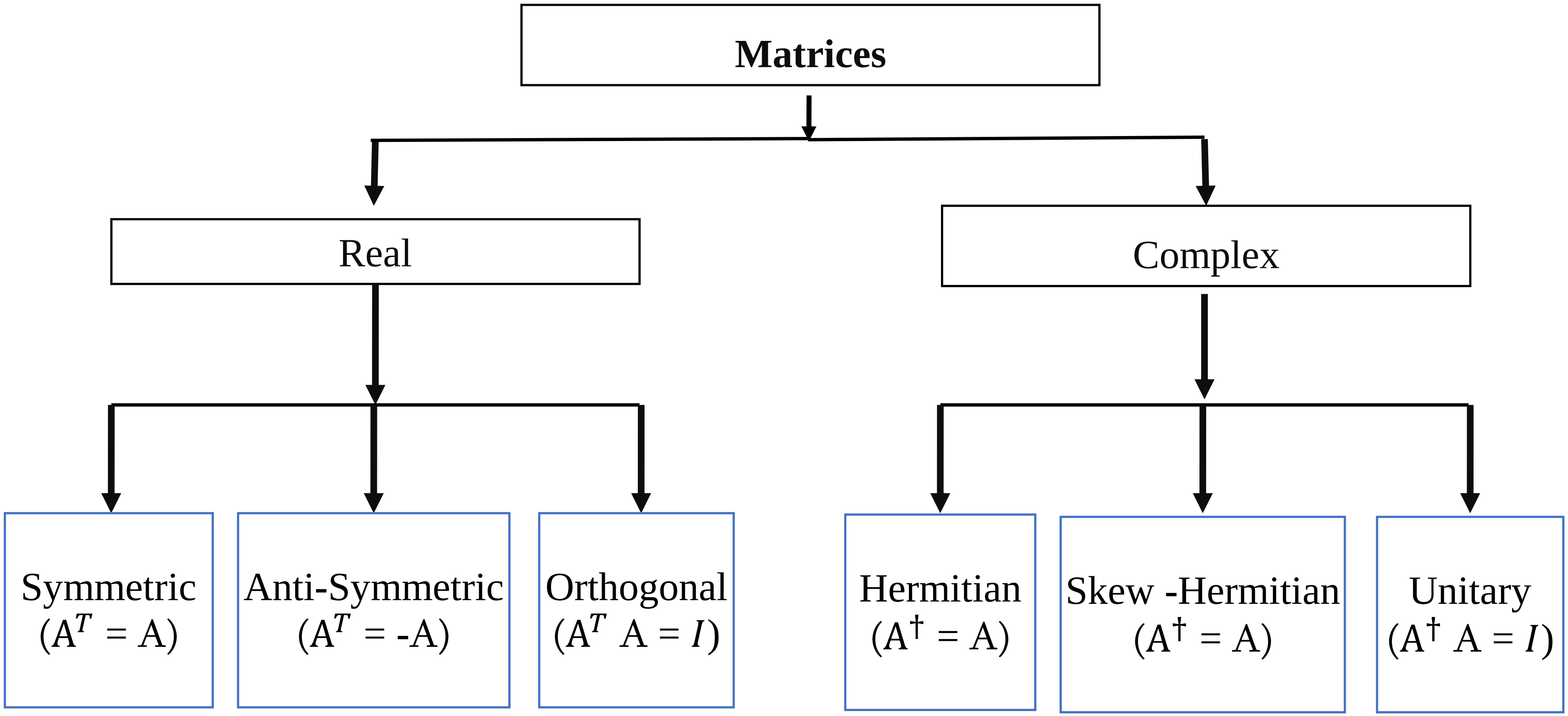
UNIT - III

MATRICES

Matrices


Matrices
Types
(1) Square
(2) Diagonal
(3) Scalar (same diagonal elements)
(4) Rows & Columns
(5) Null
(6) Upper & Lower triangles
(7) Periodic ($A^{k\dagger}$ = A, k =periodic)
(8) Idempotent (A^2 = A)
(9) Nilpotent (A^p = 0, p = index) (Trace & determinant always zero)
(10) Involutory (A^2 = I)
(11) Adjoint

Further Types



Properties of Hermitian, Skew – Hermitian & Unitary matrices

Hermitian matrix :

1. Principal diagonal elements of Hermitian matrices are always **real.**

2. Maximum number of real elements for the most general $n \times n$ **Hermitian matrix** is n**.**

3. If A is Hermitian matrix, the iA & - iA will be Anti-Hermitian.

4. If A is Hermitian matrix, kA will be Hermitian.

Skew – Hermitian matrix :

1. Principal diagonal elements of Skew-Hermitian are either zero or imaginary.

2. Maximum number of real elements for the most general $n \times n$ **Skew – Hermitian matrix** is 0**.**

Unitary matrix :

1. Each row and column of an Unitary matrix is a normalized vector.

2. Any two rows or any two column of an Unitary matrix are orthogonal to each other.

3. If A & B are Unitary matrices then AB& BA will be always unitary.

Orthogonal

1. Det(A) = ± 1
2. $A^T = A^{-1}$
3. Each row and column of an orthogonal matrix is a normalized vector.
4. Any two rows or any two column of an orthogonal matrices are orthogonal to each other.
5. If A & B are orthogonal matrices then AB& BA will be always orthogonal.

Adjoint

1. $(Adj\ \ A)\, A = A\, (Adj\ \ A) = |\, A\, |\, I$
2. If A is a square matrix of order n having determinant m then $|Adj\ \ A| = m^{n-1}$

Rank

1. The rank of a null matrix is zero.
2. The rank of a non-zero matrix is ≥ 1.
3. The rank of any non-singular matrix of order n is n .
4. The rank of a matrix that results from the product of two matrices cannot exceed the rank of either of matrix.

Rank of a Matrix

Example 1 : **Find the rank of the following matrix**

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 3 & 0 \\ 3 & 1 & 2 \end{pmatrix}$$

Ans It is a square matrix. Find the determinant of it

$$|A| = \begin{vmatrix} 2 & 1 & 1 \\ 0 & 3 & 0 \\ 3 & 1 & 2 \end{vmatrix} = 2(6) - 1(0) + (-9) = 12 - 9 \quad \neq 0$$

Rank of matrix $= 3$

Example 2 : **Find the rank of the following matrix**

$$A = \begin{pmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 10 & 3 & 9 & 7 \\ 16 & 4 & 12 & 15 \end{pmatrix}$$

Step 1

$$|A| = 6\begin{vmatrix} 2 & 6 & -1 \\ 3 & 9 & 7 \\ 4 & 12 & 15 \end{vmatrix} - 1\begin{vmatrix} 4 & 6 & -1 \\ 10 & 9 & 7 \\ 16 & 12 & 15 \end{vmatrix} + 3\begin{vmatrix} 4 & 2 & -1 \\ 10 & 3 & 7 \\ 16 & 4 & 15 \end{vmatrix} - 8\begin{vmatrix} 4 & 2 & 6 \\ 10 & 3 & 9 \\ 16 & 4 & 12 \end{vmatrix}$$

$$= 6(\quad) - 1(\quad) + 3(\quad) - 8(\quad)$$

$$= 0$$

$|A| = 0 \quad \therefore$ The rank should be less than 4.

Step 2 Now consider are 3×3 matrices from A.

$$A = \begin{pmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 10 & 3 & 9 & 7 \\ 16 & 4 & 12 & 15 \end{pmatrix}$$

$$A_1 = \begin{pmatrix} 6 & 1 & 3 \\ 4 & 2 & 6 \\ 10 & 3 & 9 \end{pmatrix} \quad A_2 = \begin{pmatrix} 6 & 1 & 8 \\ 4 & 2 & -1 \\ 10 & 3 & 7 \end{pmatrix} \quad A_3 = \begin{pmatrix} 1 & 3 & 8 \\ 2 & 6 & -1 \\ 3 & 9 & 7 \end{pmatrix} \quad A_{10} = \begin{pmatrix} 6 & 3 & 8 \\ 4 & 6 & -1 \\ 10 & 9 & 7 \end{pmatrix}$$

$$A_4 = \begin{pmatrix} 4 & 2 & 6 \\ 10 & 3 & 9 \\ 16 & 4 & 12 \end{pmatrix} \quad A_5 = \begin{pmatrix} 4 & 2 & -1 \\ 10 & 3 & 7 \\ 16 & 4 & 15 \end{pmatrix} \quad A_6 = \begin{pmatrix} 2 & 6 & -1 \\ 3 & 9 & 7 \\ 4 & 12 & 15 \end{pmatrix} \quad A_{11} = \begin{pmatrix} 6 & 3 & 8 \\ 10 & 9 & 7 \\ 16 & 12 & 15 \end{pmatrix}$$

$$A_7 = \begin{pmatrix} 6 & 1 & 3 \\ 4 & 2 & 6 \\ 16 & 4 & 12 \end{pmatrix} \quad A_8 = \begin{pmatrix} 6 & 1 & 8 \\ 4 & 2 & -1 \\ 16 & 4 & 15 \end{pmatrix} \quad A_9 = \begin{pmatrix} 1 & 3 & 8 \\ 2 & 6 & -1 \\ 4 & 12 & 15 \end{pmatrix} \quad A_{12} = \begin{pmatrix} 4 & 6 & -1 \\ 10 & 9 & 7 \\ 16 & 12 & 15 \end{pmatrix}$$

$$|A_i| = 0, \qquad i = 1,2,\dots,12$$

$\therefore$ All determinants vanish. $\therefore$ The rank should not be 3.

Step 3 Now consider are 2×2 matrices from A.

$$A_1 = \begin{pmatrix} 6 & 1 \\ 4 & 2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 6 & 3 \\ 4 & 6 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 6 & 8 \\ 4 & -1 \end{pmatrix} \dots\dots\dots,$$

$$|A_1| = 12 - 4 = 8 \neq 0$$

The **determinant** value is **not zero**.

$\therefore$ Rank of the given matrix = 2.

Example 3 :

Find the rank of the matrix

$$A = \begin{pmatrix} 1 & -1 & 3 & 6 \\ 1 & 3 & -3 & -4 \\ 5 & 3 & 3 & 11 \end{pmatrix}$$

The given matrix is of the order 3×4. (not a square matrix).

Hence Rank cannot be 4. The rank can be **3** or **2** or **1**.

Step 1

Consider are 3×3 matrices from A.

$$A_1 = \begin{pmatrix} 1 & -1 & 3 \\ 1 & 3 & -3 \\ 5 & 3 & 3 \end{pmatrix} \quad A_2 = \begin{pmatrix} 1 & 3 & 6 \\ 1 & -3 & -4 \\ 5 & 3 & 11 \end{pmatrix} \quad A_3 = \begin{pmatrix} -1 & 3 & 6 \\ 3 & -3 & -4 \\ 3 & 3 & 11 \end{pmatrix}$$

$$|A_1| = 1(9+9) + 1(3+15) + 3(3-15)$$

$$= 18 + 18 - 36 \quad = 0$$

$$|A_2| = 1(-33+12) - 3(11+20) + 6(3+15)$$

$$= -21 - 93 + 108 \quad \neq 0$$

Rank is 3

Rank of the Matrix

Properties

(i) The rank of a null matrix is zero.

(ii) The rank of every non-zero matrix is ≥ 1.

(iii) The rank of every n -square non-singular matrix is n.

(iv) (a) The rank of any $m \times n$ matrix is $\leq m$ if $m \leq n$.

(b) The rank of any $m \times n$ matrix is $\leq n$ if $n \leq m$.

(iv) The rank of a product of two matrices cannot exceed the rank of either of matrix, that is

$$\text{rank (AB)} \leq \text{rank (A)}; \quad \text{rank (AB)} \leq \text{rank (B)}$$

Characteristic EQUATIONS, EIGENVALUES & Eigenvectors

- Consider the solutions of the homogeneous system of algebraic equations

$$AX = \lambda X$$

A is an nxn matrix

$$(A - \lambda I)X = 0$$

tells us that X is a solution of a homogeneous system of equations with coefficient matrix $(A-\lambda I)$

- Let $P_n(\lambda)$ be the polynomial of degree 'n' in λ defined by the determinant

$$P_n(\lambda) = \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix}$$

- The polynomial $P_n(\lambda)$ is called the characteristic polynomial of A

- The associated polynomial equation $P_n(\lambda)=0$ is the characteristic equation of A

- The characteristic equation of A is of degree n in λ, it will have 'n' roots. The roots are called Eigenvalues.

- The set of all Eigenvalues λ_1 , λ_2 … λ_n of A is called the spectrum of A.

- An Eigenvector of an n x n matrix A corresponding to an Eigenvalue $\lambda=\lambda_j$ is a non zero n-element column-vector $X_{i,}$ that satisfy the matrix

$$AX_i = \lambda_j X_i$$

Eigenvalues and Eigenvectors

Example 1 :

Find eigenvalues and normalized eigenvectors of the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Determination of Eigenvalues

Step 1 **Characteristic equation** $|A - \lambda \mathrm{I}| = 0$

$$A - \lambda \mathrm{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} = \begin{pmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{pmatrix}$$

$$|A - \lambda \mathrm{I}| = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{vmatrix}$$

$$|A - \lambda \mathrm{I}| = 0 \Longrightarrow (1-\lambda)[(1-\lambda)^2 - 1] = (1-\lambda)[1 + \lambda^2 - 2\lambda - 1] = 0$$

$$= (1-\lambda)(\lambda^2 - 2\lambda) = 0$$

$$= (1-\lambda)\,\lambda(\lambda - 2) = 0$$

Eigenvalues $\lambda_1 = 0, \quad \lambda_2 = 1, \quad \lambda_3 = 2$

Determination of Eigenvectors

Eigenvalue equation $(A-\lambda \mathrm{I})\mathrm{X}=0$

$$\begin{pmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=0$$

(i) Construction of Eigenvector for $\lambda_1=0$

Substitute $\lambda_1=0$ in the above equation and determine the value of (x,y,z). The associated column matrix is called eigenvector for $\lambda_1=0$

Substituting $\lambda_1=0$ in the above equation

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow \left.\begin{aligned} x=0 \\ y+z=0 \\ y+z=0 \end{aligned}\right\} \Longrightarrow x=0,\ z=-y$$

Let us choose $y=1, \quad \therefore\ z=-1$

Eigenvector

$$\mathrm{X}_1=\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

Normalized Eigenvector

$$\mathrm{X}_1=\sqrt{0+1^2+(-1)^2}=\sqrt{2} \qquad \mathrm{X}_1=\frac{1}{|\mathrm{X}_1|}.\mathrm{X}_1\ ; \quad \mathrm{X}_1=\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

(ii) Construction of Eigenvector for $\lambda_1 = 1$

Substituting $\lambda_2 = 1$ in the equation

$$\begin{pmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \implies \left.\begin{matrix} z = 0 \\ y = 0 \end{matrix}\right\} \implies x \text{ arbitrary.}$$

Let us choose $x = 1$

Eigenvector

$$X_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Normalized Eigenvector

$$X_2 = \sqrt{1^2 + 0^2 + 0^2} = 1$$

$$X_2 = \frac{1}{|X_2|} . X_2 ;$$

$$X_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

(iii) Construction of Eigenvector for $\lambda_3 = 2$

Substituting $\lambda_3 = 2$ in the equation

$$\begin{pmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$x = 0, \quad -y + z = 0, \quad y - z = 0 \Longrightarrow x = 0, \quad z = y = 1 \quad \text{(say)}$$

Normalized Eigenvector

$$X_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

Given matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Eigenvalues

$$\lambda = 0, 1, 2$$

Normalized Eigenvectors

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

Example 2 :

Find eigenvalues and eigenvectors of

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Characteristic equation : $\lambda^2 - 2\lambda\cos\theta + 1 = 0$

Eigenvalues (complex) : $\lambda = \cos\theta \pm i\sin\theta = e^{\pm i\theta}$

$$\lambda_1 = e^{i\theta} \;; \qquad \lambda_2 = e^{-i\theta}$$

Eigenvectors (complex) :

Equation $(\cos\theta - \lambda)x - \sin\theta\, y = 0\,;$

$$\sin\theta\, x - (\cos\theta - \lambda)y = 0\,;$$

$\lambda_1 = \cos\theta + i\sin\theta \implies y = -ix$

$\therefore x = 1 \qquad y = -i$

$\lambda_1 = \cos\theta - i\sin\theta \implies y = ix$

$\therefore x = 1 \qquad y = i$

Eigenvectors

$$\begin{pmatrix} 1 \\ -i \end{pmatrix}, \qquad \begin{pmatrix} 1 \\ i \end{pmatrix}$$

Eigenvalues & Eigenvectors

1. Sum of the eigenvalues of a matrix is equal to trace of the matrix.

2. Product of the eigenvalues of a matrix = determinant.

3. Any square matrix A and its transpose have **same eigenvalues**.

4. If $\lambda_1, \lambda_2, \lambda_3$ are the eigenvalues of A, then

(a) Eigenvalues of kA are $k\lambda_1, k\lambda_2, k\lambda_3 \ldots.$

(b) Eigenvalues of the matrix A^{-1} will be $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3} \ldots$

(c) Eigenvalues of the matrix A^m will be $\lambda_1^m, \lambda_2^m, \lambda_3^m \ldots$

5. Eigenvalues of a real symmetric/ Hermitian matrix are always **real**.

6. Eigenvalues of a Skew-Hermitian matrix is either zero or pure imaginary.

7. Eigenvalues of real orthogonal matrix / unitary matrix are of unit modulus.

8. Eigenvalues of a diagonal / upper triangular / lower triangular matrix are the principal diagonal.

Eigenvalues & Eigenvectors

9. Two eigenvectors corresponding to two distinct eigenvalues of Hermitian matrix and a Unitary matrix are orthogonal to each other.

10. Eigenvalues of a nilpotent matrix are always **zero**.

11. Eigenvalues of an idempotent matrix are either 0 or unity.

12. Consider a $n \times n$ matrix having all elements equal to 1. One of the eigenvalues of the matrix will be equal to order of the matrix and all other eigenvalues are zero, that n,0,0,0…

13. Consider a $n \times n$ matrix having rows and columns which are scalar multiple of a particular row & column respectively. One of the eigenvalues of the matrix will be equal to the trace of the matrix and all eigenvalues are zero, (i.e.,) trace,0,0,0….

Cayley-Hamilton theorem

Statement

"Every square matrix satisfies its own characteristic equation"

- If $|A-\lambda I| = (-1)^n(\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \ldots\ldots + a_n)$ be the characteristic polynomial of n x n matrix $A = a_{ij}$, then the matrix equation,

$$X^n + a_1 X^{n-1} + a_2 X^{n-2} + \ldots\ldots\ldots\ldots + a_n I = 0$$

is satisfied by X = A i.e.

$$\boxed{A^n + a_1 A^{n-1} + a_2 A^{n-2} + \ldots\ldots\ldots\ldots + a_n I = 0}$$

Proof

- Since the elements of $A - \lambda I$ are at most of the first degree in λ, the elements of adj $(A - \lambda I)$ are at most degree (n-1) in λ. Thus, adj $(A - \lambda I)$ may be written as a matrix polynomial in λ, given by λ,

$$\text{adj}(A - \lambda I) = B_0 \lambda^{n-1} + B_2 \lambda^{n-2} + \ldots + B_{n-1}$$

where $B_{0,}$ $B_{1,}$ . . . , B_{n-1} are n x n matrices, their elements being polynomial in λ

- As we know that $(A-\lambda\, I)\ adj(A-\lambda\, I)=\left|A-\lambda\, I\right| I$

$$(A-\lambda\, I)\ (B_0\lambda^{n-1}+B_2\lambda^{n-2}+........+B_{n-1})$$
$$=(-1)^n\ (\lambda^n+a_1\lambda^{n-1}+a_2\lambda^{n-2}+........+a_n)\, I$$

- Equating coefficients of like powers of λ on both sides we get

$$-I\, B_0=(-1)^n\, I$$
$$A B_0-I\, B_1=(-1)^n\, a_1 I$$
$$A B_1-I\, B_2=(-1)^n\, a_2 I$$
$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$
$$A B_{n-1}=(-1)^n\, a_n I$$

On multiplying the equation by A^n , A^{n-1} , . . . , I respectively and adding, we obtain

$$0 = (-1)^n \, (A^n + a_1 A^{n-1} + a_2 A^{n-2} + \ldots\ldots\ldots\ldots + a_n \, I)$$

$$(A^n + a_1 A^{n-1} + a_2 A^{n-2} + \ldots\ldots\ldots\ldots + a_n \, I) = 0$$

Proved

Example

Verify the Cayley - Hamilton theorem for the matrix

$$A = \begin{pmatrix} 2 & 1 \\ 5 & 2 \end{pmatrix}$$

Characteristic polynomial

$$P(\lambda) = \lambda^2 - 4\lambda - 1$$

$$P(A) = A^2 - 4A - I$$

$$P(A) = \begin{pmatrix} 9 & 4 \\ 20 & 9 \end{pmatrix} - 4\begin{pmatrix} 2 & 1 \\ 5 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Cayley – Hamilton theorem
Applications

Application 1 : Using Cayley-Hamilton theorem, we can find inverse of the given matrix.

Example : Prove that the matrix $A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ satisfies its own eigenvalue equation and hence find A^{-1}.

Answer: (i) Characteristic equation

$$\lambda^3 - \lambda^2 - 5\lambda + 5 = 0$$ **(Home work)**

(ii) Check

$$A^3 - A^2 - 5A + 5I = 0$$ **(Home work)**

Inverse : $$I = \frac{1}{5}(-A^3 + A^2 + 5A)$$

Pre-multiplying this equation by A^{-1}, we obtain $A^{-1} = \frac{1}{5}(-A^2 + A + 5I)$

$$A^{-1} = \frac{1}{5}\left\{\begin{pmatrix} -5 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + 5\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right\} = \frac{1}{5}\begin{pmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$ **(H.W.)**

Application of Determinants
Cramer's Rule

The solution of the following equations,

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

is given by,

$$x = \frac{D_1}{D} \quad \text{where,} \quad D_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}, \qquad D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$y = \frac{D_2}{D} \quad \text{where,} \quad D_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$$

$$z = \frac{D_3}{D} \quad \text{where,} \quad D_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

Example : Solve the following system of equations using Cramer's rule,

$5x - 7y + z = 11, \qquad 6x - 8y - z = 15, \qquad 3x + 2y - 6z = 7.$

$$D = \begin{vmatrix} 5 & -7 & 1 \\ 6 & -8 & -1 \\ 3 & 2 & -6 \end{vmatrix} = 55$$

$$D_1 = \begin{vmatrix} 11 & -7 & 1 \\ 15 & -8 & -1 \\ 7 & 2 & -6 \end{vmatrix} = 55$$

$$D_2 = \begin{vmatrix} 5 & 11 & 1 \\ 6 & 15 & -1 \\ 3 & 7 & -6 \end{vmatrix} = -55$$

$$D_3 = \begin{vmatrix} 5 & -7 & 11 \\ 6 & -8 & 15 \\ 3 & 2 & 7 \end{vmatrix} = -55$$

$$x = \frac{D_1}{D} = \frac{55}{55} = 1, \qquad y = -1, \qquad z = -1$$

Cross – Check:

$5x - 7y + z = 11 \Rightarrow 5(1) - 7(-1) + (-1) = 5 + 7 - 1 = 11$

$6x - 8y - z = 15 \Rightarrow 6(1) - 8(-1) - (-1) = 6 + 8 + 1 = 15$

$3x + 2y - 6z = 7 \Rightarrow 3(1) + 2(-1) - 6(-1) = 3 - 2 + 6 = 7$

Diagonalisation of a Matrix

Theorem

If a square matrix A of order n has n linearly independent eigenvectors, then a matrix P (composed of the eigenvectors of A) can be found such that $P^{-1}AP$ is a diagonal matrix

- Let us consider a matrix A of order 3

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

- Let λ_1, λ_2, λ_3 be its eigenvalues and X_1, X_2, X_3 the corresponding eigenvectors

where

$$X_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \quad X_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \quad X_3 = \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix}$$

Construct the matrix P composed of the eigenvectors of A

$$P = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$$

Whose columns are the eigenvectors of A

Consider the diagonal matrix of A

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

λ_1, λ_2 and λ_3 are the eigenvalues of A

Then diagonalisation is given by

$$A\,P = P\,D$$

Since $P^{-1} P = I$

$$P^{-1} A\,P = P^{-1} P\,D$$

$$\boxed{P^{-1} A\,P = D}$$

D is the Diagonal matrix

Example

Let us consider the matrix
$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 1 \\ 2 & 0 & 2 \end{bmatrix}$$

Eigenvectors of matrix A are

$$X_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad X_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \quad X_3 = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$$

Hence the matrix P is

$$P = \begin{bmatrix} 0 & 0 & -1 \\ 1 & -1 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

Inverse of matrix P is given by

$$P^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}$$

As we know that diagonal matrix D is given by

$$D = P^{-1} A P$$

$$D = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 1 \\ 2 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 \\ 1 & -1 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Remarks

- An n x n matrix can be diagonalized provided it possesses ‘n’ linearly independent eigenvectors

- A symmetric matrix can always be diagonalized

- The diagonalizing matrix for a real n x n matrix A may contain complex elements. This is because although the characteristic polynomials of A has real coefficients, its zeros either will be real or will occur in complex conjugate pairs

- A diagonalizing matrix is not unique, because its form depends on the order in which the eigenvectors are used to form its columns

Application
Powers of a Matrix

We can obtain powers of a matrix with the help of diagonalized matrix.

Recall, $D = P^{-1} A P$

- D: Diagonal matrix
- A: Given matrix
- P: Eigenvector matrix

$$D^2 = DD = P^{-1} A P P^{-1} A P = P^{-1} A^2 P$$

($P P^{-1} = I$)

$$D^3 = DD^2 = P^{-1} A P P^{-1} A^2 P = P^{-1} A^3 P$$

($P P^{-1} = I$)

$\vdots$

$$D^n = P^{-1} A^n P$$

$$PD^n = PP^{-1} A^n P = A^n P$$

$$PD^n P^{-1} = A^n PP^{-1} = A^n \Rightarrow A^n = P D^n P^{-1}$$

- P: Eigenvector matrix
- D^n: Diagonal matrix raised to power n

Example : $A = \begin{pmatrix} \frac{4}{3} & \frac{\sqrt{2}}{3} \\ \frac{\sqrt{2}}{3} & \frac{5}{3} \end{pmatrix}$, Calculate A^{50}

Step 1: Eigenvalues $\lambda_1 = 1, 2$ **(H.W.)**

Step 2: Eigenvector $\begin{pmatrix} \sqrt{2} \\ -1 \end{pmatrix} \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}$ **(H.W.)**

Step 3: $P = \begin{pmatrix} \sqrt{2} & 1 \\ -1 & \sqrt{2} \end{pmatrix}$ $P^{-1} = \begin{pmatrix} \frac{\sqrt{2}}{3} & \frac{-1}{3} \\ \frac{1}{3} & \frac{\sqrt{2}}{3} \end{pmatrix}$ **(H.W.)**

Step 4: Diagonal matrix $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ **(H.W.)**

Step 5: $A^{50} = PD^{50}P^{-1}$

$$= \begin{pmatrix} \sqrt{2} & 1 \\ -1 & \sqrt{2} \end{pmatrix} \begin{pmatrix} 1^{50} & 0 \\ 0 & 2^{50} \end{pmatrix} \begin{pmatrix} \frac{\sqrt{2}}{3} & \frac{-1}{3} \\ \frac{1}{3} & \frac{\sqrt{2}}{3} \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 2^{50} + 2 & (2^{50} - 1)\sqrt{2} \\ (2^{50} - 1)\sqrt{2} & 2^{51} + 1 \end{pmatrix}$$ **(H.W.)**

Orthogonality of Eigenvectors

Demonstration: Consider the matrix $A = \begin{pmatrix} 0 & -1 & 0 \\ -1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

The Eigenvalues are $\lambda_1 = 0, \lambda_2 = 1 \ \ and \ \lambda_3 = -2$ **(H.W.)**

The Eigenvectors are $k_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, k_2 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} and \ k_3 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ **(H.W.)**

Orthogonality

$$k_1^T k_2 = \begin{pmatrix} 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = -1 + 1 = 0$$

$$k_1^T k_2 = \begin{pmatrix} 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = -1 + 1 = 0$$

$$k_2^T k_3 = \begin{pmatrix} -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = -1 + 2 - 1 = 0$$

Differentiation and Integration of matrix

$Let\ \ the\ \boldsymbol{n} \times \boldsymbol{1}\ column\ \ vector$

$$x(t) = [x_1(t), x_2(t), \dots, x_n(t)]^T$$

$x_i(t)\ for\ \boldsymbol{i} = \boldsymbol{1}, \boldsymbol{2}, \dots\dots, \boldsymbol{n}$

$Let\ \boldsymbol{m} \times \boldsymbol{n}\ matrix$

$$G(t) = \left[g_{ij}(t)\right]$$

$g_{ij}(t)\ with\ i = 1,2, \dots\dots, m\ \ \ and\ j = 1,2, \dots\dots, n.$

$Derivatives\ of\ \boldsymbol{x}(\boldsymbol{t}) and\ \boldsymbol{G}(\boldsymbol{t}) with\ respect\ to\ \boldsymbol{t}\ are\ defined\ as$

$$\frac{dx(t)}{dt} = \begin{bmatrix} dx_1/dt \\ dx_2/dt \\ \vdots \\ dx_n/dt \end{bmatrix}$$

$$\frac{d\boldsymbol{G}(t)}{dt} = \begin{bmatrix} dg_{11}/dt & dg_{2}/dt & \cdots & dg_{1n}/dt \\ dg_{21}/dt & dg_{22}/dt & \cdots & dg_{2n}/dt \\ \vdots & \vdots & \vdots & \vdots \\ dg_{m1}/dt & dg_{m2}/dt & \cdots & dg_{mn}/dt \end{bmatrix}$$

$\boldsymbol{A}$ *is constant matrix*

$$then\ \frac{\boldsymbol{dA}}{\boldsymbol{dt}} = 0 \qquad in\ particular\ \frac{\boldsymbol{dI}}{\boldsymbol{dt}} = 0$$

defintion of matrix multiplication

$$\frac{d[\boldsymbol{AG(t)}]}{dt} = \boldsymbol{A}\frac{d\boldsymbol{G}(t)}{dt}$$

Example : Find $\frac{d[AG(t)]}{dt}$ and $\frac{d^2[AG(t)]}{dt^2}$ if

$$A = \begin{bmatrix} 1 & -3 \\ 2 & 4 \end{bmatrix} \text{ and } G = \begin{bmatrix} \sin t & \cos t \\ -\operatorname{caos} t & \sin t \end{bmatrix}$$

Solution :

$$\frac{d}{dt}[AG(t)] = A\frac{dG(t)}{\text{dt}} = \begin{bmatrix} 1 & -3 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$$

$$= \begin{bmatrix} \cos t - 3\sin t & -\sin t - 3\cos t \\ 2\cos t - 4\sin t & -2\sin t + 4\cos t \end{bmatrix}$$

$$\frac{d^2}{dt^2}[AG(t)] = \frac{d}{dt}\left(\frac{d[AG(t)]}{dt}\right) = A\frac{d^2G(t)}{\text{dt}^2}$$

$$= \begin{bmatrix} -\sin t - 3\cos t & -\cos t + 3\sin t \\ -2\sin t + 4\cos t & -2\cos t - 4\sin t \end{bmatrix}$$

If $G(t)$ and H(t) are conformable for addition, then

$$\frac{d}{dt}[G(t) + H(t)] = \frac{dG(t)}{dt} + \frac{dH(t)}{dt}$$

Furthermore, If $G(t)$ and H(t) are conformable for the product $G(t)H(t)$, then

$$\frac{d}{dt}[G(t)H(t)] = \frac{dG(t)}{dt}H(t) + G(t)\frac{dH(t)}{dt}$$

A less obvious result is that if $G(t)$ is a nonsingular $n \times n$ matrix, then

$$\frac{dG^{-1}(t)}{dt} = -G^{-1}(t)\frac{dG(t)}{dt}G^{-1}(t)$$

Proof :

By differentiating the product $G(t)G^{-1}(t)$

$$\frac{d}{dt}[G(t)G^{-1}(t)] = \frac{dG(t)}{dt}G^{-1}(t) + G(t)\frac{dG^{-1}(t)}{dt} \longrightarrow \quad (1)$$

We Know that $G(t)G^{-1}(t) = I$

$$\frac{d}{dt}[G(t)G^{-1}(t)] = \frac{d}{dt}[I] = 0 \longrightarrow \quad (2)$$

From 1 and 2

$$\frac{dG(t)}{dt}G^{-1}(t) + G(t)\frac{dG^{-1}(t)}{dt} = 0$$

Rearranging the above equation

$$\frac{dG^{-1}(t)}{dt} = -G^{-1}(t)\frac{dG(t)}{dt}G^{-1}(t)$$

Example : $Find\,\dfrac{dG^{-1}(t)}{dt}\quad if$

$$G(t) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$$

Solution :

$Differentiate$

$$G^{-1}(t) = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$$

$$\frac{dG^{-1}(t)}{dt} = \begin{bmatrix} -\sin t & -\cos t \\ \cos t & -\sin t \end{bmatrix}$$

$then$

$$\frac{dG(t)}{dt} = \begin{bmatrix} -\sin t & -\cos t \\ -\cos t & -\sin t \end{bmatrix}$$

$By\ simplifying\ the\ above\ expressions\ for\ \ \boldsymbol{G^{-1}(t)}\ and\ d\boldsymbol{G}(t)/dt$

$$\frac{dG^{-1}(t)}{dt} = -G^{-1}(t)\frac{dG(t)}{dt}\ G^{-1}(t)$$

$finally\ we\ get$

$$\boxed{\frac{dG^{-1}(t)}{dt} = \begin{bmatrix} -\sin t & -\cos t \\ \cos t & -\sin t \end{bmatrix}}$$

Definition

If $\boldsymbol{A}(\boldsymbol{t}) = [\boldsymbol{a_{ij}}(\boldsymbol{t})]$ *is an* $\boldsymbol{m} \times \boldsymbol{n}$ *matrix , with* ***i* = *1,2,. . . ,m*** *and* ***j* = *1,2,…..,n***

Indefinite integral of the element in the **i** th row and **j** th column of **A(t)** is $\int \mathbf{a_{ij}}(\mathbf{t})\mathbf{dt}$

Indefinite integral of $\boldsymbol{A}(\boldsymbol{t})$ *is*

$$\int \boldsymbol{A}(t)dt = \left[\int a_{ij}(t)dt\right]$$

Definite integral of ***A(t)*** between limits ***t* = *a and t* = *b***

So that

$$\int_a^b \boldsymbol{A}(t)dt = \left[\int_a^b a_{ij}(t)dt\right]$$

Example : $Find\ A(t)\ dt \quad if \quad A(t) = \begin{bmatrix} 2\sin t & \cos t \\ -3\cos t & \sin t \end{bmatrix}$

Solution :

$$\int \boldsymbol{A}(t)\, dt = \begin{bmatrix} -2\cos t + C_1 & \sin t + C_2 \\ -3\sin t + C_3 & -\cos t + C_4 \end{bmatrix}$$

So

$$A(t) = \begin{bmatrix} 2\sin t & \cos t \\ -3\cos t & \sin t \end{bmatrix} + C$$

$$C = \begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix} \quad is\ an\ arbitary\ constant$$

THEOREM

Let all the eigenvalues $\boldsymbol{\lambda_1},\ldots\ldots,\boldsymbol{\lambda_n}$ *of the* $\boldsymbol{n}\times\boldsymbol{n}$ *real matrix* $\boldsymbol{A}$ *be real and distinct and* $\boldsymbol{E_1},\ldots\ldots,\boldsymbol{E_n}$ *are eigenvectors belonging to* $\boldsymbol{\lambda_1},\ldots,\boldsymbol{\lambda_n}$ *respectively. Then a general solution is*

$$X(t) = c_1 e^{\lambda_1 t} E_1 + \ldots + c_n e^{\lambda_n t} E_n,$$

$c_1,\ldots,c_n$ *being arbitrary constants,*

Example : To solve

$$x'_1 = x_1 + 3x_2$$

$$x'_2 = x_1 - x_2$$

We compute the eigenvalues λ_1, λ_2 of the matrix

$$A = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix}$$

Corresponding eigenvectors belonging to $\lambda_1 = 2,\ \lambda = -2\ are\ \boldsymbol{E_1} = \begin{pmatrix} \mathbf{3} \\ \mathbf{1} \end{pmatrix}$, $\boldsymbol{E_2} = \begin{pmatrix} \mathbf{-1} \\ \mathbf{1} \end{pmatrix}$. The general solution of this first order linear differential equation is

$$X(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1 e^{2t} \begin{pmatrix} 3 \\ 1 \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

which can be rewritten as

$$x_1(t) = \ 3c_1e^{2t} \ - c_2e^{-2t}$$

$$x_2(t) = \ c_1e^{2t} + c2\ e^{-2t}$$

Homework

Solve the following systems of first order linear differential equations

1) $x'_1 = x_1 + 6x_2 \quad ; \quad x'_2 = 5x_1 + 2x_2$

2) $x'_1 = x_1 - 2x_2 \quad ; \quad x'_2 = x_1 - x_2$

3) $x'_1 = x_1 - x_2 + 4x_3 \quad ; \quad x'_2 = 3x_1 + 2x_2 - x_3 \ ; \quad x'_3 = 2x_1 + x_2 - x_3$

Example : Find the general solution of the system of equations

$$\frac{dx_1}{dt} = x_1 + x_2 \quad , \qquad \frac{dx_2}{dt} = \quad x_2 - x_1$$

Solution :

Hint : $x_1\left(\frac{\pi}{2}\right) = 1$, $x_2\left(\frac{\pi}{2}\right) = 2$

In matrix $\frac{dx}{dt} = Ax, \ \ with \quad x = [x_1, x_2]^T$

Where

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

The eigenvalues and eigenvectors of A are

$$\lambda_1 = 1 + i \, , \qquad x_1 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

$$\lambda_2 = 1 - i \, , \qquad x_2 \quad = \begin{bmatrix} i \\ 1 \end{bmatrix}$$

So as the vectors $\boldsymbol{e^{-it}x_i}$ $with$ $\boldsymbol{i = 1, 2}$ are linearly independent solutions,

$$\phi(t) = \begin{bmatrix} -ie^{(1+i)t} & ie^{(1-i)t} \\ e^{(1+i)t} & e^{(1-i)t} \end{bmatrix}$$

Thus the general solution $\boldsymbol{x(t) = \phi(t)C}$ $becomes$

$$x(t) = \begin{bmatrix} -ie^{(1+i)t} & ie^{(1-i)t} \\ e^{(1+i)t} & e^{(1-i)t} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$$

$$= \begin{bmatrix} -iC_1e^{(1+i)t} & iC_2e^{(1-i)t} \\ C_1e^{(1+i)t} & C_2e^{(1-i)t} \end{bmatrix}$$

$where\ C_1\ and\ C_2\ are\ complex\ numbers$

Let us set $C_1 = a + ib \quad and \quad C_2 = a - ib$, then the general solution becomes

$$x(t) = \begin{bmatrix} 2ae' \sin t + 2be' \cos t \\ 2ae' \cos t \ - 2be' \sin t \end{bmatrix}$$

Both $\boldsymbol{a}$ and $\boldsymbol{b}$ are arbitrary constants , so we set $\boldsymbol{k_1 = 2a \quad and}$ $\boldsymbol{k_2 = 2b}$, then general solutions becomes,

$$x_1(t) = e^t(k_1 \sin t + k_2 \cos t) \ \ and \ \ x_2(t) = e^t(k_1 \cos t \ - k_2 \sin t)$$

To satisfy the initial conditions

$$x_1\left(\frac{\pi}{2}\right) = 1, \qquad x_2\left(\frac{\pi}{2}\right) = 0, \qquad t = \frac{\pi}{2} \ in \ general \ solution$$

Initial condition $x_1\left(\frac{\pi}{2}\right) = 1$: $1 = e^{\pi/2}\, k_1$ $k_1 = e^{-\pi/2}$

Initial condition $x_2\left(\frac{\pi}{2}\right) = 2$: $2 = -e^{\pi/2}\, k_2$ $k_2 = -2e^{-\pi/2}$

Then the solution of initial-value problem is found to be

$$x_1(t) = e^{\left(t-\frac{\pi}{2}\right)}(\sin t\ - 2\cos t\ )$$

$$x_2(t) = e^{\left(t-\frac{\pi}{2}\right)}(\cos t + 2\sin t). \qquad where\ t \geq \frac{\pi}{2}$$

Example : Find the general solution of the following third-order differential equation by converting it to a first-order system:

$$\frac{d^3y}{dt^3} + \frac{d^2y}{dt^2} + \frac{dy}{dt} + y = 0$$

Solution : Introduce the two new dependent variables z_1 *and* z_2, *by setting*

$$\frac{dy}{dt} = z_1 \qquad and \qquad \frac{d^2y}{dt^2} = \frac{dz_1}{dt} = z_2$$

Third order equation replaced by equivalent first-order system

$$\frac{dy}{dt} = z_1, \qquad \frac{dz_1}{dt} = z_2$$

$$\frac{dz_2}{dt} + z_2 + z_1 + y = 0$$

When written in matrix form, this system becomes

$$\frac{dz}{dt} = Az$$

$$with \quad z = \begin{bmatrix} y(t) \\ z_1(t) \\ z_2(t) \end{bmatrix}, \qquad A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{bmatrix}$$

The eigenvalues and eigenvectors of A are

$$\lambda_1 = -1\,; x_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \qquad \lambda_2 = i \;;\; x_2 = \begin{bmatrix} -1 \\ -i \\ 1 \end{bmatrix}$$

$$\lambda_3 = -i \;\; ; x_3 = \begin{bmatrix} -1 \\ i \\ 1 \end{bmatrix}$$

As the vectors $\boldsymbol{e^{-\lambda_i t} x_i}$ $\quad with\ \boldsymbol{i = 1, 2, 3}$ are solutions , a fundamental matrix is

$$\phi(t) = \begin{bmatrix} e^{t} & -e^{it} & -e^{-it} \\ -e^{-t} & -ie^{it} & ie^{-it} \\ e^{-t} & e^{it} & e^{-it} \end{bmatrix}$$

when the general solution becomes

$$\begin{bmatrix} y \\ z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} e^{t} & -e^{it} & -e^{-it} \\ -e^{-t} & -ie^{it} & ie^{-it} \\ e^{-t} & e^{it} & e^{-it} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix}$$

Y(t) of the original third-order differential equation is needed

$$y(t) = C_1 e^{-t} + C_2 e^{it} + C_3^{-it}$$

C_2, C_3 *are complex conjugates, so setting* $C_2 = a + ib$ *and* $C_3 = a - ib$ and a and b are arbitrary constants.

It results

$$y(t) = C_1 e^{-t} + 2b \cos t \ - 2b \sin t$$

Writing $\mathbf{C_2}$ **in place of 2a** *and* $\mathbf{C_3}$ **in place of** $-\mathbf{2b}$, *then the general solution is*

$$y(t) = C_1 e^{-t} + C_2 \cos t + C_3 \sin t$$

Solving for $\boldsymbol{z_1}$ *and* $\boldsymbol{z_2}$ *will give* $\boldsymbol{\frac{dy}{dt}}$ ***and*** $\boldsymbol{\frac{d^2t}{dt^2}}$ by determination of ***y(t)***

Spectral Decomposition

Suppose $A = PDP^{-1}$ where the columns of P are orthonormal eigen vectors $u_1 \dots \dots \dots u_n . of\ A$ and the corresponding eigenvalues $\lambda_1 \dots \dots \dots \lambda_n$ are in diagonal matrix D.

$then\ P^{-1} = P^T$

$$A = PDP^T = [u_1 \quad \dots \quad u_n]_{1\times n} \begin{bmatrix} \lambda_1 & & 0 \\ 0 & \ddots & \lambda_n \end{bmatrix}_{n\times n} \begin{bmatrix} u_1^T \\ \vdots \\ u_n^T \end{bmatrix}_{n\times 1}$$

$$= [\lambda_1 u_1 \quad \dots \quad \lambda_n u_n] \begin{bmatrix} u_1^T \\ \vdots \\ u_n^T \end{bmatrix}$$

Using the column-row expansion of a product , we can write

$$A = \lambda_1 u_1 u_1^T + \lambda_2 u_2 u_2^T + \quad \dots \quad \lambda_n u_n u_n^T$$

Spectral decomposition of A

Example:

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}.$$

Its eigenvalues are $\lambda = 2,4$ and the corresponding unit vectors are

$$\boldsymbol{u}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \quad \text{and} \quad \boldsymbol{u}_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}.$$

Therefore,

$$\boldsymbol{u}_1\boldsymbol{u}_1^T = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

And

$$\boldsymbol{u}_2\boldsymbol{u}_2^T = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

So

$$\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} = 2\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} + 4\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

is the spectral decomposition of A.

Example : Construct a spectral decomposition of the matrix ***A*** that has the orthogonal diagonalization.

$$A = \begin{bmatrix} 7 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} \begin{bmatrix} 8 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$$

Solution : Denote the columns of P by u_1 *and* u_2.

$$A = 8u_1 u_1^T + 3u_2u_2^T$$

To verify this decomposition of A , compute

$$u_1u_1^T = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$$

$$= \begin{bmatrix} 4/5 & -2/5 \\ 2/5 & 1/5 \end{bmatrix}$$

$$u_2 u_2^T = \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} \begin{bmatrix} \frac{-1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}$$

$$= \begin{bmatrix} 1/5 & -2/5 \\ -2/5 & 4/5 \end{bmatrix}$$

and

$$8u_1 u_1^T + 3u_2 u_2^T = \begin{bmatrix} 32/5 & 16/5 \\ 16/5 & 8/5 \end{bmatrix} + \begin{bmatrix} 3/5 & -6/5 \\ -6/5 & 12/5 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 2 \\ 2 & 4 \end{bmatrix}$$

$$= \boldsymbol{A}$$

Homework

i) Write the spectral decomposition $A = PDP^*$ $\quad if$

$a)$ $\quad A = \begin{pmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{pmatrix}$ $\qquad b)$ $\quad A = \begin{pmatrix} 6 & -2 & -1 \\ -2 & 6 & -1 \\ -1 & -1 & 5 \end{pmatrix}$

ii) Classify (positive definite, negative definite , or indefinite, etc) the quadratic form $q(x) = x^*Ax \quad if\ A =$

$a)$ $\begin{pmatrix} 2 & 3 \\ 3 & -6 \end{pmatrix}$ $\qquad b)$ $\begin{pmatrix} 9 & -4 \\ -4 & 3 \end{pmatrix}$

$c)$ $\begin{pmatrix} -5 & 2 \\ 2 & -2 \end{pmatrix}$ $\qquad d)$ $\begin{pmatrix} 3 & 2 & 0 \\ 2 & 2 & 2 \\ 0 & 2 & 1 \end{pmatrix}$

THEOREM

A quadratic form $\boldsymbol{q}(\boldsymbol{x}) = \boldsymbol{x}^{*}\boldsymbol{A}\boldsymbol{x}$ *with* $\boldsymbol{A}$ *symmetric is*

i. positive definite if and only if all the eigenvalues of A are positive.

ii. negative definite if and only if all the eigenvalues of A are negative.

iii. indefinite if and only if neither ***i)*** *nor* ***ii)*** *holds.*

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