Introduction to Linear Vector Space

Sets

A set is a collection of elements having certain common properties.

Example:

1. A set of integers $\{..., -4, -3, -2, -1, 0, 1, 2, 3, 4, ...\}$
2. A set of rational numbers $\{...., -1.5, -0.5, 0, 0.5, 1.5, ...\}$
3. A set of even numbers $\{..., -6, -4, -2, 0, 2, 4, 6, ...\}$

Group

Suppose the elements of a set G are denoted by $\{a, b, c, ...\}$. An operation combining any two of them is denoted by, say $a \otimes b$. The set G becomes a group if the following properties are satisfied.

1. Closure: If $a, b \in G$, then $a \otimes b \in G$.
2. Associativity: If $a, b, c \in G$ then

$$a \otimes (b \otimes c) = (a \otimes b) \otimes c$$

3. Identity: For every element $a \in G$ $\exists$ one $e \in G$ such that

$$e \otimes a \ = \ a \otimes e \ = \ a.$$

4. Inverse element: For every $a \in G$, $\exists$ a member $a^{-1} \in G$ such that

$$a \otimes a^{-1} = a^{-1} \otimes a = e \qquad a^{-1} \ = \text{inverse of } a.$$

Group: Example

The set of all integers constitute a group with respect to addition.

Binary Operation:

$$G = \{\cdots, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, \cdots\}$$
$$\otimes = addition$$

1. Closure: Let us consider any two elements, $a = -2$, $b = 3$

$$-2 + 3 = 1 \in G$$

2. Associativity: Let us consider $a, b, c \in G$,

$$\begin{aligned} a = -1, \quad b &= 2, \quad c = 3 \\ -1 + (2 + 3) &= (-1 + 2) + 3 \\ 4 &= 4 \in G. \end{aligned}$$

Group: Example

3. Identity: 0 is the identity element.

$$a \otimes e = e \otimes a = a$$
$$-1 + 0 = 0 + (-1) = -1 \ \in \ G.$$

4. Inverse: For every $a \in G$, $\exists\, a^{-1} \in G \ \Rightarrow \ a \otimes a^{-1} \ = \ e$

$$a = 1, \quad a^{-1} = -1$$
$$a + a^{-1} = 1 - 1 = 0 = e \ \in G, \quad a^{-1} = -a$$

The set of integers forms a group with respect to addition.

Abelian Group

The Group G satisfying the above four properties is called the Abelian Group with respect to the operation $\otimes$ if

$$a \otimes b = b \otimes a$$

Example:

1. The set of integers form an Abelian group with respect to the operation of addition.
2. (i) The of set of real numbers
   (ii) The set of rational numbers
   (iii) The set of complex numbers
   are also examples for Abelian **groups** with respect to addition.

Field

Definition

Suppose $\mathbb{F}$ is a non-empty set equipped with two binary operations called addition and multiplication and denoted by ' $+$ and '.' respectively for all $a, b \in \mathbb{F}$ we have $a + b \in \mathbb{F}$ and $a.b \in \mathbb{F}$. Then this algebraic structure $(\mathbb{F}, +, .)$ is called a field, if the following postulates are satisfied.

1. Addition is commutative, that is

$$a + b = b + a \quad \forall\ a, b\ \in\ \mathbb{F}.$$

2. Addition is associative, that is

$$(a + b) + c = a + (b + c) \quad \forall\ a, b, c\ \in\ \mathbb{F}.$$

3. $\exists$ an element denoted by 0 (called zero) in $\mathbb{F}$ such that

$$a + 0 = a \quad \forall\ a \in \mathbb{F}.$$

4. To each element a in $\mathbb{F}$ there exists an element $-a$ in $\mathbb{F}$ such that

$$a + (-a) = 0.$$

5.Multiplication is commutative, that is

$$a.b = b.a \quad \forall\ a, b \in \mathbb{F}.$$

6. Multiplication is associative, that is

$$(a.b).c = a.(b.c) \quad \forall\ a, b, c \in \mathbb{F}.$$

7. $\exists$ a non-zero element denoted by 1 (called one) in $\mathbb{F}$ such that

$$a.1 = a \quad \forall\, a \in \mathbb{F}.$$

8. To every non-zero element a in $\mathbb{F}$ there corresponds an element a^{-1} (or $\frac{1}{a}$) in $\mathbb{F}$ such that

$$a.a^{-1} = 1.$$

9. Multiplication is distributive with respect to addition, that is

$$a.(b + c) = a.b + a.c \quad \forall\, a, b, c \in \mathbb{F}$$

Field: Example

The set R of all real numbers is a field, the addition and multiplication of real numbers being the two field compositions. Consider the real numbers $a = 1, b = 0.5$ and $c = 2$ $a, b, c \in \mathbb{F}$

Property:1

$$\begin{aligned} a + b &= b + a \quad \forall\, a, b \in \mathbb{F}. \\ 1 + 0.5 &= 0.5 + 1 \\ 1.5 &= 1.5 \end{aligned}$$

Property:2

$$\begin{aligned} (a + b) + c &= a + (b + c) \quad \forall\, a, b, c \in \mathbb{F}. \\ (1 + 0.5) + 2 &= 1 + (2.5 + 2) \\ 1.5 + 2 &= 1 + 2.5 \\ 3.5 &= 3.5 \end{aligned}$$

Example of Field-contd

Property:3

$$\begin{aligned} a + 0 &= a \quad \forall\ a \in \mathbb{F}. \\ 1 + 0 &= 1 \\ 1 &= 1 \end{aligned}$$

Property:4

The additive inverse $-a = -1$

$$\begin{aligned} a + (-a) &= 0 \quad \forall\ a, b \in \mathbb{F}. \\ 1 + (-1) &= 0 \end{aligned}$$

Property:5

$$\begin{aligned} a.b &= b.a \quad \forall\ a, b \in \mathbb{F}. \\ 1.(0.5) &= (0.5).1 \\ 0.5 &= 0.5 \end{aligned}$$

Example of Field-contd

Property:6

$$
\begin{aligned}
(a.b).c &= a.(b.c) \quad \forall\, a, b, c \in \mathbb{F}. \\
1.((0.5).2) &= (1.(0.5)).2 \\
1.1 &= (0.5).2 \\
1 &= 1
\end{aligned}
$$

Property:7

$$
\begin{aligned}
a.1 &= a \quad \forall\, a \in \mathbb{F}. \\
1.1 &= 1 \\
1 &= 1
\end{aligned}
$$

Example of Field-contd

Property:8

The multiplicative inverse $a^{-1} = \frac{1}{1} = 1$

$$\begin{aligned} a.a^{-1} &= 1 \\ 1.1 &= 1 \\ 1 &= 1 \end{aligned}$$

Property:9

$$\begin{aligned} a.(b+c) &= a.b + a.c \quad \forall\, a, b, c \in \mathbb{F} \\ 1.(0.5+2) &= 1.0.5 + 1.2 \\ 1.2.5 &= 0.5 + 2 \\ 2.5 &= 2.5 \end{aligned}$$

Example of Field-contd

Example: 2

The set Q of all rational numbers is a field, the addition and multiplication of rational numbers being the two field compositions. Since $Q \subset R$, therefore the field of rational numbers is a subfield of the field of real numbers. The rational numbers 0 is the **zero element** of this field and the rational number 1 is the **unity** of the field.

Vector space

Internal composition

Let A be any set. If $a * b \in A \; \forall \, a, b \in A$ and $a * b$ is unique then $*$ is said to be an *internel composition* in the set A. Here both a and b are the elements of the set A.
Example: i) Addition of two matrices, (ii) Addition of two complex numbers.

External composition

Let V and $\mathbb{F}$ be any two sets. If $a \otimes \alpha \in V$ for all $a \in \mathbb{F}$ and for all $\alpha \in V$ and $a \otimes \alpha$ is unique then $\otimes$ is said to be an *external composition* in V over $\mathbb{F}$. Here a is an element of the set $\mathbb{F}$ and α is an element of the set V and the resulting element $a \otimes \alpha$ is an element of the set V.
Example: i) Multiplication of a matrix with a scalar,
(ii) Multiplication of a complex number with a real number.

Vector space

Definition:

Let $(F, +, .)$ be a field. The elements of $\mathbb{F}$ will be called scalars. Let V be a non-empty set whose elements will be called vectors. Then V be a vector space over the field $\mathbb{F}$, if

1. There is defined an internal composition in V called addition of vectors and denoted by '$+$'. Also for this composition V is the abelian group, that is

   (i) $\alpha + \beta \in V$ *for all* $\alpha, \beta \in V$

   (ii) $\alpha + \beta = \beta + \alpha$ *for all* $\alpha, \beta \in V$

   (iii) $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$ *for all* $\alpha, \beta, \gamma \in V$

   (iv) *To every vector* $\alpha \in V$ *there exists a vector* $-\alpha \in V$ *such that*

$$\alpha + (-\alpha) = 0$$

Vector space

2. There is an external composition in V over $\mathbb{F}$ called scalar multiplication and denoted multiplicatively i.e., $a\alpha \in V$ for all $a \in \mathbb{F}$ and for all $\alpha \in V$. In other words V is closed with respect to scalar multiplication.

3. The two compositions that is, scalar multiplication and addition of vectors satisfy the following postulates:

(i) $a(\alpha + \beta) = a\alpha + a\beta \ \forall a \in \mathbb{F}$ and $\forall \alpha, \beta \in V$.

(ii) $(a + b)\alpha = a\alpha + b\alpha \ \forall a, b \in \mathbb{F}$ and $\forall \alpha \in V$.

(iii) $(ab)\alpha = a(b\alpha) \ \forall a, b \in \mathbb{F}$ and $\forall \alpha \in V$.

(iv) $1\alpha = \alpha \ \forall \alpha \in V$ and 1 is the unity element of the field $\mathbb{F}$.

When V is a vector space over the field $\mathbb{F}$ we shall say that $V(\mathbb{F})$ is a vector space. If the field $\mathbb{F}$ is understood we can simply say that V is the vector space. If $\mathbb{F}$ is the field R of real numbers, V is called a Real Vector Space; If $\mathbb{F}$ is C, it is Complex Vector Space.

Example-Vector space

Example

The set V of all $m \times n$ matrices with their elements as real number is a vector space over the field F of real numbers with respect to addition of matrices as addition of vector and multiplication of a matrix by a scalar. Here we consider the vector space 1×2 matrices. Let

$$\alpha = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 2 & 1 \end{pmatrix}, \quad \gamma = \begin{pmatrix} 1 & 1 \end{pmatrix} \quad \alpha, \beta, \gamma \in V$$

Property-1(i)

$$\alpha + \beta \in V \; \forall \alpha, \beta \in V$$

$$\begin{pmatrix} 1 & 0 \end{pmatrix} + \begin{pmatrix} 2 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 1 \end{pmatrix} \in V$$

Example-contd

Property-1(ii)

$$\alpha + \beta = \beta + \alpha \ \forall \alpha , \beta \in V$$

$$\begin{pmatrix} 1 & 0 \end{pmatrix} + \begin{pmatrix} 2 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 1 \end{pmatrix}$$

Property-1(iii)

$$\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma \ \forall \alpha, \beta , \gamma \in V$$

$$\begin{pmatrix} 1 & 0 \end{pmatrix} + \left[\begin{pmatrix} 2 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 \end{pmatrix} \right] = \left[\begin{pmatrix} 2 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \end{pmatrix} \right] + \begin{pmatrix} 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \end{pmatrix} + \begin{pmatrix} 3 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 2 \end{pmatrix}$$

Example-contd

Property:1(iv)

$$\alpha + (-\alpha) = 0 - \alpha \in V.$$

$$-\alpha = \begin{pmatrix} -1 & 0 \end{pmatrix}$$

$$\alpha + (-\alpha) = \begin{pmatrix} 1 & 0 \end{pmatrix} + \begin{pmatrix} -1 & 0 \end{pmatrix} = 0$$

Property-2

$$a \in \mathbb{F}; \quad \alpha \in V; \quad a\alpha \in V.$$

$$a\alpha = 5\begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 5 & 0 \end{pmatrix} \in V \qquad (\textit{where } a = 5)$$

Property-3(i)

$$a(\alpha+\beta) = a\alpha + a\beta \; \forall a \in \mathbb{F} \;\; and \; \forall \alpha, \beta \in V.$$

$$5[\begin{pmatrix}1 & 0\end{pmatrix} + \begin{pmatrix}2 & 1\end{pmatrix}] = 5\begin{pmatrix}1 & 0\end{pmatrix} + 5\begin{pmatrix}2 & 1\end{pmatrix}$$

$$5\begin{pmatrix}3 & 1\end{pmatrix} = \begin{pmatrix}5 & 0\end{pmatrix} + \begin{pmatrix}10 & 5\end{pmatrix}$$

$$\begin{pmatrix}15 & 5\end{pmatrix} = \begin{pmatrix}15 & 5\end{pmatrix}$$

Property-3(ii)

$$(a+b)\alpha = a\alpha + b\alpha \; \forall a, b \in \mathbb{F} \; and \; \forall \alpha \in V.$$

$$(5+2)\begin{pmatrix}1 & 0\end{pmatrix} = 5\begin{pmatrix}1 & 0\end{pmatrix} + 2\begin{pmatrix}1 & 0\end{pmatrix}$$

$$7\begin{pmatrix}1 & 0\end{pmatrix} = \begin{pmatrix}5 & 0\end{pmatrix} + \begin{pmatrix}2 & 0\end{pmatrix}$$

$$\begin{pmatrix}7 & 0\end{pmatrix} = \begin{pmatrix}7 & 0\end{pmatrix}$$

Property-3(iii)

$$(ab)\alpha = a(b\alpha)\ \forall a, b \in \mathbb{F}\ \textit{and}\ \forall \alpha \in V. \quad (\text{let } b = 2)$$

$$(5 \times 2)\begin{pmatrix} 1 & 0 \end{pmatrix} = 5[2\begin{pmatrix} 1 & 0 \end{pmatrix}]$$

$$10\begin{pmatrix} 1 & 0 \end{pmatrix} = 5\begin{pmatrix} 2 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 10 & 0 \end{pmatrix} = \begin{pmatrix} 10 & 0 \end{pmatrix}$$

Property-3(iii)

$$1\alpha = \alpha\ \forall \alpha \in V\ \textit{and 1 is the unity element of the field}\ \mathbb{F}.$$

$$1\begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

Vector sub-space

Let V be a vector space over the field $\mathbb{F}$ and let $W \subseteq V$. Then W is called a subspace of V if W itself is a vector space over $\mathbb{F}$ with respect to the operations of vector addition and scalar multiplication in V.

Example

The set of all $n \times n$ diagonal matrix is a proper subspace S_n of the vector space $V_{n\times n}$ formed by the set of all $n \times n$ matrices.

Linear dependence

Definition

A set of vectors $x_1, x_2, ..., x_n$ in a vector space V over a field $\mathbb{F}$ is said to be linearly dependent if there exist scalars $\alpha_1, \alpha_2, ..., \alpha_n$ not all zero,in $\mathbb{F}$ such that

$$\sum_{i=1}^{n} \alpha_i x_i = \alpha_1 x_1 + \alpha_2 x_2 + ..., +\alpha_n x_n = 0. \quad (1)$$

If on the other hand, (1) implies that $\alpha_i = 0$ for each i,then the set of vectors $x_1, x_2, x_3..., x_n$ is said to be linearly independent.

Linear dependence: Example

The vectors

$$x_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad x_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \quad x_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad x_4 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

are linearly dependent since

$$x_1 + 3x_2 + 6x_3 - x_4 = 0.$$
$$x_4 = x_1 + 3x_2 + 6x_3$$

x_4 can be expressed interms of x_1, x_2 and x_3.
If we consider x_1, x_2 and x_4 then

$$\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_4 = 0 \qquad (2)$$

if and only if $\alpha_1 = \alpha_2 = \alpha_3 = 0$.

Linear combination

Let X be a vector in a vector space V over a field $\mathbb{F}$. If there exist vectors $x_1, x_2, x_3 ..., x_n$ in V and scalars $\alpha_1, \alpha_2, ..., \alpha_n$ in $\mathbb{F}$ such that

$$X = \alpha_1 x_1 + \alpha_2 x_2 + ..., + \alpha_n x_n$$

then X is said to be a linear combination of the vectors $x_1, x_2, x_3, ..., x_n$.

Example:

$$X = \begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix} \qquad \textit{then} \qquad X = 3x_1 + 2x_2 + 2x_3 + x_4$$

where

$$x_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad x_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \quad x_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad x_4 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

Linear combination: contd

Alternatively, we can represent X as

$$X = 5x_1 + 8x_2 + 14x_3 - x_4$$

$$X = 4x_1 + 5x_2 + 8x_3.$$

In fact X admits infinitely many representations in terms of x_1, x_2, x_3 and x_4 which are linearly dependent.

But representation of X in terms of x_1, x_2 and x_3 which are linearly independent is unique.

$$X = 4x_1 + 5x_2 + 8x_3.$$

$$x_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad x_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \quad x_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad x_4 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

Basis and Dimensions

Basis

In a vector space V over a field $\mathbb{F}$, a basis is a set of linearly independent vectors $\{x_1, x_2, x_3..., x_n\}$ given in a definite order such that every vector X in V is a linear combination of the vectors in this set.

Dimensions

A dimension in a vector space V is the maximum number of linearly independent vectors in the space.
The set of there independent vectors is called a basis for that space.

Example:

The unit vectors $\hat{i}$, $\hat{j}$, $\hat{k}$ are linearly independent vectors in the Euclidean space. Dimensions $= 3$.

$$\hat{i} = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^T, \quad \hat{j} = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}^T, \quad \hat{k} = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^T,$$

Inner product space

Definition

Let $v(F)$ be a vector space where $\mathbb{F}$ is either the field of real numbers or the field of complex numbers. An inner product on v is a function from $v \times v$ into $\mathbb{F}$ which assigns to each ordered pair of vectors X, Y in V a scalar $\langle X, Y \rangle$ in such a way that

(i) $\langle X, Y \rangle = \overline{\langle Y, X \rangle}$

(ii) $\langle aX + bY, Z \rangle = a\langle X, Z \rangle + b\langle Y, Z \rangle$

(iii) $\langle X, X \rangle \geq 0$ and $\langle X, X \rangle = 0 \Rightarrow X = 0$ for any $X, Y, Z \in v$ and $a, b \in \mathbb{F}$.

The vector space v is then said to be an inner product space with respect to that specified inner product defined on it.

Inner product space: Example

On $v_n(c)$ there is an inner product which we call the standard inner product.
If $X = (X_1, X_2, \ldots, X_n), \quad Y = (Y_1, Y_2, \ldots, Y_n) \in v_n(c)$ then we define
$\langle X, Y \rangle = X_1\overline{Y_1} + X_2\overline{Y_2} + \ldots + X_n\overline{Y_n} = \sum_{i=1}^{n} X_i\overline{Y_i}$

Proof:

$$
\begin{aligned}
\text{(i)} \quad \langle X, Y \rangle &= \overline{\langle Y, X \rangle} \\
\langle Y, X \rangle &= Y_1\overline{X_1} + Y_2\overline{X_2} + \ldots + Y_n\overline{X_n} \\
\overline{\langle Y, X \rangle} &= \overline{Y_1\overline{X_1} + Y_2\overline{X_2} + \ldots + Y_n\overline{X_n}} \\
&= \overline{Y_1\overline{X_1}} + \overline{Y_2\overline{X_2}} + \ldots + \overline{Y_n\overline{X_n}} \\
&= \overline{Y_1}\overline{\bar{X}_1} + \overline{Y_2}\overline{\bar{X}_2} + \ldots + \overline{Y_n}\overline{\bar{X}_n} \\
&= X_1\overline{Y_1} + X_2\overline{Y_2} + \ldots + X_n\overline{Y_n} \\
\overline{\langle Y, X \rangle} &= \langle X, Y \rangle
\end{aligned}
$$

Inner product space: Example: contd

(ii) Let $Z = (Z_1, Z_2, \ldots, Z_n) \in v_n(c)$

$$\begin{aligned} aX + bY =& a(X_1, X_2, \ldots, X_n) + b(Y_1, Y_2, \ldots, Y_n) \\ =& aX_1 + bY_1, aX_2 + bY_2, \ldots, aX_n + bY_n \end{aligned}$$

$$\begin{aligned} \langle aX + bY, Z \rangle =& (aX_1 + bY_1)\overline{c_1} + \ldots + (aX_n + bY_n)\overline{c_n} \\ =& (aX_1\overline{c_1} + \ldots + aX_n\overline{c_n}) + (bY_1\overline{c_1} + \ldots + bY_n\overline{c_n}) \\ =& a(X_1\overline{c_1} + \ldots + X_n\overline{c_n}) + b(Y_1\overline{c_1} + \ldots + Y_n\overline{c_n}) \\ =& a\langle X, Z \rangle + b\langle Y, Z \rangle \end{aligned}$$

Inner product space: Example: contd

$$\text{(iii)} \quad \langle X, X\rangle = X_1\overline{X_1} + \ldots + X_n\overline{X_n} = |X_1|^2 + \ldots + |X_n|^2$$

$\overline{X_n}$ is a complex number.

$$\therefore \quad |X_n|^2 \geq 0.$$

$$\therefore \quad |X_1|^2 + \ldots + |X_n|^2 > 0$$

$$\text{Each} \quad |X_i|^2 = 0 \Rightarrow X = 0$$

Example:2

If $X = (X_1, X_2)$, $Y = (Y_1, Y_2) \in v_2(\mathbb{F})$.
By defining $\langle X, Y\rangle = X_1Y_1 - X_2Y_1 - X_1Y_2 + 4X_2Y_2$. Check all postulates of an inner product.

Norm or length of a vector in an inner product space

Let v be an inner product space. If $X \in v$, then the norm of the vector X, written as $\|X\|$ is defined as the positive square root of $\langle X, X\rangle$, that is

$$\|X\| = \sqrt{\langle X, X\rangle}$$

Example:

Consider a vector $X = a\hat{i} + b\hat{j} + c\hat{k}$.
The inner product of the vector $\langle X, X\rangle = a^2 + b^2 + c^2$.
Norm or length of a vector $\|X\| = \sqrt{a^2 + b^2 + c^2}$

Example:

Consider a matrix

$$X = \begin{pmatrix} 2 & 1 & 1 \end{pmatrix} \tag{1}$$

The inner product of the matrix $\langle X, X\rangle = 2^2 + 1^2 + 1^2 = 6$
Norm or length of the matrix $\|X\| = \sqrt{6}$

Unit Vector:

Let v be an inner product space. If $X \in v$ such that $\|X\| = 1$ then X is called a unit vector.

In an inner product space a vector is called a unit vector if its length is 1.

Example:

$$X = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \; Y = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \tag{2}$$

The matrices X and Y are found to have unit norm. Thus they are unit vectors.

Schwarz Inequality

In an inner product space $v(F)$ we can prove

$$||X, Y|| \leq \|X\| \, \|Y\|$$

The inequality holds when $X = 0$.

Application:

Cauchy-Schwartz inequality can be applied to obtain a solution for a complex problem.
(i) If x, y, z are three positive real numbers such that $x + y + z \leq 3$ then we can prove $\frac{1}{x} + \frac{1}{y} + \frac{1}{z}$ is always greater than or equal to 3.

Triangle inequality

If X, Y are vectors in an inner product space v, then

$$\|X + Y\| \leq \|X\| + \|Y\|.$$

Normed Vector Space

Let $v(\mathbb{F})$ be a vector space where $\mathbb{F}$ is either the field of real numbers or the field of complex numbers. Then v is said to be a normed vector space if to each vector X there corresponds a real number denoted by $\|X\|$ called the norm of X in such a manner that

$$\begin{array}{rl} (i) & \|X\| \geq 0 \; and \; \|X\| = 0 \;\Rightarrow X = 0. \\ (ii) & \|aX\| = |a|.\|X\| \quad \forall\, a \in \mathbb{F}. \\ (iii) & \|X+Y\| \leq \|X\| + \|Y\| \quad \forall\, X, Y \in v. \end{array}$$

Distance in an inner product space

Let $v(\mathbb{F})$ be an inner product space. Then we define the distance $\langle X, Y\rangle$ between two vectors X and Y by

$$d(X, Y) = \|X - Y\| = \sqrt{\langle X - Y, X - Y\rangle}$$

Orthogonality

Let X and Y be vectors in an inner product space V. Then X is said to be orthogonal to Y if

$$\langle X, Y\rangle = 0$$

Example

Let us consider the matrices

$$X = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \quad Y = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

The inner product of the matrices

$$\langle X, Y\rangle = 2(1) + 1(-1) + 1(-1) = 0$$

Orthogonal set

Let S be a set of vectors in an inner product space V. Then S is said to be an orthogonal set provided that any distinct vectors in S are orthogonal.

Example

Consider the matrices

$$X = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \; Y = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \; Z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

These matrices forms orthogonal set as they satisfy

$$\langle X, Y \rangle = \langle Y, Z \rangle = \langle X, Z \rangle = 0$$

Orthonormal set

Let S be a set of vectors in an inner product space V. Then S is said to be an orthonormal set if
(i) $X \in S \Rightarrow \|X\| = 1$ that is $\langle X, X \rangle = 1$ and
(ii) $X, Y \in S$ and $X \neq Y \Rightarrow \langle X, Y \rangle = 0$.
Thus an orthonormal set in an orthogonal set with the additional property that each vector in it is of length 1. In other words a set consisting of mutually orthogonal unit vectors is called an orthonormal set.

Example:

consider the matrices

$$X = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}, \ Y = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}, \ Z = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}.$$

These matrices found to have unit norm and they are orthogonal to each other. Thus forms the orthonormal set.

Ex.1:

The vectors

$$X = \begin{pmatrix} 1 & i & 0 \end{pmatrix}^T \qquad and \quad Y = \begin{pmatrix} i & 1 & 0 \end{pmatrix}^T$$

are orthogonal.

$$\langle X, Y \rangle = x_1 \bar{y_1} + x_2 \bar{y_2} + x_3 \bar{y_3} = 1(-i) + i(1) + 0 = 0$$

Ex. 2:

Find (i)$\langle X, Y \rangle$, (ii)$\langle Y, X \rangle$, (iii)$\|X\|$, (iv)$\|Y\|$ and (v) the normalized X and Y are orthogonal, where

$$X = \begin{pmatrix} 2 \\ i \\ 3 \end{pmatrix}, \qquad Y = \begin{pmatrix} 1+2i \\ 2i \\ 1 \end{pmatrix}$$

Answers

$$(i)\ \langle X, Y\rangle = x_1\bar{y_1} + x_2\bar{y_2} + x_3\bar{y_3} = 2(1-2i) + i(-2i) + 3\times 1$$
$$= 2 - 4i + 2 + 3 = 7 - 4i$$

$$(ii)\ \langle Y, X\rangle = y_1\bar{x_1} + y_2\bar{x_2} + y_3\bar{x_3} = (1+2i)2 + 2i(-i) + 1\times 3$$
$$= 2 + 4i + 2 + 3 = 7 + 4i$$

$$(iii)\ \|X\| = \sqrt{\langle X, X\rangle} = \sqrt{x_1\bar{x_1} + x_2\bar{x_2} + x_3\bar{x_3}}$$
$$= \sqrt{2\times 2 + i(-i) + 3\times 3} = \sqrt{4+1+9} = \sqrt{14}$$

$$(iv)\ \|Y\| = \sqrt{10}$$

(v) normalized X and Y

$$\frac{X}{\|X\|} = \frac{1}{\sqrt{14}}\begin{pmatrix}2\\ i\\ 3\end{pmatrix} = \begin{pmatrix}\frac{2}{\sqrt{14}}\\ \frac{i}{\sqrt{14}}\\ \frac{3}{\sqrt{14}}\end{pmatrix}$$

$$\frac{Y}{\|Y\|} = \begin{pmatrix}\frac{1+2i}{\sqrt{10}}\\ \frac{2i}{\sqrt{10}}\\ \frac{1}{\sqrt{10}}\end{pmatrix}$$

(vi) $d(X, Y) = \|(X - Y)\| = \sqrt{|x_1 - y_1|^2 + |x_2 - y_2|^2 + |x_3 - y_3|^2}$

$$= \sqrt{|1 - 2i|^2 + |-i|^2 + 2^2} = \sqrt{10}.$$

Properties

Any orthonomal set of vectors in an inner product space is linearly independent.

orthonormal basis

A basis of an inner product space that consists of mutually orthogonal unit vectors is called an orthonormal basis.

Orthonormal Set

The set of vectors

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

is an orthonormal set with respect to innner product defined above.

$$\langle e_1 e_2 \rangle = 1 \times 0 + 0 \times 1 + 0 \times 0 = 0$$

$$\langle e_1 e_3 \rangle = 0, \quad \langle e_2 e_3 \rangle = 0$$

Gram-Schmidt process of orthogonalization

Given a set of n linearly independent vectors in an inner product space v_n, it is always possible to construct an orthonormal basis by what is known as the G-S process.
(An orthonomal basis has several advantages).
For a set of m linearly independent vectors $x_1, x_2, ..., x_m$ in an inner product space v_n there exists a set of orthonormal vectors $y_1, y_2, ..., y_m$ defined by

$$\begin{aligned} y_1 &= x_1 \\ y_k &= x_k - \sum_{i=1}^{k-1} \frac{\langle x_k, y_i \rangle}{\langle y_i, y_i \rangle} y_i, \quad k = 2, ..., m. \end{aligned} \tag{3}$$

Example

Use the Gram-Schmidt orthonormalization process to determine an orthonormal basis in $\mathbb{R}^3$ for the given set of independent vectors.

$$x_1 = \begin{pmatrix} 1 & 0 & 1 \end{pmatrix}^T, \quad x_2 = \begin{pmatrix} -1 & 1 & 0 \end{pmatrix}^T, \quad x_3 = \begin{pmatrix} -3 & 2 & 0 \end{pmatrix}^T.$$

We have,

$$y_1 = x_1, \quad y_2 = x_2 - \frac{\langle x_2, y_1 \rangle}{\langle y_1, y_1 \rangle},$$

$$y_3 = x_3 - \frac{\langle x_3, y_1 \rangle}{\langle y_1, y_1 \rangle} y_1 - \frac{\langle x_3, y_2 \rangle}{\langle y_2, y_2 \rangle} y_2.$$

Example: Contd

$$y_1 = x_1 = \begin{pmatrix}1 & 0 & 1\end{pmatrix}^T, \quad \langle y_1, y_1 \rangle = 2.$$

$$\begin{aligned} y_2 &= x_2 - \frac{\langle x_2, y_1 \rangle}{\langle y_1, y_1 \rangle} y_1 = \begin{pmatrix}-1 & 1 & 0\end{pmatrix}^T \times \left(\frac{1}{2}\right) \begin{pmatrix}1 & 0 & 1\end{pmatrix}^T \\ &= \frac{1}{2} \begin{pmatrix}-1 & 2 & 1\end{pmatrix}^T, \qquad \langle y_2, y_2 \rangle = \frac{3}{2}. \end{aligned}$$

$$\begin{aligned} y_3 &= x_3 - \frac{\langle x_3, y_1 \rangle}{\langle y_1, y_1 \rangle} y_1 - \frac{\langle x_3, y_2 \rangle}{\langle y_2, y_2 \rangle} y_2 \\ &= \begin{pmatrix}-3 & 2 & 0\end{pmatrix}^T + \frac{3}{2} \begin{pmatrix}1 & 0 & 1\end{pmatrix}^T - \frac{7}{3} \begin{pmatrix}-\frac{1}{2} & 1 & \frac{1}{2}\end{pmatrix}^T \\ &= \frac{1}{3} \begin{pmatrix}-1 & -1 & 1\end{pmatrix}^T, \qquad \langle y_3, y_3 \rangle = \frac{1}{\sqrt{3}}. \end{aligned}$$

Example: Contd

Orthonormal basis

$$e_1 = \frac{y_1}{||y_1||} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 \end{pmatrix}^T$$

$$e_2 = \frac{y_2}{||y_2||} = \frac{1}{2}\sqrt{\frac{2}{3}} \begin{pmatrix} -1 & 2 & 1 \end{pmatrix}^T = \frac{1}{\sqrt{6}} \begin{pmatrix} -1 & 2 & 1 \end{pmatrix}^T$$

$$e_3 = \frac{y_3}{||y_3||} = \frac{1}{3}\sqrt{3} \begin{pmatrix} -1 & -1 & 1 \end{pmatrix}^T = \frac{1}{\sqrt{3}} \begin{pmatrix} -1 & -1 & 1 \end{pmatrix}^T$$

Example 2:

Using Gram-Schimdt process, orthogonalize the basis
$x_1 = (1, 2, 2)$, $x_2 = (-1, 0, 2)$, $x_3 = (0, 0, 1)$.

$$\begin{aligned}
y_1 &= x_1 = (1, 2, 2), & \|y_1\| = 3 \\
y_2 &= x_2 - \frac{\langle x_2, y_1 \rangle}{\langle y_1, y_1 \rangle} y_1 = \left(-\frac{1}{3}, -\frac{2}{3}, \frac{4}{3}\right), & \|y_2\| = 2 \\
y_3 &= x_3 - \frac{\langle x_3, y_1 \rangle}{\langle y_1, y_1 \rangle} y_1 - \frac{\langle x_3, y_2 \rangle}{\langle y_2, y_2 \rangle} y_2 = \left(\frac{2}{9}, -\frac{2}{9}, \frac{1}{9}\right), & \|y_3\| = \frac{1}{3}
\end{aligned}$$

Orthonormal basis

$$\begin{aligned} w_1 &= \frac{y_1}{\|y_1\|} = \frac{1}{3}(1,2,2) \\ w_2 &= \frac{y_2}{\|y_2\|} = \frac{1}{3}(-2,-1,2) \\ w_3 &= \frac{y_3}{\|y_3\|} = \frac{1}{3}(2,-2,1) \end{aligned}$$

Properties of Gram-Schimdt process

1. $V_k = x_k - (\alpha_1 x_1 + \ldots + \alpha_{k-1} x_{k-1})$, $1 \leq k \leq n$.
2. The span of $v_1, \ldots, v_k$ is same as the span of $x_1, \ldots, x_k$.
3. v_k is orthogonal to $x_1, \ldots, x_{k-1}$.

Realization in physics

Let us consider a particle in a box of width $2L$ extending from $x = -L$ to $x = L$.
The eigenstates are

$$|n\rangle = \psi_n(x) = \sqrt{\frac{1}{2L}} e^{\frac{in\pi x}{L}}$$

These eigenstates have orthonormal properties.

$$\int_{-L}^{L} \psi_n^*(x)\psi_n(x) = 1 \Rightarrow \langle n|n\rangle = 1$$

and

$$\int_{-L}^{L} \psi_n^*(x)\psi_n(x) = 0,\ n \neq m \Rightarrow \langle n|m\rangle = 0$$

The linear combination of all wave functions

$$
\begin{aligned}
f(x) &= \sum b_n \psi_n(x) = \frac{1}{\sqrt{2L}} \sum_n b_n e^{\frac{in\pi x}{L}}, \; n = 0, \pm 1, \pm 2 \\
&= \sum_{n=0}^{\infty} b_n |n\rangle
\end{aligned}
$$

Thus the eigenfunction $\psi_n(x)$ (or $|n\rangle$)

1. are normalized to unity ($\langle n|n\rangle = 1$)
2. are orthogonal to each other ($\langle n|m\rangle = 0, n \neq m$)
3. are able to express any function $|f(x)\rangle$ in terms of their linear combination.

The eigenfunction $|n\rangle$ may be treated as basis in the hypothetical finite/infinite dimensional space - linear vector space.