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UNIT - IV

VECTOR CALCULUS

Properties of $\vec{\nabla}$ operator

If ϕ and ψ are differentiable scalar functions and $\vec{A}$ and $\vec{B}$ are differentiable vector point functions, then

(i) $\vec{\nabla}(\phi + \psi) = \vec{\nabla}\phi + \vec{\nabla}\psi$

(ii) $\vec{\nabla} . \left(\vec{A} + \vec{B}\right) = \vec{\nabla}. \vec{A} + \vec{\nabla}. \vec{B}$

(iii) $\vec{\nabla} \times \left(\vec{A} \times \vec{B}\right) = \vec{\nabla} \times \vec{A} + \vec{\nabla} \times \vec{B}$

(iv) $\vec{\nabla}. \left(\phi\, \vec{A}\right) = \left(\vec{\nabla}\, \phi\right). \vec{A} + \phi\, (\vec{\nabla} . \vec{A})$

(v) $\vec{\nabla} \times \left(\phi\, \vec{A}\right) = \left(\vec{\nabla}\, \phi\right) \times \vec{A} + \phi(\vec{\nabla} \times \vec{A})$

(vi) $\vec{\nabla}\left(\vec{A} . \vec{B}\right) = \left(\vec{B} . \vec{\nabla}\right)\vec{A} + \left(\vec{A} . \vec{\nabla}\right)\vec{B} + \vec{B} \times \left(\vec{\nabla} \times \vec{A}\right) + \vec{A} \times (\vec{\nabla} \times \vec{B})$

(vii) $\vec{\nabla} . \left(\vec{A} \times \vec{B}\right) = \vec{B} . \left(\vec{\nabla} \times \vec{A}\right) - \vec{A} . (\vec{\nabla} \times \vec{B})$

(viii) $\vec{\nabla} \times \left(\vec{A} \times \vec{B}\right) = \left(\vec{B} . \vec{\nabla}\right)\vec{A} - \vec{B}\left(\vec{\nabla} . \vec{A}\right) - \left(\vec{A}. \vec{\nabla}\right)\vec{B} + \vec{A}\left(\vec{\nabla} . \vec{B}\right)$

Proof:(iv) $\vec{\nabla}.(\phi\,\vec{A}) = (\vec{\nabla}\,\phi).\vec{A} + \phi\,(\vec{\nabla}\,.\vec{A})$

Let $\vec{A} = A_1\,\hat{\imath} + A_2\,\hat{\jmath} + A_3\,\hat{k}$ $\qquad$ $\vec{\nabla}\,.(\phi\vec{A}) = \vec{\nabla}\,.(\phi\,A_1\,\hat{\imath} + \phi\,A_2\,\hat{\jmath} + \phi\,A_3\,\hat{k})$

$$= \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).(\phi A_1\hat{\imath} + \phi\,A_2\hat{\jmath} + \phi\,A_3\,\hat{k})$$

$$= \frac{\partial}{\partial x}(\phi\,A_1) + \frac{\partial}{\partial y}(\phi\,A_2) + \frac{\partial}{\partial z}(\phi\,A_3)$$

$$= \frac{\partial\phi}{\partial x}A_1 + \phi\frac{\partial A_1}{\partial x} + \frac{\partial\phi}{\partial y}A_2 + \phi\frac{\partial A_2}{\partial y} + \frac{\partial\phi}{\partial z}A_3 + \phi\frac{\partial A_3}{\partial z}$$

$$= \frac{\partial\phi}{\partial x}A_1 + \frac{\partial\phi}{\partial y}A_2 + \frac{\partial\phi}{\partial z}A_3 + \phi\left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right)$$

$$= \left(\hat{\imath}\frac{\partial\phi}{\partial x} + \hat{\jmath}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right).(\hat{\imath}\,A_1 + \hat{\jmath}A_2 + \hat{k}\,A_3) + \phi\left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right)$$

$$= \left(\hat{\imath}\frac{\partial\phi}{\partial x} + \hat{\jmath}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right).(\hat{\imath}\,A_1 + \hat{\jmath}A_2 + \hat{k}\,A_3) + \phi\left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).(A_1\hat{\imath} + A_2\hat{\jmath} + A_3\,\hat{k})$$

$$\boxed{\vec{\nabla}.(\phi\,\vec{A}) = (\vec{\nabla}\,\phi).\vec{A} + \phi\,(\vec{\nabla}\,.\vec{A})}$$

Second order $\vec{\nabla}$ operator

(i) $\vec{\nabla}\,.\left(\vec{\nabla}\phi\right) = \nabla^2\phi$

(H.W)

(ii) $\vec{\nabla} \times \left(\vec{\nabla}\,\phi\right) = 0$

(iII) $\vec{\nabla}\,.\left(\,\vec{\nabla}\,\phi \times \vec{\nabla}\,\phi\right) = 0$

Second Order $\vec{\nabla}$ Operator

If ϕ is a differentiable scalar point function and $\vec{V}$ a differentiable vector point function, then $\vec{\nabla}\phi$ and $\vec{\nabla} \times \vec{V}$ a vector point functions and $\vec{\nabla}.\vec{V}$ is a scalar point function. Hence, (i) div and $curl$ of $\vec{\nabla}\phi$ or $\vec{\nabla} \times \vec{V}$ and (ii) $grad$ $\vec{\nabla}.\vec{V}$ can be defined.

Let us derive some important formulae, applying the operator twice.

(i) $div\ \vec{\nabla}\phi$ or $\vec{\nabla}.(\vec{\nabla}\,\phi)$

$$\vec{\nabla}\,.(\vec{\nabla}\phi) = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\left(\hat{\imath}\frac{\partial \phi}{\partial x} + \hat{\jmath}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}\right)\phi$$

$$= \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial}{\partial z^2}\right)\phi$$

$$= \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\phi$$

$$= \nabla^2\phi$$

Thus $$\boldsymbol{\vec{\nabla}.(\vec{\nabla}\,\phi) = \vec{\nabla}^2\phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}}$$

(ii) $curl\ \vec{\nabla}\phi$ or $\vec{\nabla} \times (\vec{\nabla}\ \phi)$

$$\vec{\nabla} \times \left(\vec{\nabla}\phi\right) = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \left(\hat{\imath}\frac{\partial \phi}{\partial x} + \hat{\jmath}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}\right)\phi$$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix}$$

$$= i\ \left(\frac{\partial^2 \phi}{\partial y \partial z} - \frac{\partial^2 \phi}{\partial z \partial y}\right) + j\ \left(\frac{\partial^2 \phi}{\partial z \partial x} - \frac{\partial^2 \phi}{\partial x \partial z}\right) + k\ \left(\frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial y \partial x}\right)$$

Examples

Evaluate $(i)\ \vec{\nabla}\,(r^n)$ and $(ii)\ \nabla^2\,(r^n)$, where $r = (x^2 + y^2 + z^2)^{\frac{1}{2}}$

$$\vec{\nabla}\, \mathrm{r}^{\mathrm{n}} = \hat{\imath}\frac{\partial}{\partial x}(r^n) + \hat{\jmath}\frac{\partial}{\partial y}(r^n) + \hat{k}\frac{\partial}{\partial z}(r^n) = \hat{\imath}\frac{d}{dr}(r^n)\frac{dr}{dx} + \hat{\jmath}\frac{d}{dr}(r^n)\frac{dr}{dy} + \hat{k}\frac{d}{dr}(r^n)\frac{dr}{dz}$$

$$= n\,\hat{\imath}\,r^{n-1}\frac{dr}{dx} + n\,\hat{\jmath}\,r^{n-1}\frac{dr}{dx} + n\,\hat{k}\,r^{n-1}\frac{dr}{dz}$$

$$= n\,r^{n-1}\left[\hat{\imath}\frac{dr}{dx} + \hat{\jmath}\frac{dr}{dy} + \hat{k}\frac{dr}{dz}\right]$$

$$\frac{dr}{dx} = \frac{d}{dr}\left((x^2 + y^2 + z^2)^{\frac{1}{2}}\right) = \frac{x}{(x^2 + y^2 + z^2)^{\frac{1}{2}}} = \frac{x}{r}$$

Similarly $\frac{dr}{dy} = \frac{y}{r}$ and $\frac{dr}{dz} = \frac{z}{r}$

$$\nabla(r^n) = n\,r^{n-1}\left(\frac{\hat{\imath}x}{r} + \hat{\jmath}\frac{y}{r} + \hat{k}\frac{z}{r}\right) = n\,r^{n-2}\left(\hat{\imath}\,x + \hat{\jmath}\,y + \hat{k}\,z\right) = n\,r^{n-2}\vec{r}$$

(ii) $\nabla^2(r^n) = \vec{\nabla}.\vec{\nabla}(r^n) = \vec{\nabla}.(n\, r^{n-2}\, \vec{r}) = \vec{\nabla}.[n\, r^{n-2}\left(\hat{\imath}\, x + \hat{\jmath}\, y + \hat{k}\, z\right)$

$$= \frac{\partial}{\partial x}(n\, r^{n-2}\, x) + \frac{\partial}{\partial y}(n\, r^{n-2}\, y) + \frac{\partial}{\partial z}(n\, r^{n-2} z)$$

$$= n\,(n+1)\, r^{n-2}$$

H.W

(iii) $\phi = 2\, z^2\, y - x\, y^2$, find $\vec{\nabla}\phi$ and the directional derivative of ϕ at (2,1,1) in the direction of $3\,\hat{\imath} + 6\,\hat{\jmath} + 2\,\hat{k}$

Ans:

$$\vec{\nabla}\,\phi = \hat{\imath}\,(-y^2) + \hat{\jmath}\,(2\,z^2 - 2\,xy) + \hat{k}(4\,zy) \qquad \vec{\nabla}\phi\Big|_{2,1,1} = -\hat{\imath} - 2\,\hat{\jmath} + \hat{k}(4)$$

Directional Derivative $\quad \vec{\nabla}\,\phi.\,\vec{U}$

Unit vector in the direction which has to be calculated

$$\vec{U} = \frac{3\,\hat{\imath} + 6\,\hat{\jmath} + 2\,\hat{k}}{\left|3\,\hat{\imath} + 6\,\hat{\jmath} + 2\,\hat{k}\right|} = \frac{3\hat{\imath} + 6\,\hat{\jmath} + 2\,\hat{k}}{7}$$

$$\vec{\nabla}\,\phi\,.\left(\frac{3\hat{\imath}+6\,\hat{\jmath}+2\,\hat{k}}{7}\right) = \left(-\hat{\imath} - 2\,\hat{\jmath} + 4\hat{k}\right).\frac{3\hat{\imath}+6\,\hat{\jmath}+2\,\hat{k}}{7} = \frac{-3-12+8}{7} = -1$$

H.W

If $\phi = x\,y\,z$, $\vec{A} = 2\,y\,z\,\hat{\imath} - x^2\,y\,\hat{\jmath} + x\,z^2\,\hat{k}$ and $\vec{B} = x^2\,\hat{\imath} + y\,z\,\hat{\jmath} - x\,y\,\hat{k}$ find the value of

(i) $\vec{A}\,.\,\vec{\nabla}\,\phi$ (ii) $\vec{A} \times \left(\vec{\nabla}\,\phi\right)$ and (iii) $\left(\vec{A}\,.\,\vec{\nabla}\right)\vec{B}$

Ans:

(i) $2\,y^2 z^2 - x^3\,y\,z + x^2\,y\,z^2$

(ii) $-\,(x^3\,y^2 - x^2 z^3)\hat{\imath} + (\,xy\,z^3 - 2\,x\,y^2 z\,)\hat{\jmath} + (\,2\,x\,y\,z^2 - x^2 y^2 z)\hat{k}$

(iii) $4\,x\,y\,z\,\hat{\imath} + (\,x\,y\,z^2 - x^2\,y\,z)\hat{\jmath} + (\,x^3\,y - 2\,y^2\,z\,)\hat{k}$

2. If $\vec{a}$ is a constant vector, show that $\vec{\nabla} \times \left(\frac{\vec{a}\times\vec{r}}{r^3}\right) = -\frac{\vec{a}}{r^3} + \frac{3\,\vec{r}}{r^5}\,(\vec{a}.\vec{r})$, where $\vec{r}$ and r have usual meanings.

Hint: $$\frac{\vec{a}\times\vec{r}}{r^3} = \frac{\hat{\imath}(a_2 z - a_3 y) + \hat{\jmath}\,(a_3 x - a_1 z) + \hat{k}\,(a_1 y - a_2 x)}{r^3}$$

$$\vec{\nabla}\times\frac{\vec{a}\times\vec{r}}{r^3} = \vec{\imath}\left[\frac{\partial}{\partial y}\left(\frac{a_1 y - a_2 x}{r^3}\right) - \frac{\partial}{\partial z}\left(\frac{a_3 x - a_1 z}{r^3}\right)\right] + \hat{\jmath}\left[\frac{\partial}{\partial z}\left(\frac{a_2 z - a_3 y}{r^3}\right) - \frac{\partial}{\partial x}\left(\frac{a_1 y - a_2 x}{r^3}\right)\right]$$

$$+\,\vec{k}\left[\frac{\partial}{\partial x}\left(\frac{a_3 x - a_1 z}{r^3}\right) - \frac{\partial}{\partial y}\left(\frac{a_2 z - a_3 y}{r^3}\right)\right]$$

1 $$= -\frac{a_1}{r^3} + \frac{3\,x\,(a_1 x + a_2 y + a_3 z)}{r^5}$$

2 $$= -\frac{a_2}{r^3} + \frac{3\,y\,(a_1 x + a_2 y + a_3 z)}{r^5}$$

3 $$= -\frac{a_3}{r^3} + \frac{3\,z\,(a_1 x + a_2 y + a_3 z)}{r^5}$$

1 + 2 + 3 $$= -\frac{\vec{a}}{r^3} + \frac{3\,\vec{r}\,(\vec{a}.\vec{r})}{r^5}$$

3. Prove that $\vec{\nabla} \times (\vec{r} \times \vec{F}) = (\vec{\nabla} . \vec{F})\vec{r} - (\vec{r} . \vec{\nabla})\vec{F} - 2\vec{F}$

Proof: $\vec{\nabla}(\vec{\nabla} . \vec{V}) = \vec{\nabla} \times (\vec{\nabla} \times \vec{V}) + \nabla^2 V$

L.H.S: $\vec{\nabla}(\vec{\nabla} . \vec{\mathrm{V}})$

Let $\vec{V} = V_1\,\hat{\imath} + V_2\,\hat{\jmath} + V_3\,\hat{k}$ Then $\vec{\nabla} . \vec{\mathrm{V}} = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}$

$$\vec{\nabla}(\vec{\nabla} . \vec{\mathrm{V}}) = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) . \left(\frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}\right)$$

$$= \left(\frac{\partial^2 V_1}{\partial x^2} + \frac{\partial^2 V_2}{\partial x \partial y} + \frac{\partial^2 V_3}{\partial x \partial z}\right)\hat{\imath} + \left(\frac{\partial^2 V_1}{\partial y \partial x} + \frac{\partial^2 V_2}{\partial y^2} + \frac{\partial^2 V_3}{\partial y \partial z}\right)\hat{\jmath} + \left(\frac{\partial^2 V_1}{\partial x \partial z} + \frac{\partial^2 V_2}{\partial z \partial y} + \frac{\partial^2 V_3}{\partial z^2}\right)\hat{k}$$

R.H.S:

$$\left(\vec{\nabla} \times \vec{\mathrm{V}}\right) = \hat{\mathrm{i}} \left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right) + \hat{j}\left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right) + \hat{k}\left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)$$

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{\mathrm{V}}\right) = \left(\frac{\partial^2 V_2}{\partial y\, \partial x} + \frac{\partial^2 V_3}{\partial z \partial x} + \frac{\partial^2 V_1}{\partial y^2} - \frac{\partial^2 V_1}{\partial z^2}\right)\hat{\imath} + \left(\frac{\partial^2 V_3}{\partial z \partial y} + \frac{\partial^2 V_1}{\partial x \partial y} - \frac{\partial^2 V_2}{\partial z^2} - \frac{\partial^2 V_2}{\partial x^2}\right)\hat{j}$$
$$+ \left(\frac{\partial^2 V_1}{\partial x \partial z} + \frac{\partial^2 V_2}{\partial z \partial y} - \frac{\partial^2 V_3}{\partial x^2} - \frac{\partial^2 V_3}{\partial y^2}\right)\hat{k}$$

$$\vec{\nabla}\left(\vec{\nabla}.\vec{\mathrm{V}}\right) = \vec{\nabla} \times \left(\vec{\nabla} \times \vec{V}\right)$$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\left(V_1\hat{\imath} + V_2\hat{j} + V_3\hat{k}\right)$$

$$= \nabla^2 V$$

Directional Derivative

1. If $F(x,y,z) = xy^2 + 3\,x^2 \;- z^3$, find $\vec{\nabla}\, F$ at $(2,-1,4)$

$\vec{\nabla}\,\mathrm{F} = (\mathrm{y}^2 + 6\,\mathrm{x})\,\hat{\imath} + 2\,x\,y\,\hat{\jmath}\; - 3\,z^2\,\hat{k}$ $\qquad \therefore\ \vec{\nabla}\,\mathrm{F}\,\Big|_{2,-1,4} = 13\,\hat{\imath}\; - 4\,\hat{\jmath}\; - 48\,\hat{\mathrm{k}}$

2. Find the directional derivative of $f(x,y) = 2\,x^2y^3 + 6\,x\,y$ at $(1,1)$ in the direction of a unit vector whose angle with the positive x −axis is $\pi/6$.

$\vec{\nabla}\,f(\mathrm{x,y}) = (\,4\,\mathrm{x}\,\mathrm{y}^3 + 6\,\mathrm{y})\hat{\imath} + (6\,x^2\,y^2 + 6\,x)\hat{\jmath}$ $\qquad \vec{\nabla}\,f(1,1) = 10\,\hat{\imath} + 12\,\hat{\jmath}$

Suppose $\hat{u} = \cos\theta\,\hat{\imath} + \sin\theta\;\hat{\jmath}$ is a unit vector in the $xy -$ plane that makes an angle θ with the positive $x -$ axis.

Now at $\frac{\pi}{6}$, $\hat{u} = \cos 30\,\hat{\imath} + \sin 30\;\hat{\jmath} = \frac{\sqrt{3}}{2}\;\hat{\imath} + \frac{1}{2}\;\hat{\jmath}$

$$\therefore\ D_u f(1,1) = (\,10\,\hat{\imath} + 12\,\hat{\jmath}\,).\left(\frac{\sqrt{3}}{2}\;\hat{\imath} + \frac{1}{2}\;\hat{\jmath}\right) = 5\sqrt{3} + 6$$

Maximum and minimum values of Directional Derivative

1. The maximum value of the directional derivative is $\|\vec{\nabla} f\|$and it occurs when $\hat{u}$ has the same direction as $\vec{\nabla}\, f$.

2. The minimum value of the directional derivative is $\|\vec{\nabla} f\|$ and it occurs when $\hat{u}$ and $\vec{\nabla} f$ have opposite directions.

Example:

Find the directional derivative of $F(x,y,z) = x\,y^2 - 4\,x^2 y + z^2$ at $(1,-1,2)$ in the direction of $6\hat{\imath} + 2\,\hat{\jmath} + 3\hat{k}$. Also determine the maximum and minimum values of directional derivative.

$\vec{\nabla}\,\mathrm{F(x,y,z)} = (\mathrm{y}^2 - 8\,\mathrm{xy})\hat{\imath} + (\,2xy\ - 4\,x^2)\hat{\jmath}\ + 2\,z\hat{k}$ $\quad\quad$ $\vec{\nabla}\mathrm{F}(1,-1,2) = 9\,\hat{\imath}\ - 6\,\hat{\jmath} + 4\,\hat{k}$

$$\hat{u} = \frac{1}{7}\left(\,6\,\hat{\imath} + 2\,\hat{\jmath} + 3\,\hat{k}\right)$$

$$D_u\,F(1,-1,2) = \left(\,9\,\hat{\imath}\ - 6\,\hat{\jmath} + 4\,\hat{k}\right).\frac{1}{7}\left(6\,\hat{\imath} + 2\,\hat{\jmath} + 3\,\hat{k}\right)\ = \frac{54}{7}$$

The maximum value of $D_u\,F(1,-1,2) = \|\vec{\nabla} f(1,-1,2)\| = \sqrt{133}$

The minimum value of $D_u\,F(1,-1,2) = \|\vec{\nabla} f(1,-1,2)\| = -\sqrt{133}$

Example 1:

Find the maximum value of the directional derivative of $\phi = x^3\, y\, z$ at the point (1,4,1).

Ans: The directional directive is maximum in the direction of $\vec{\nabla}\,\phi$ and the maximum value is $|\vec{\nabla}\phi|$

$\vec{\nabla}\phi = 12\,\hat{\imath} + \hat{\jmath} + 4\,\hat{k},$ $\quad |\vec{\nabla}\,\phi| = \sqrt{161}$ **(H.W)**

Example 2:

In what direction from $(3,1,-2)$ in the directional derivative of $\phi = x^2y^2z^4$ maximum? Find also the magnitude of this maximum?

Ans: $\phi = x^2\,y^2z^4$ $\quad \vec{\nabla}\phi\Big|_{3,1,-2} = 96\left(\hat{\imath} + 3\,\hat{\jmath} - 3\,\hat{k}\right).$ **(H.W)**

The directional derivative is maximum in the direction $96\left(\hat{\imath} + 3\,\hat{\jmath} - 3\hat{k}\right)$.

Maximum value $|\vec{\nabla}\phi| = 96\,\sqrt{19}$

Example 3:

In what direction from $(1,1,-2)$ is the directional derivative of $\phi = x^2 - 2y^2 + 4z^4$ maximum? Also find the maximum directional derivative.

Ans: $\vec{\nabla}\phi|_{1,1,-2} = 2(\,\hat{\imath} - 2\,\hat{\jmath} - 8\,\hat{k}),$ $\quad |\vec{\nabla}\,\phi| = 2\sqrt{69}$ **(H.W)**

Example 4:

What is the greatest rate of increase of $\phi = x\,y\,z^2$ at (1,0,3)?

Ans: The greatest rate of increase is the maximum value of the directional derivative and it is equal to $|\vec{\nabla}\,\phi|$.

Ans: 9 **(H.W)**

Example 5:

Find the angle between the surfaces $x^2 + y^2 + z^2 = 9$ and $x^2 + y^2 - z = 3$ at the point $(2,-1,2)$.

Ans: The given surfaces are $x^2 + y^2 + z^2 = 9$ and $x^2 + y^2 - z = 3$.
$P(2,-1,2)$ is the common point.

$$\vec{\nabla}\,\mathrm{f}\Big|_{2,-1,2} = 4\,\hat{\imath} - 2\,\hat{\jmath} + 4\hat{\mathrm{k}} \qquad \vec{\nabla}\mathrm{g}\Big|_{2,-1,2} = 4\,\hat{\imath} - 2\hat{\jmath} - \hat{\mathrm{k}}$$

Let θ be the angle. Then $\cos\theta = \dfrac{\vec{\nabla}f.\vec{\nabla}g}{|\vec{\nabla}f|.|\vec{\nabla}g|} = \dfrac{8}{3\sqrt{21}}$

$$\boxed{\theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)}$$

Example:

Find the angle between the normal to the surface $x\,y = z^2$ at the points $(1,4,2)$ and $(-3,-3,3)$.

Ans: $\phi = x\,y - z^2 \qquad \vec{\nabla}\phi_1\Big|_{1,4,2} = 4\,\hat{\imath} + \hat{\jmath} - 4\,\hat{k} \qquad \vec{\nabla}\,\phi_2\Big|_{-3,-3,3} = -3\,\hat{\imath} - 3\,\hat{\jmath} - 6\,\hat{k}$

$$\cos\theta = \frac{\vec{\nabla}\phi_1.\vec{\nabla}\,\phi_2}{|\vec{\nabla}\phi_1||\vec{\nabla}\phi_2|} = \frac{1}{\sqrt{22}}$$

Example:

Find the directional derivative of the function $\phi = x\,y^2 + y\,z^3$ at the point $(2,-1,1)$ in the direction of the normal to the surface $x \log z - y^2 + 4 = 0$ at the point $(-1,2,1)$.

Ans: $\vec{\nabla}\phi\Big|_{2,-1,1} = \hat{\imath} - 3\,\hat{\jmath} - 3\,\hat{k} \qquad \vec{\nabla}f\Big|_{-1,2,1} = -4\,\hat{\jmath} - \hat{k}$

$$\text{D.D} = \vec{\nabla}\phi.\frac{\vec{a}}{|\vec{a}|} = \left(\hat{\imath} - 3\,\hat{\jmath} - 3\,\hat{k}\right).\frac{(-4\,\hat{\jmath} - \hat{k})}{\sqrt{17}} = \frac{15}{17}$$

Tangent Plane

Let P(x_0, y_0, z_0) be a point on the graph of $F\ (x, y, z) = c$, where $\vec{\nabla} F$ is not 0. The tangent plane at P is that plane through P that is perpendicular to $\vec{\nabla} F$ evaluated at P.

Equation of a tangent plane

Let $P(x_0, y_0, z_0)$ be a point on the graph of $F(x, y, z) = c$, where $\vec{\nabla} F$ is not 0. Then an equation of the tangent plane at P is

$$\boxed{F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0.}$$

Example 1 : Find an equation of the tangent plane to the graph of $x^2 - 4\,y^2 + z^2 = 16$ at (2,1,4).

Ans: Recall $F(x, y, z) = c \;\rightarrow F = x^2 - 4\,y^2 + z^2$

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0.$$

$$\vec{\nabla}\,\mathrm{F} = (\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z})(x^2 - 4\,y^2 + z^2)$$

$= 2\,\mathrm{x}\,\hat{\imath} - 8\,\mathrm{y}\,\hat{\jmath} + 2\,\mathrm{z}\,\hat{\mathrm{k}}$ and $\vec{\nabla}\,F(2,1,4) = 4\,\hat{\imath} - 8\,\hat{\jmath} + 8\,\hat{k} \neq 0$

Then $F_x = 2x,\quad F_y = -8y,\quad F_z = 2z.$

$F_x(2,1,4) = 4,\qquad F_y = (2,1,4) = -8,\qquad F_z(2,1,4) = 8$

$$4(x - 2) - 8(y - 1) + 8\,(z - 4) = 0$$

or

$$4x - 8y + 8z = 32 \quad \rightarrow \quad \boxed{\boldsymbol{x - 2y + 2z = 8}}$$

Example 2 : Find an equation of the tangent plane to the graph at $z = \frac{x^2}{2} + \frac{y^2}{2} + 4$ at $(1, -1, 5)$.

Ans:

Recall: $F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0.$

$\vec{\nabla}\text{F} = \text{x}\,\hat{\text{i}} + \text{y}\,\hat{\text{j}} - \hat{\text{k}}$ and $\vec{\nabla}F(1, -1,5) = \hat{\imath} - \hat{\jmath} - \hat{k}$

$F_x\,(1, -1,5) = 1 \qquad F_y(1, -1,5) = 1 \qquad F_z(1, -1,5) = -1$

$$(x - 1) + (y + 1) - (z - 5) = 0$$

$$x + y - z = -3$$

$$\downarrow$$

Equation of Tangent Plane: $\boxed{-x + y + z = 3}$

Example: 6 Find the equation of the tangent plane and the equation of the normal to the surface $x^2 - 4y^2 + 3z^2 + 4 = 0$ at the point $(3, 2, 1)$.

$$\vec{\nabla}\phi = 2x\,\hat{\imath} - 8\,y\,\hat{\jmath} + 6\,z\,\hat{k}$$

$$\phi_x = 2x\ , \qquad \phi_y = -8\,y, \qquad \phi_z = 6\,z$$

$$\phi_x\Big|_{3,2,1} = 6, \qquad \phi_y\Big|_{3,2,1} = -16, \phi_z\Big|_{3,2,1} = 6$$

Equation of the tangent plane $\quad 6\,(x - 3) - 16\,(y - 2) + 6\,(z - 1) = 0$

Equation of the normal at (x_0, y_0, z_0) is

$$\frac{x - x_0}{\phi_x} = \frac{y - y_0}{\phi_y} = \frac{z - z_0}{\phi_z}$$

Equation of the normal at (3,2,1) is

$$\frac{(x-3)}{3} = \frac{y-2}{-8} = \frac{z-1}{3}$$

Example: 7

1. Find the equation of the tangent plane and normal to the surface $x^2 + y^2 + z^2 = 25$ at $(4,0,3)$.

$$\vec{\nabla}\,\phi\Big|_{4,0,3} = 8\,\hat{\imath} + 6\,\hat{k}$$

Equation of tangent plane $4\,x + 3\,z = 25$ **(H.W)**

Equation of the normal to the surface at $(4,0,3)$ is

$$\frac{x-4}{4} = \frac{y}{0} = \frac{z}{3}$$ **(H.W)**

2. If the of $\phi = a(\,x+y) + b(y+z) + c(z+x)$ has a maximum value 12 at $(1,2,1)$ in the direction parallel to the line $\frac{x-1}{1} = \frac{y-2}{2} = \frac{z-1}{3}$, find the value of a, b, c.

Ans. $a = 0$ $b = \pm\frac{24}{\sqrt{14}}$ $c = \pm\frac{12}{\sqrt{14}}$ **(H.W)**

Home work:

Find an equation of the tangent plane to the graph of the given equation at the indicated point.

1. $x^2 + y^2 + z^2 = 9$; $(-2, 2, 1)$

2. $x^2 - y^2 - 3z^2 = 5$; $(6, 2, 3)$

3. $z = 25 - x^2 - y^2$; $(3, -4, 0)$

4. $z = \cos(2x + y)$; $\left(\frac{\pi}{2}, \frac{\pi}{4}, -\frac{1}{\sqrt{2}}\right)$

5. $z = \ln(x^2 + y^2)$; $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$

Homework

1. Show that $\vec{\nabla} . [f(r)\vec{r}] = 3\, f(r) + r\, f'(r)$ where $\vec{r} = x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}$

2. Find a unit vector perpendicular to the surface $x^2 + y^2 + z^2 = 11$, at the point $(4{,}2{,}3)$.

   Hint: Determine $\vec{\nabla}\phi$ then evaluate at the point 4,2,3. Then find directional derivative

3. Find the angle between the surfaces $x^2 + y^2 + z^2 = 9$ and $x^2 + y^2 - z = 3$ at the point $(2, -1{,}2)$.

   Hint: $\vec{\nabla}\,\phi_1, \vec{\nabla}\,\phi_2$ at the point $(2, -1{,}2)$

   Then $(\vec{\nabla}\phi_1).(\vec{\nabla}\,\phi_2) = |\vec{\nabla}\,\phi_1|\ |\vec{\nabla}\,\phi_2| \cos\theta$,

   $\boxed{\text{Ans:}\theta = \cos^{-1}\left(\frac{3}{3\sqrt{21}}\right)}$

   from the above find θ.

4. Calculate the directional derivative of the function $\phi = xy^2 + y\,z^3$ at the point $(2, -1{,}1)$ in the direction of $(\hat{\imath} + 2\,\hat{\jmath} + 2\,\hat{k})$.

   $\boxed{\text{Ans:} -\frac{11}{3}}$

5. Find the directional derivative of $\phi = x^2 - 2y^2 + 4z^2$ at (1,1,-1) in the direction $(2\,\hat{\imath} + \hat{\jmath} - \hat{k})$. In what direction is the directional derivative from the point $(1{,}1, -1)$ is maximum and what is its value?

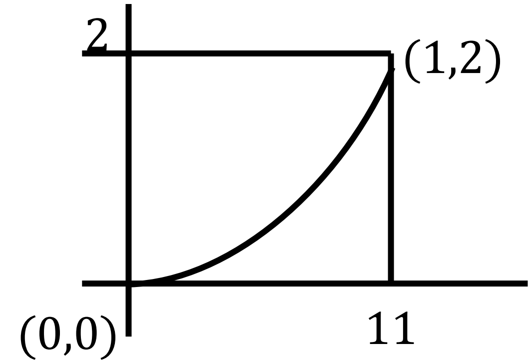
6. Find the constants a and b, so that the surface $ax^2 - b\,y\,z = (a+2)x$ will be orthogonal to the surface $4x^2y + z^3 = 4$ at the point $(1, -1{,}2)$.

Line Integral - Examples

Example: 1

If $\vec{F} = 3\,x\,y\,\hat{\imath} - y^2\,\hat{\jmath}$, evaluate $\int_c \vec{F}\,.\,d\vec{r}$, where c is the arc of the parabola $y = 2\,x^2$ from (0,0) to (1,2).

Given $\quad \vec{F} = 3x\,y\,\hat{\imath} - y^2\,\hat{\jmath}, \quad \vec{r} = x\,\hat{\imath} + y\,\hat{\jmath}, \quad d\vec{r} = dx\,\hat{\imath} + dy\,\hat{\jmath},$



$$y = 2\,x^2 \qquad \therefore\ d\vec{r} = dx\,\hat{\imath} + 4\,x\,\hat{\jmath}$$

$$\int \vec{F}\,.\,d\vec{r} = \int [3x(2x^2)\hat{\imath} - (\,4x^4)\hat{\jmath}].\,[dx\,\hat{\imath} + 4x\,\hat{\jmath}]$$

$$= \int_0^1 (6\,x^3 - 16\,x^5\,)dx$$

$$\boxed{\int \vec{F}\,.d\vec{r} \quad = \quad -\frac{7}{6}}$$

Example: 2

Line Integral - Examples

If $\vec{F} = (3\,x^2 + 6\,y)\hat{\imath}\; - 14yz\,\hat{\jmath} + 20\,x\,z^2\,\hat{k}$, evaluate $\int \vec{F}\,.\,d\vec{r}$ from (0,0,0) to (1,1,1) along the curve c by $x = t,\, y = t^2, z = t^3,$

$$x = t,\, y = t^2, z = t^3 \quad \Longrightarrow \quad dx = dt, \qquad dy = 2t\,dt, \qquad dz = 3\,t^2\,dt.$$

$$\vec{F} = (\,3\,t^2 + 6\,t^2)\hat{\imath} - 14\,(t^2)(t^3)\hat{\jmath} + 20\,(t)(t^6)\hat{k}$$

$$= 9\,t^2\,\hat{\imath} - 14\,t^5\,\hat{\jmath} + 20\,t^7\,\hat{k}$$

$$\vec{r} = \;x\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k} \quad = t\,\hat{\imath} + t^2\hat{\jmath} + t^3\,\hat{k}$$

$$d\vec{r} = \left(\vec{\imath} + 2\,t\,\hat{\jmath} + 3\,t^2\hat{k}\right)dt$$

$$\vec{F}\,.\,d\vec{r} = 9\,t^2\; - 28\,t^6 + 60\,t^9$$

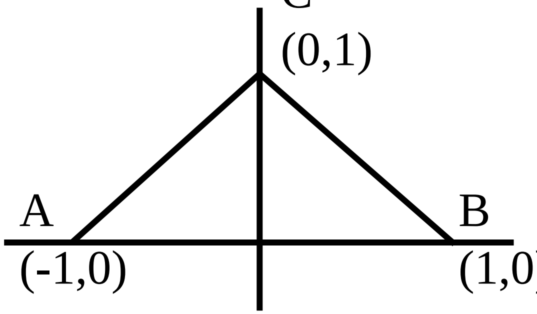
$$\boxed{\int_0^1 \vec{F}\,.\,d\vec{r} = \int_0^1 (9\,t^2\; - 28\,t^6 + 60\,t^9)\,dt \quad = 5}$$

Line Integral - Examples

Example: 3

Evaluate the line integral $\int y^2\,dx - x^2\,dy$ around the triangle whose vertices are $(1,0), (0,1), (-1,0)$ in the positive sense.

Equation of straight line $\quad \dfrac{y - y_1}{y_1 - y_2} = \dfrac{x - x_1}{x_1 - x_2}$



Eq. of AB is $y = 0$.

Eq. of BC is $\quad \dfrac{y-0}{0-1} = \dfrac{x-1}{1-0} \rightarrow \quad y = -x + 1$

Eq. of CA is $\quad \dfrac{y-1}{1-0} = \dfrac{x-0}{0+1} \rightarrow \quad y = x + 1$

$$\int_c \vec{F}\,.\,d\vec{r} = \int_{AB} y^2\,dx - x^2\,dy + \int_{BC} y^2\,dx - x^2\,dy + \int_{CA} y^2\,dy - x^2\,dy$$

$\Downarrow$ 0 $\quad\quad \Downarrow$ -2/3 $\quad\quad \Downarrow$ 0

$$\boxed{\int_c \vec{F}\,.\,d\vec{r} = -2/3}$$

Example: 4

Show that the vector field $\vec{f}$, where $\vec{f} = (y + y^2 + z^2)\,\hat{\imath} + (x + z + 2\,xy\,)\,\hat{\jmath} + (y + 2\,x\,z\,)\,\hat{k}$ is conservative and find its scalar potential.

$$\vec{\nabla} \times \vec{f} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y + y^2 + z^2 & x + z + 2\,x\,y & y + 2\,x\,z \end{vmatrix} = 0$$

$\therefore F$ is conservative.

- Potential: $\vec{F} = \vec{\nabla}\,\phi(x, y, z)$

$$(y + y^2 + z^2)\hat{\imath} + (\,x + z + 2\,x\,y\,)\hat{\jmath} + (\,y + 2\,x\,z\,)\hat{k} = \frac{\partial \phi}{\partial x}\,\hat{\imath} + \frac{\partial \phi}{\partial y}\,\hat{\jmath} + \frac{\partial \phi}{\partial z}\,\hat{k}$$

$$\frac{\partial \phi}{\partial x} = y + y^2 + z^2 \Longrightarrow 1 \qquad \frac{\partial \phi}{\partial y} = x + z + 2\,x\,y \Longrightarrow 2 \qquad \frac{\partial \phi}{\partial z} = y + 2\,x\,z \Longrightarrow 3$$

Integrating 1 $\Longrightarrow$ $\phi = x\,y + x\,y^2 + x\,z^2 + \phi_1\,(y, z) \Longrightarrow 4$

Arbitrary function

Substituting 3 in 2

$$x + 2\,x\,y + \frac{\partial \phi}{\partial y} = x + z + 2\,x\,y \implies \frac{\partial \phi_1}{\partial y} = z \xrightarrow{\text{Integrating}} \phi_1 = y\,z + \phi_2\,(z)$$

Arbitrary function

5

Substituting 5 in $\phi = x\,y + x\,y^2 + x\,z^2 + \phi_1\,(y, z)$

$$\phi = x\,y + x\,y^2 + x\,z^2 + y\,z + \phi_2\,(z) \xrightarrow{\text{Differentiating}} \frac{\partial \phi}{\partial z} = 2\,x\,z + y + \frac{\partial \phi_2}{\partial z}$$

$$2\,x\,z + y + \frac{\partial \phi_2}{\partial z} = y + 2\,x\,z \implies \frac{\partial \phi_2}{\partial z} = 0 \xrightarrow{\text{Integrating}} \phi_2 = c$$

$$\therefore \phi = x\,y + x\,y^2 + x\,z^2 + y\,z + c$$

Example: 5

Show that $\vec{F} = (2\,x\,y + z^3)\hat{\imath} + x^2\,\hat{\jmath} + 3\,x\,z^2\,\hat{k}$ is a conservative vector field. Find the scalar potential and the work done in moving an object in this field from $(1,-2,1)$ to $(3,1,4)$.

$$\phi = x^2\,y + x\,z^3 + c \qquad \textbf{(H.W)}$$

$$\vec{F} = \vec{\nabla}\,\phi \quad \Longrightarrow \quad \vec{F} = \hat{\imath}\frac{\partial\phi}{\partial x} + \hat{\jmath}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\ ; \quad \vec{r} = x\,\hat{\imath} + y\,\hat{\jmath} + \hat{z}\ ; \ d\vec{r} = dx\,\hat{\imath} + dy\,\hat{\jmath} + dz\,\hat{k}$$

$$\vec{F}\,.\,d\vec{r} = \left(\hat{\imath}\frac{\partial\phi}{\partial x} + \hat{\jmath}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right).\left(dx\,\hat{\imath} + dy\,\hat{\jmath} + dz\,\hat{k}\right) \ = \frac{\partial\phi}{\partial x}\,dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}\,dz = \boldsymbol{d\phi}$$

$$\int \vec{F}\,.\,d\vec{r} = \int d\,\phi\,(x, y\,z) \xRightarrow{\text{Integrating}} \phi(x,y,z)$$

Work done by the Force from $(1,-2,1)$ to $(3,1,4)$ is

$$\phi(x,y,z)\Big|_{1,-2,1}^{3,1,4} = x^2\,y + x\,z^3\,\Big|_{1,-2,1}^{3,1,4} = 202$$

Example:

Show that surfaces $5x^2 - 2\,yz - 9x = 0$ and $4\,x^2y + z^3 - 4 = 0$ are orthogonal at $(1,-1,2)$.

Ans: $\vec{\nabla}\,\mathrm{f}\Big|_{1,-1,2} = \hat{\imath} - 4\,\hat{\jmath} + 2\,\hat{\mathrm{k}}$ $\qquad$ $\vec{\nabla}\mathrm{g}\Big|_{1,-1,2} = -8\,\hat{\imath} + 4\,\hat{\jmath} + 12\,\hat{\mathrm{k}}$

$\vec{\nabla}\mathrm{f}.\vec{\nabla}\mathrm{g} = 0$ The surfaces are orthogonal.

Example:

Find a and b if the surfaces $a\,x^2 - b\,y\,z = (a+2)x$ and $4x^2y + z^3 = 4$ cut orthogonally at $(1,-1,2)$.

Ans:

$\vec{\nabla}\mathrm{f}\Big|_{1,-1,2} = (\mathrm{a}-2)\hat{\imath} - 2\,b\,\hat{\jmath} + b\,\hat{k},$ $\qquad$ $\vec{\nabla}g\Big|_{1,-1,2} = -8\,\hat{\imath} + 4\,\hat{\jmath} + 12\,\hat{k}$

$\vec{\nabla}\mathrm{f}.\,\vec{\nabla}\mathrm{g} = 0 \quad \Longrightarrow \quad 2a - b = 4$

Since $(1,-1,2)$ is a point on the surface $f = 0$, we get $a + 2\,b - (a+2) = 0$

$$\boxed{a = \frac{5}{2}, \qquad b = 1}$$

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