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UNIT - I

ORDINARY DIFFERENTIAL EQUATION

An equation containing the derivative of one or more dependent variables, with respect to one or more independent variables, is said to be a differential equation.

Differential Equation (DE)

Ordinary Differential Equation (ODE)

If an equation contains only ordinary derivatives of one or more dependent variables (y) with respect to a single independent variable (x), it is said to be an ordinary differential equation.

Example 1 :

$$\frac{dy}{dx} + 5y = e^x$$

1st derivative

Dependent variable **(one)**

Independent variable **(one)**

Example 2 :

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} + 6y = 0$$

2nd derivative

Example 3 :

$$(2x-3)\frac{d^3y}{dx^3} - (6x-7)\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} - 4y = 8$$

3rd derivative

2nd derivative

Independent variable

Dependent variable

Differential Equation (DE)

2. Partial Differential Equation (PDE)

An equation involving the partial derivatives of one or more dependent variables (u) with respect to two or more independent variables (x,t) is called a partial differential equation.

Example 1 :

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Dependent variable (u)

Independent variables (more than one- (x, y))

Example 2 :

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} - 2\frac{\partial u}{\partial t}$$

Example 3 :

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

Order of Equation = 2 (Highest derivative)
No. of dependent variable =1 ($u(t, x)$)
No. of independent variables = 2 (t, x)

All the above equations are second order **PDE**s

ORDER AND DEGREE OF DIFFERENTIAL EQUATIONS

- Differential equations are classified on the basis of two features (i) Order and (ii) Degree

Differential Equation (DE)

Order

- Order is the highest derivative that appear in the equation

Example 1:

$$5\frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right)^3 + 2y = e^x + 10$$

1^{st} derivative

2^{nd} derivative

Highest derivative = Order of the equation = **2**

Example 2:

$$x\frac{dy}{dx} + \left(\frac{dy}{dx}\right)^{-1} = e^x + 10 \Rightarrow x\frac{dy}{dx} + \frac{1}{\frac{dy}{dx}} = e^x + 10$$

Multiply by $\frac{dy}{dx}$

$$x\left(\frac{dy}{dx}\right)^2 + 1 = (e^x + 10)\frac{dy}{dx}$$

1^{st} derivative → **Highest** derivative = Order of the equation = **1**

Example 3:

Radical

$$\left(\frac{d^3y}{dx^3}\right)^{\frac{1}{2}} = \left[1 + \frac{dy}{dx} + \left(\frac{dy}{dx}\right)^2\right]^{\frac{2}{3}}$$

To estimate the order or degree of a D. E. radical should be removed first.

Multiply powers by 6,

$$\left(\frac{d^3y}{dx^3}\right)^{\frac{1}{2}(6)} = \left[1 + \frac{dy}{dx} + \left(\frac{dy}{dx}\right)^2\right]^{\frac{2}{3}(6)}$$

$$\left(\frac{d^3y}{dx^3}\right)^{3} = \left[1 + \frac{dy}{dx} + \left(\frac{dy}{dx}\right)^2\right]^{4}$$

Only 1st derivative

Third derivative

Highest derivative = Order of the equation = **3**

SYSTEM OF EQUATIONS

Example 4:

x & *y* Dependent variables

$$\frac{dx}{dt} = 5x + 3y \qquad \frac{dy}{dt} = 2x - 7y$$

t independent variable

Name: Two coupled 1st order equation / System of 1st order equations

Example 5:

$$\frac{dx}{dt} = 5x + 3y + 4z \qquad \frac{dy}{dt} = 2x - 7y - 13z \qquad \frac{dz}{dt} = x + y + z$$

Name: Three coupled 1st order equation

Note: In physics, $\dot{x}$, $\dot{y}$, $\dot{z}$ (dot notation) is used instead of $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$

ORDER AND DEGREE OF DIFFERENTIAL EQUATIONS

- Differential equation are classified on the basis of two features (i) Order and (ii) Degree

Differential Equation (DE)

Order | **Degree**

Degree

- Power of the highest derivative that appear in the Differential Equation

Power = Degree

<u>Example 1:</u>

$$5\left(\frac{d^2y}{dx^2}\right)^4 - \left(\frac{dy}{dx}\right)^3 + 2y = e^x + 10$$

Highest derivative

<u>Degree = 4</u> (Power of the highest derivative)

Example 1: Find the order and degree of the differential equation

$$\frac{d^2y}{dx^2} - \left[1 + \left(\frac{dy}{dx}\right)\right]^{3/2} = 0$$

Answer : Let us rewrite the equation in the following form

$$\frac{d^2y}{dx^2} = \left[1 + \left(\frac{dy}{dx}\right)\right]^{3/2}$$

radical

To determine the degree the radical should be removed

Let us square the equation on both sides

$$\left(\frac{d^2y}{dx^2}\right)^2 = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^3$$

On expanding

Power of the highest derivative = **<u>Degree = 2</u>**

$$\left(\frac{d^2y}{dx^2}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^3 + 3\left(\frac{dy}{dx}\right)^2 + 3\left(\frac{dy}{dx}\right)^4$$

Highest derivative = **<u>Order = 2</u>**

Example 2: Find the order and degree of the differential equation

$$x\frac{\mathrm{d}y}{\mathrm{d}x}+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^{-1}=y^2$$

Rewriting

$$x\frac{dy}{dx}+\frac{1}{\frac{dy}{dx}}=y^2$$

Simplifying

$$x\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2+1=y^2\frac{\mathrm{d}y}{\mathrm{d}x}$$

or

Power of the highest derivative = **Degree = 2**

$$x\left(\frac{dy}{dx}\right)^2-y^2\frac{dy}{dx}+1=0$$

Highest derivative = **Order = 1**

HOME WORK

State the order and Degree of the following Differential equations

1. $$\left(\frac{d^2y}{dx^2}\right)^4 + 3\left(\frac{dy}{dx}\right)^6 + 4 = 0$$

2. $$\left(\frac{dy}{dx}\right)^2 + 2\left(\frac{dx}{dy}\right)^2 = xy$$

3. $$\left(\frac{d^3y}{dx^3}\right)^{1/2} = \left[1 + \frac{dx}{dy} + \left(\frac{dy}{dx}\right)^2\right]^{2/3}$$

4. $$\frac{d^2y}{dx^2} - \sqrt{x + \frac{dy}{dx}} = 1$$

5. $$\frac{dx}{dt} = x + 3y \;;\; \frac{dy}{dt} = 5x + 3y$$

FORMATION OF DIFFERENTIAL EQUATION

Aim: To construct an DE from the given equation with two variables

Example:

Consider a trigonometric equation $y = a\sin(x + b)$, where a and b are parameters. Construct a DE free of those constants.

Answer:

$$y = a\sin(x + b)$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = a\cos(x + b)$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\boxed{a\sin(x + b)}$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -y$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + y = 0$$

From solution to differential equation

From differential equation to solution

Lesson: By eliminating two constants we end up at 2nd order equation

FORMATION OF DIFFERENTIAL EQUATION

Example 2 :

Consider the algebraic equation $x^2+y^2+2gx+2fy+c=0$. Construct a differential equation free of those constants. Assume y is a function of x.

Need to be eliminated

Answer: $$x^2+y^2+2gx+2fy+c=0 \longrightarrow (1)$$

$$\frac{d}{dx}(x^2+y^2+2gx+2fy+c)=0 \Rightarrow 2x+2y\frac{dy}{dx}+2g+2f\frac{dy}{dx}=0 \longrightarrow (2)$$

$$2\frac{d}{dx}\left(x+y\frac{dy}{dx}+g+f\frac{dy}{dx}\right)=0 \Rightarrow 1+\left(\frac{dy}{dx}\right)^2+y\frac{d^2y}{dx^2}+f\frac{d^2y}{dx^2}=0 \longrightarrow (3)$$

$$\frac{d}{dx}\left(1+\left(\frac{dy}{dx}\right)^2+y\frac{d^2y}{dx^2}+f\frac{d^2y}{dx^2}\right)=0 \Rightarrow \frac{3}{\cancel{2}}\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right)+\cancel{\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right)}+y\frac{d^3y}{dx^3}+f\frac{d^3y}{dx^3}=0 \longrightarrow (4)$$

From (3) we express

$$f=\frac{-1}{\frac{d^2y}{dx^2}}\left[1+\left(\frac{dy}{dx}\right)^2+y\frac{d^2y}{dx^2}\right] \longrightarrow (5)$$

Substituting f in (4)

$$3\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right)+y\frac{d^3y}{dx^2}-\frac{\frac{d^3y}{dx^3}}{\frac{d^2y}{dx^2}}\left[1+\left(\frac{dy}{dx}\right)^2+y\frac{d^2y}{dx^2}\right]=0$$

$$\Downarrow$$

$$\left[3\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right)+\cancel{y\frac{d^3y}{dx^3}}\right]\frac{d^2y}{dx^2}-\left[1+\left(\frac{dy}{dx}\right)^2+\cancel{y\frac{d^2y}{dx^2}}\right]\frac{d^3y}{dx^3}=0$$

$$\Downarrow$$

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]\frac{d^3y}{dx^3}-3\frac{dy}{dx}\left(\frac{d^2y}{dx^2}\right)^2=0$$

3rd order equation

The given expression is the general solution of this equation

Differential Equation (DE)

Linear ODE

The dependent variable (y) and its derivatives $(y', y'' \ldots \ldots)$ should appear linearly (no powers/no multiplication in dependent variables)

Example 1 :

$$x\frac{\mathrm{d}y}{\mathrm{d}x} + x^3 y = 0$$

Independent variable (Power 1)

Dependent variable (Power 1)

Power of derivative 1

Example 2 :

$$(2x^3 - 3)\frac{\mathrm{d}^3 y}{\mathrm{d}x^3} - (6x^2 - 7)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 4x\frac{\mathrm{d}y}{\mathrm{d}x} - 4y = 8$$

Only independent variable

Product term (independent variable with derivative)

derivative term

Differential Equation (DE)

Linear ODE

The dependent variable (y) and its derivatives $(y', y'' \dots\dots)$ appear linearly

Nonlinear ODE

The dependent variable (y) and its derivatives $(y', y'' \dots\dots)$ appear in powers or in products

Example 1 :

$$2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + x\frac{\mathrm{d}y}{\mathrm{d}x} - y = 0$$

derivative has powers

Example 2 :

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + y^3 = 0$$

dependent variable has powers

Example 3 :

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + y\frac{\mathrm{d}y}{\mathrm{d}x} + y = 0$$

Dependent variable and its derivative appears in multiplication

Identify Linear and non-linear ODE among the following:

1. $$\frac{d^2\theta}{dt^2} + \sin\theta = 0$$

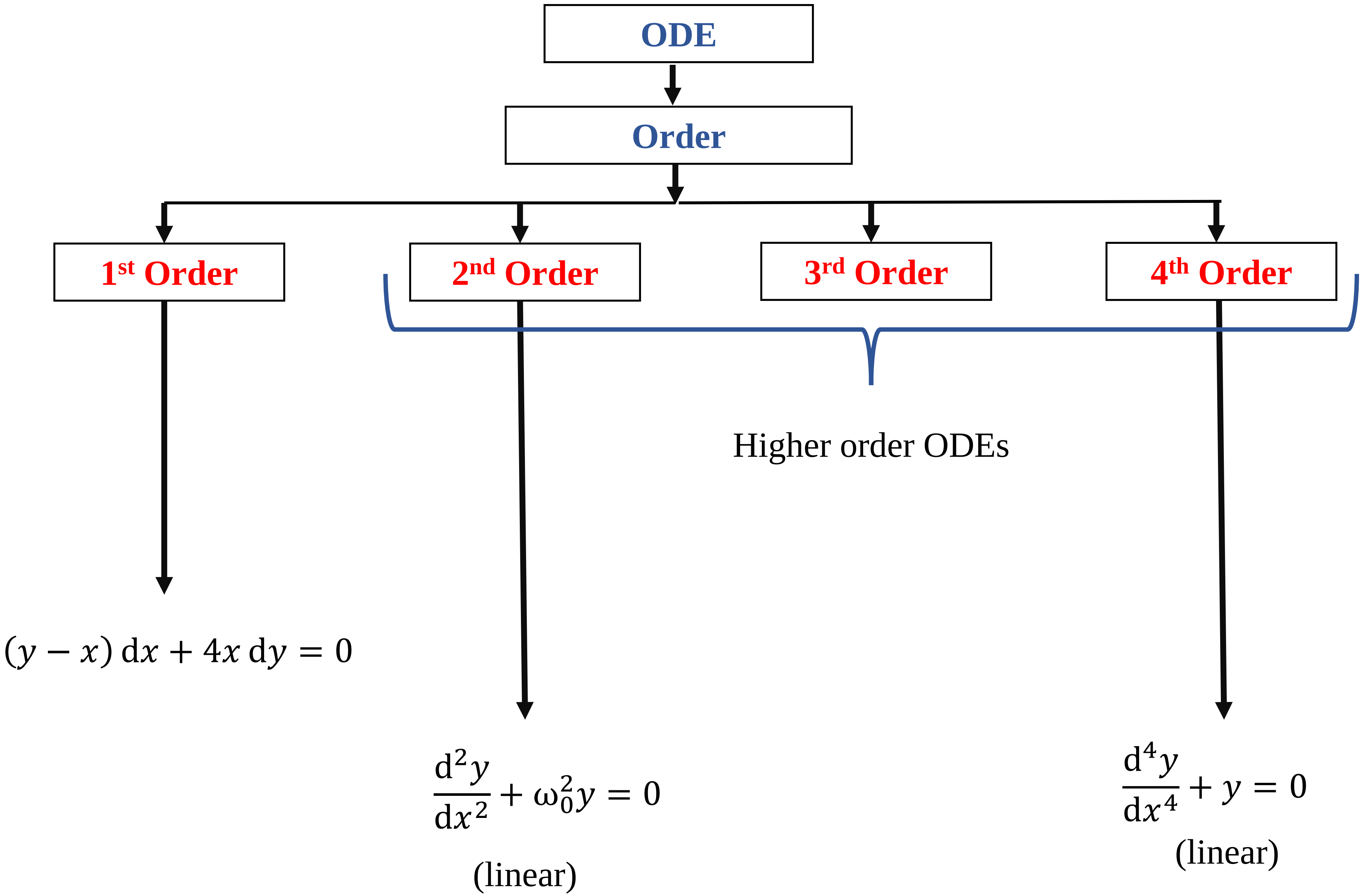
2. $$\frac{d^2x}{dt^2} + \alpha\frac{dx}{dt} + \beta t - x^3 = 0$$

3. $$\frac{d^2x}{dt^2} + \frac{x}{\sqrt{x^2+y^2}} = 0\ , \qquad \frac{d^2y}{dt^2} + \frac{y}{\sqrt{x^2+y^2}} = 0$$

4. $$\frac{d^2x}{dt^2} + x\frac{dx}{dt} + x^3 = 0$$

5. $$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \sin t = 0$$

6. $$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + t^3 + \mathrm{e}^t = 0$$



How to solve these equations and obtain their solutions?

GENERAL SOLUTION OF A DIFFERENTIAL EQUATION

A solution of a DE is called as general solution if it contains as many arbitrary constants as the order of the DE.

Example 1:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2$$

1st Order ODE

Solution:

$$y = 2x + c$$

One arbitrary constant (GS)

Example 2:

$$\frac{d^3y}{dx^3} = 0$$

3rd Order ODE

Solution:

Three arbitrary constants (GS)

$$y = c_1 x^2 + c_2 x + c_3$$

PARTICULAR SOLUTION OF A DIFFERENTIAL EQUATION

A particular solution of a DE can be obtained by giving particular values to the arbitrary constants in the general solution of the DE.

Example 1:

$$\frac{dy}{dx} = 2$$

GS $y = 2x + c$

PS $y = 2x$ (c=0)

(Check the answer)

Example 2:

$$\frac{d^3y}{dx^3} = 0$$

GS $y = c_1x^2 + c_2x + c_3$

PS

i. $y = x^2$ $(c_1 = 1, c_2 = c_3 = 0)$

ii. $y = x$ $(c_l = 0, c_2 = 1, c_3 = 0)$

iii. $y = c_3$ $(c_1 = c_2 = 0)$

iv. $y = x^2 + x$ $(c_1 = c_2 = 1, c_3 = 0)$

v. $x^2 + a$ $(c_1 = 1, c_2 = 0, c_3 = 0)$

All five solutions are Particular Solutions

Note:

General solution : **Unique**

Particular solution : **Many**

Note : Sometimes you may not get the particular solution from the general solution by changing whatever the value of the constants. Such particular solution is called **singular solution.**

SINGULAR SOLUTIONS

Example 1: $\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = 4y$

$y = (x+c)^2$ General solution

$y = 0$ $\boxed{\text{Particular solution}}$

Singular Solution → Cannot be deduced from GS

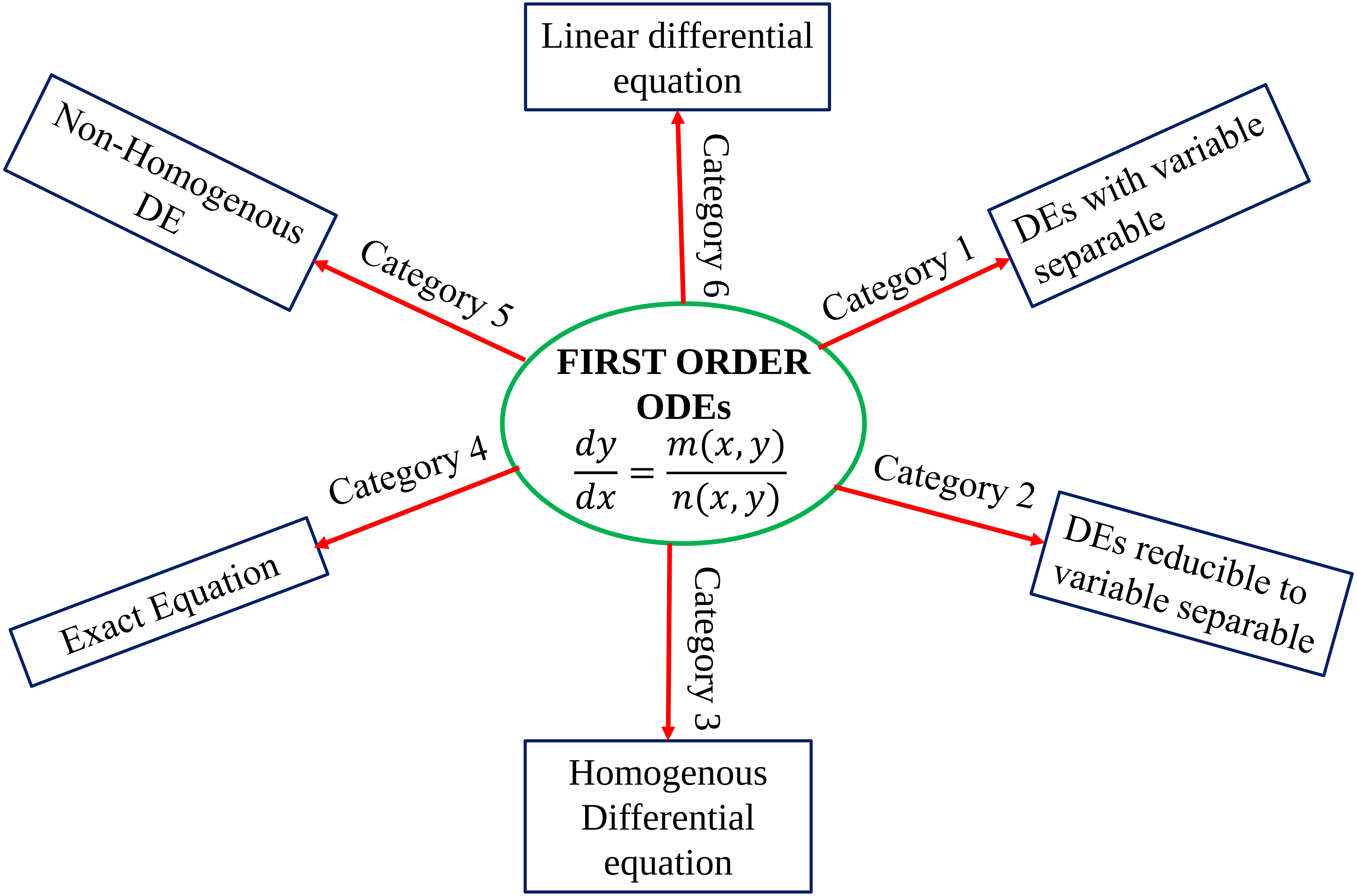
Example 2: $y\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 - 2x\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) + y = 0$ (Order 1)

$y^2 = 2cx - x^2$ General solution

$y = +x$ or $-x$ $\boxed{\text{Particular solution}}$

Singular Solution → Cannot be deduced from GS

SOLUTION OF FIRST ORDER AND FIRST DEGREE ODEs



Category 1: Differential equations with variable separable

Form 1: $f_1(x)\,\mathrm{d}x + f_2(y)\,\mathrm{d}y = 0$

Some function which involves only x

Another function which involves only y

Form 2: $\dfrac{\mathrm{d}y}{\mathrm{d}x} = p(x)\,q(y)$

In product form

Only a function of y

Only a function of x

- In the above two cases, we can take x terms on one side and y terms on other side.

Form 1: $$\int f_1(x)\ \mathrm{d}x = -\int f_2(y)\ \mathrm{d}y$$

Form 2: $$\int \frac{\mathrm{d}y}{q(y)} = \int p(x)\,dx$$

- Each term can be integrated separately.

Example 1: Solve the given DE by separation of variables

$$x\frac{dy}{dx} = 4y$$

Answer :

$$x\frac{dy}{dx} = 4y \Rightarrow \frac{dy}{dx} = \frac{4y}{x} \Rightarrow \frac{dy}{y} = \frac{4}{x}dx$$

Function of x only

Function of y only

Integrating on both sides

$$\int \frac{dy}{y} = 4\int \frac{dx}{x} + \log c$$

$$\log y = 4\log x + \log c$$

$$e^{\log y} = e^{4\log x + \log c} = e^{\log x^4 + \log c} = e^{\log x^4} e^{\log c}$$

$$y = x^4 \times c$$

$$y = cx^4$$

Cross-check:

$$\frac{dy}{dx} = c(4x^3) \quad \text{We know} \quad c = \frac{y}{x^4}$$

$$\frac{dy}{dx} = \frac{y}{x^4} \times 4x^3 \Rightarrow \frac{dy}{dx} = \frac{4y}{x} \Rightarrow \boxed{x\frac{dy}{dx} = 4y}$$ Given equation

<u>Example 2:</u> Solve the given DE $x^2 y' = y(1-x)$ $x^2 \frac{\mathrm{d}y}{\mathrm{d}x} = y(1-x)$

Multiply by dx $x^2 \frac{\mathrm{d}y}{\cancel{\mathrm{d}x}} \cancel{\mathrm{d}x} = y(1-x)dx$ $x^2\,\mathrm{d}y = y(1-x)\,\mathrm{d}x$

Function of x only

$$\frac{\mathrm{d}y}{y} = \frac{1-x}{x^2}\mathrm{d}x \Rightarrow \frac{\mathrm{d}y}{y} = \left(\frac{1}{x^2} - \frac{1}{x}\right)\mathrm{d}x$$

Function of y only

Integrating on both sides

$$\int \frac{\mathrm{d}y}{y} = \int \frac{\mathrm{d}x}{x^2} - \int \frac{\mathrm{d}x}{x} \Rightarrow \log y = -\frac{1}{x} - \log x + \log c \Rightarrow \log y + \log x - \log c = -\frac{1}{x}$$

$$\mathrm{e}^{(\log y + \log x - \log c)} = \mathrm{e}^{-1/x} \Rightarrow \mathrm{e}^{\log y}\mathrm{e}^{\log x}\mathrm{e}^{-\log c} = \mathrm{e}^{-1/x} \Rightarrow y\,x\frac{1}{c} = \mathrm{e}^{-1/x}$$

$$y = \frac{c}{x}\mathrm{e}^{-1/x}$$

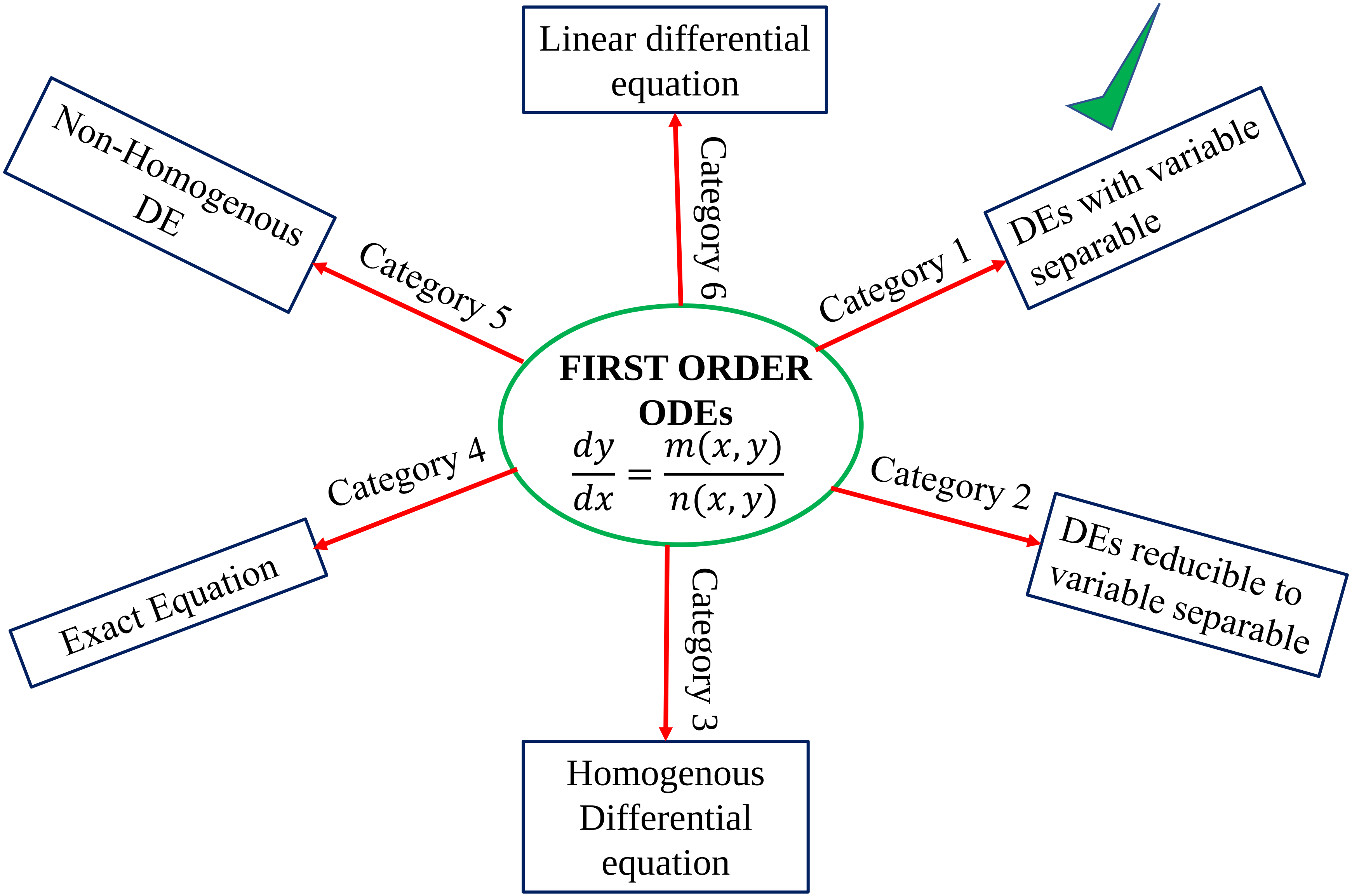
<u>Cross-check:</u> $y = \frac{c}{x}\mathrm{e}^{-1/x}$

$$\frac{dy}{dx} = \frac{-c}{x^2}e^{(-1/x)} + \frac{c}{x}e^{(-1/x)}\left(\frac{1}{x^2}\right) \Rightarrow \frac{dy}{dx} = \frac{-c}{x^2}e^{(-1/x)} + \frac{c}{x^3}e^{(-1/x)}$$

We know $ce^{(-1/x)} = xy$ $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-xy}{x^2} + \frac{xy}{x^3} \Rightarrow \frac{-y}{x} + \frac{y}{x^2} = \frac{y}{x^2}(1-x)$

$$\boxed{x^2\frac{\mathrm{d}y}{\mathrm{d}x} = y(1-x)}$$

SOLUTION OF FIRST ORDER AND FIRST DEGREE ODEs



Category 2: Differential equations reducible to variables separable

Suppose a given DE is given in the form

$$\boxed{\frac{dy}{dx} = f(ax+by+c)}$$ For $a \neq 0$ *and* $b \neq 0$

We can't separate x & y variables

Ex: $\frac{dy}{dx} = 1 + e^{x-y}$

$$\boxed{\frac{dy}{dx} = f(ax+by+c)}$$ For $a = 0$ *or* $b = 0$

We can separate the variables

Ex: $\frac{dy}{dx} = 1 + e^{-y}$

It can be reduced to variables separable form as follows

Let us define, $v = ax + by + c$

$$\frac{dv}{dx} = a + b\frac{dy}{dx} \Rightarrow b\frac{dy}{dx} = \frac{dv}{dx} - a \Rightarrow \frac{dy}{dx} = \frac{1}{b}\frac{dv}{dx} - \frac{a}{b}$$

$\therefore$ The given ODE becomes

$$\frac{1}{b}\frac{dv}{dx} - \frac{a}{b} = f(v) \Rightarrow \frac{1}{b}\frac{dv}{dx} = f(v) + \frac{a}{b} \Rightarrow \frac{1}{\cancel{b}}\frac{dv}{dx} = \frac{bf(v)+a}{\cancel{b}}$$

$$\boxed{\frac{dv}{a+bf(v)} = dx}$$ Separable form

Example 1: Solve the given DE $\frac{dy}{dx} - e^{x-y} = 1$

Answer :

$$\frac{dy}{dx} = 1 + e^{x-y}$$

Not in separable form

$$v = x - y \Rightarrow \frac{dv}{dx} = 1 - \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = 1 - \frac{dv}{dx}$$

$$\therefore \cancel{1} - \frac{dv}{dx} = \cancel{1} + e^{v} \Rightarrow \frac{dv}{dx} = -e^{v}$$

$$\frac{dv}{e^{v}} = -\,dx$$

Upon integrating,

$$\int e^{-v}\, dv = -\int dx \Rightarrow -e^{-v} = -x + c$$

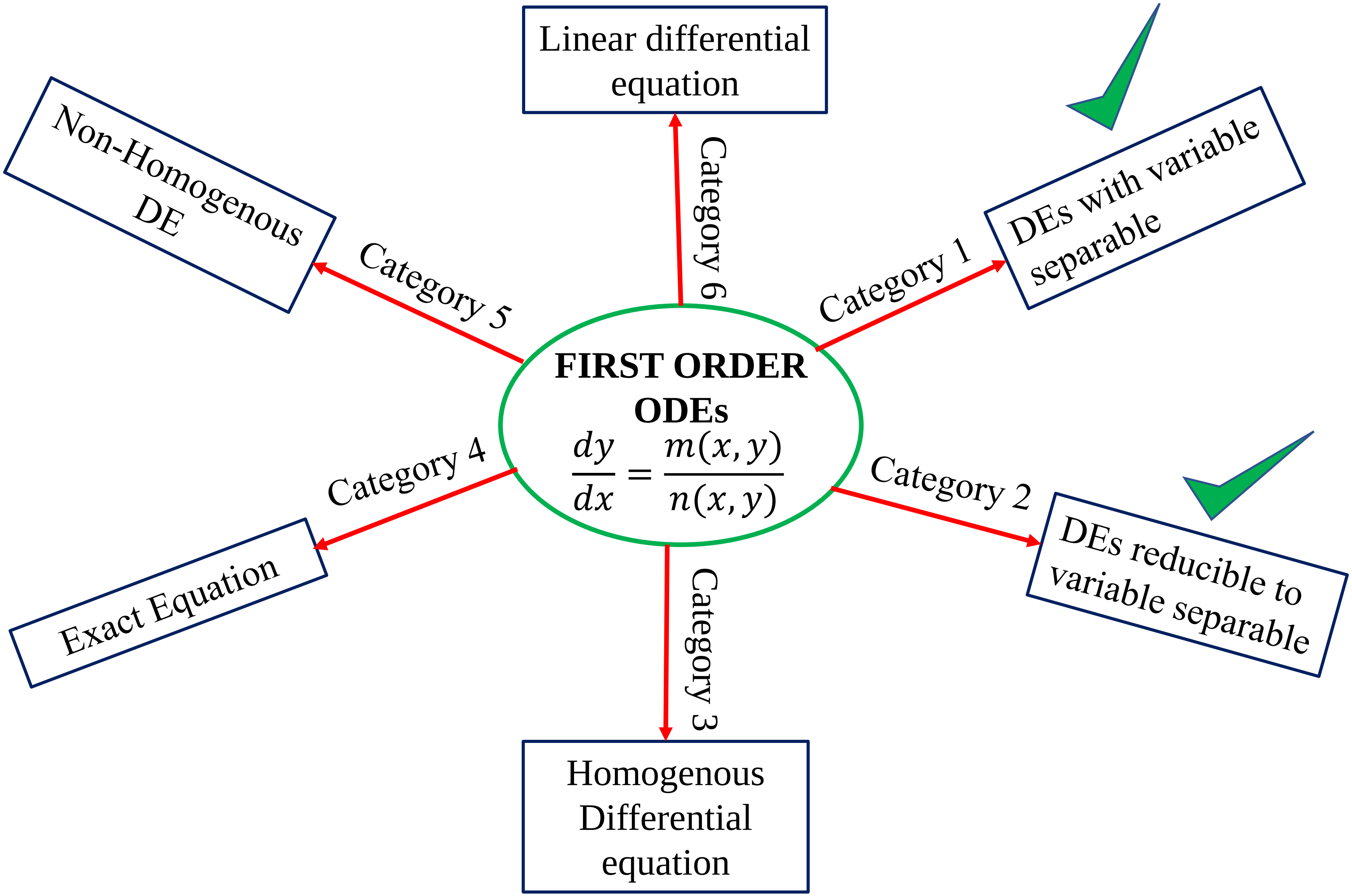
$$x - e^{-v} = c$$

$$\boxed{x - e^{y-x} = c}$$

Implicit Solution
(y can't be expressed in terms of x)

H.W : Cross-check the answer

SOLUTION OF FIRST ORDER AND FIRST DEGREE ODEs



Category 3: Homogenous Differential equations (HDE)

A Differential Equation $M(x,y)dx + N(x,y)dy = 0$ is called a homogenous DE if $M(x,y)$ and $N(x,y)$ are both homogenous functions of the same degree in *x* and *y*.

Example 1: $(x^2 - 2xy)\,\mathrm{dx} + (x^2 - 3xy + 2y^2)\,\mathrm{dy} = 0$

1. It is of the form $M(x,y)dx + N(x,y)dy = 0$ with $M = x^2 - 2xy$

$N = x^2 - 3xy + 2y^2$

2. $x^2 - 2xy$

Degree of *x* =1

Degree of *y* = 1

Degree 2   Total degree 1+1 = 2

3. $x^2 - 3xy + 2y^2$ Degree of each term 2

4. Given DE is Homogenous Differential equation of degree 2

Example 2: $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2 y}{x^3 + y^3}$ Degree of Nr. =3

Degree of Dr. =3

Given equation is HDE of Degree 3

Method of solving: By substituting *y=xv(x)* in the given equation we can be brought it to separable form.

Degree = 3

Example 1: $(x^2 - 2xy)\,\mathrm{d}y + (x^2 - 3xy + 2y^2)\,\mathrm{d}x = 0$

Answer: $(x^2 - 2xy)\dfrac{\mathrm{d}y}{\mathrm{d}x} + (x^2 - 3xy + 2y^2) = 0 \Rightarrow \dfrac{\mathrm{d}y}{\mathrm{d}x} = -\dfrac{(x^2 - 3xy + 2y^2)}{x^2 - 2xy}$ (1)

Substituting $y = xv$ & $\dfrac{\mathrm{d}y}{\mathrm{d}x} = v + x\dfrac{\mathrm{d}v}{\mathrm{d}x}$ in (1)

$$v + x\frac{\mathrm{d}v}{\mathrm{d}x} = -\frac{x^2 - 3x(xv) + 2(x^2v^2)}{x^2 - 2(vx)} \Rightarrow v + x\frac{\mathrm{d}v}{\mathrm{d}x} = -\frac{x^2 - 3x^2v + 2(x^2v^2)}{x^2 - 2(vx)}$$

$$v + x\frac{\mathrm{d}v}{\mathrm{d}x} = -\frac{\cancel{x}^2(1 - 3v + 2v^2)}{\cancel{x}^2(1 - 2v)} \quad = -\frac{(1 - 3v + 2v^2)}{(1 - 2v)}$$

$$x\frac{\mathrm{d}v}{\mathrm{d}x} = -\frac{(1 - 3v + 2v^2)}{(1 - 2v)} - v \quad = -\frac{(1 - \overset{2}{\cancel{3}}v + \cancel{2v^2} + \cancel{v} - \cancel{2v^2})}{1 - 2v}$$

$$x\frac{\mathrm{d}v}{\mathrm{d}x} = -\frac{(1-2v)}{(1-2v)} \Rightarrow x\frac{\mathrm{d}v}{\mathrm{d}x} = -1$$

$$x\frac{\mathrm{d}v}{\cancel{\mathrm{d}x}}\cancel{\mathrm{d}x} = -\,\mathrm{d}x \Rightarrow x\,\mathrm{d}v = -\,\mathrm{d}x \Rightarrow \int \mathrm{d}v = -\int \frac{\mathrm{d}x}{x}$$

$$v = -\log x + c \Rightarrow \frac{y}{x} = -\log x + c \Rightarrow \boxed{y = x(c - \log x)}$$

<u>**H.W :** Cross-check the answer</u>

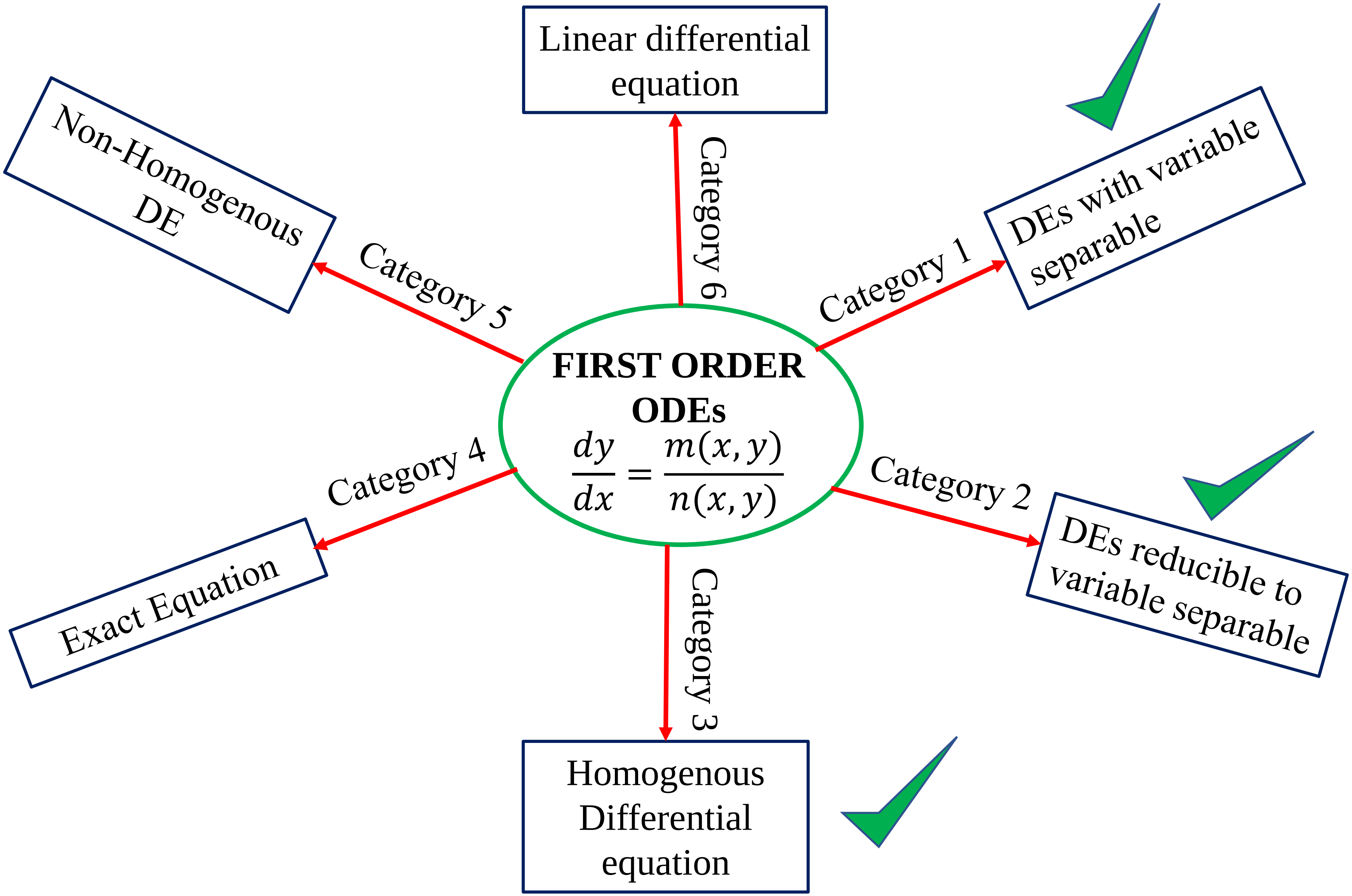
H.W : Solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = (2x + 4y + 1)^2 + \frac{1}{2}$$

Answer :

$$2x + 4y + 1 = \tan(4x + 1)$$

SOLUTION OF FIRST ORDER AND FIRST DEGREE ODEs



Category 4: Exact Equation

A first order differential equation of the form $M(x,y)dx + N(x,y)dy = 0$ is said to be exact form if it satisfies the condition $M_y = N_x$

If the expression $Mdx + Ndy = 0$ is exact, there exists some function f(x,y) such that,

(i) Condition for Exact Equation

Differentials cannot be zero

$$Mdx + Ndy = df(x,y) \Rightarrow Mdx + Ndy = \frac{\partial f}{\partial x}\mathrm{dx} + \frac{\partial f}{\partial y}\mathrm{dy} \Rightarrow \underbrace{\left(M - \frac{\partial f}{\partial x}\right)}_{=\mathbf{0}}\mathrm{dx} + \underbrace{\left(N - \frac{\partial f}{\partial y}\right)}_{=\mathbf{0}}\mathrm{dy} = 0$$

$$\frac{\partial f}{\partial x} = M(x,y); \quad \frac{\partial f}{\partial y} = N(x,y) \xrightarrow{\text{Cross differentiating}} \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial M}{\partial y} \quad ; \quad \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial N}{\partial x}$$

Since, $\dfrac{\partial^2 f}{\partial y \partial x} = \dfrac{\partial^2 f}{\partial x \partial y}$

$$\boxed{\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

Condition for Exact Equation

(ii) Method of Integration

$$\frac{\partial f}{\partial x} = M(x,y) \Rightarrow f = \int M(x,y)\,\mathrm{dx} + F(y)$$

Unknown function

Then, $\dfrac{\partial f}{\partial y} = \dfrac{\partial}{\partial y}\displaystyle\int M\,\mathrm{dx} + \dfrac{\partial F}{\partial y}$

Substituting into, $\frac{\partial f}{\partial y} = N(x,y)$, we find, $\dfrac{\partial}{\partial y}\displaystyle\int M\,\mathrm{dx} + \dfrac{\partial F}{\partial y} = N(x,y)$

Integration constant

$$\frac{\partial F}{\partial y} = N(x,y) - \frac{\partial}{\partial y}\int M\,\mathrm{dx} \Rightarrow F = \int\left[N(x,y) - \frac{\partial}{\partial y}\left(\int M\,\mathrm{dx}\right)\right]\mathrm{dy} + c$$

$$\boxed{f = \int M\,\mathrm{dx} + \int\left[N - \frac{\partial}{\partial y}\int M\,\mathrm{dx}\right]\mathrm{dy} + c}$$

This f is called as integral $\dfrac{df}{dx} = 0$

Example:

$$2xy\,dx + (x^2 - 1)\,dy = 0$$

Exact condition is satisfied

$$M = 2xy \qquad N = (x^2 - 1)$$

$$\frac{\partial M}{\partial y} = 2x \qquad \frac{\partial N}{\partial x} = 2x$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

The given equation is exact, $2xy\,dx + (x^2 - 1)\,dy = \mathrm{d}f$

$$\boxed{\mathrm{d}f = \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}dy}$$

Equating coefficient of the differential dx, $\frac{\partial f}{\partial x} = 2xy \longrightarrow$ (A)

Equating coefficient of the differential dy, $\frac{\partial f}{\partial y} = (x^2 - 1) \longrightarrow$ (B)

Integrating (A), $\frac{\partial f}{\partial x} = 2xy \xRightarrow{\text{Multiplying by } dx} \frac{\partial f}{\cancel{\partial x}}\cancel{dx} = 2xy\,dx$

$$\int \partial f = \int 2xy\,\mathrm{d}x \xRightarrow{\text{Integrating}} f = yx^2 + F(y) \longrightarrow \text{(C)}$$

Unknown function to be determined

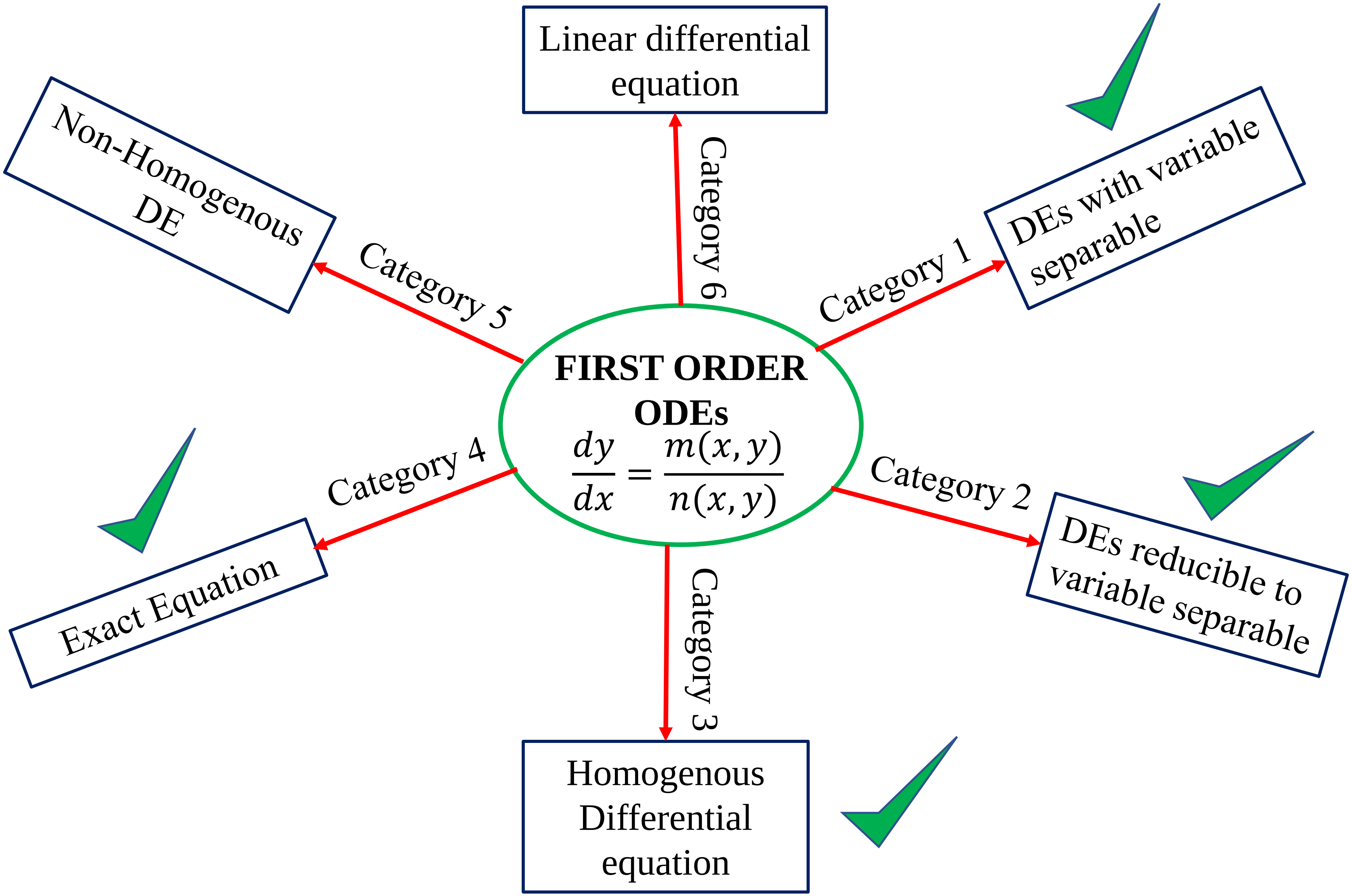
Substituting (C) in (B), $\cancel{x^2} + \frac{\partial F}{\partial y} = \cancel{x^2} - 1 \Rightarrow \frac{\partial F}{\partial y} = -1$

Multiplying by dy $\quad \frac{\partial F}{\cancel{\partial y}}\cancel{dy} = -dy \Rightarrow \int \partial F = -\int \mathrm{d}y \xRightarrow{\text{Integrating}} F = -y + c$

$$\boxed{\therefore f = yx^2 - y + c}$$

H. W: Check $\frac{\partial f}{\partial x}$ **=0**

SOLUTION OF FIRST ORDER AND FIRST DEGREE ODEs



Category 5: Non-Homogenous Differential Equations

Consider differential equations of the form,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$$

a_i, b_i, c_i = constants

If c_1 , c_2 = 0 We know how to solve.

If c_1 , c_2 $\neq 0$ How to solve?

Constants

Introducing the new variable u & v

$$x = u + h \qquad y = v + k$$

$$dx = du \qquad dy = dv$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}v}{\mathrm{d}u}$$

$$\frac{\mathrm{d}v}{\mathrm{d}u} = \frac{a_1(u+h) + b_1(v+k) + c_1}{a_2(u+h) + b_2(v+k) + c_2}$$

$$\frac{\mathrm{d}v}{\mathrm{d}u} = \frac{a_1 u + b_1 v + (a_1 h + b_1 k + c_1)}{a_2 u + b_2 v + (a_2 h + b_2 k + c_2)}$$

Constant.

We can choose h & k such that the rounded terms become zero

We can make these constants as zero.

Unknown constants

$$a_1 \boxed{h} + b_1 \boxed{k} + c_1 = 0 \longrightarrow (1) \qquad a_2h + b_2k + c_2 = 0 \longrightarrow (2)$$

Known constants

$$\boxed{\begin{aligned} a_1h + b_1k &= -c_1 \\ a_2h + b_2k &= -c_2 \end{aligned}}$$

Two equations

Two unknowns

Solving we get h and k

$$a_1 h + b_1 k + c_1 = 0 \longrightarrow (1) \qquad a_2 h + b_2 k + c_2 = 0 \longrightarrow (2)$$

Multiply (1) by b_2 $\qquad a_1 b_2 h + b_1 \cancel{b_2} k = -c_1 b_2$

Multiply (2) by b_1 $\qquad a_2 b_1 h + \cancel{b_1} b_2 k = -c_2 b_1$

((1) - (2)) $\qquad (a_1 b_2 h - a_2 b_1) h = c_2 b_1 - c_1 b_2$

$$\boxed{h = \frac{c_2 b_1 - c_1 b_2}{a_1 b_2 - a_2 b_1}} \longrightarrow (3)$$

Substituting in (1)

$$a_1 \times \frac{c_2 b_1 - c_1 b_2}{a_1 b_2 - a_2 b_1} + b_1 k = -c_1$$

$$b_1 k = -c_1 - a_1 \frac{(c_2 b_1 - c_1 b_2)}{a_1 b_2 - a_2 b_1} = \frac{-a_1 \cancel{c_1} b_2 + c_1 a_2 b_1 - a_1 c_2 b_1 + \cancel{a_1} c_1 b_2}{a_1 b_2 - a_2 b_1}$$

$$b_1 k = \frac{b_1 (c_1 a_2 - a_1 c_2)}{a_1 b_2 - a_2 b_1}$$

$$\boxed{k = \frac{c_1 a_2 - a_1 c_2}{a_1 b_2 - a_2 b_1}} \longrightarrow (4)$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2} \xrightarrow[x = u + h;\ y = v + k]{\text{Transformed to}} \frac{\mathrm{d}v}{\mathrm{d}u} = \frac{a_1u + b_1v}{a_2u + b_2v}$$ (Homogenous equation)

Substituting $v(u) = t\,u$; $\frac{\mathrm{d}v}{\mathrm{d}u} = t + u\frac{\mathrm{d}t}{\mathrm{d}u}$

$$t + u\frac{\mathrm{d}t}{\mathrm{d}u} = \frac{a_1u + b_1\,t\,u}{a_2u + b_2\,t\,u} \quad = \frac{a_1 + b_1t}{a_2 + b_2t}$$

$$u\frac{\mathrm{d}t}{\mathrm{d}u} = \frac{a_1 + b_1t}{a_2 + b_2t} - t \quad = \frac{a_1 + b_1t - a_2t - b_2t^2}{a_2 + b_2t} \boxed{= \frac{a_1 + (b_1 - a_2)t - b_2t^2}{a_2 + b_2t}}$$

Function of *t* alone

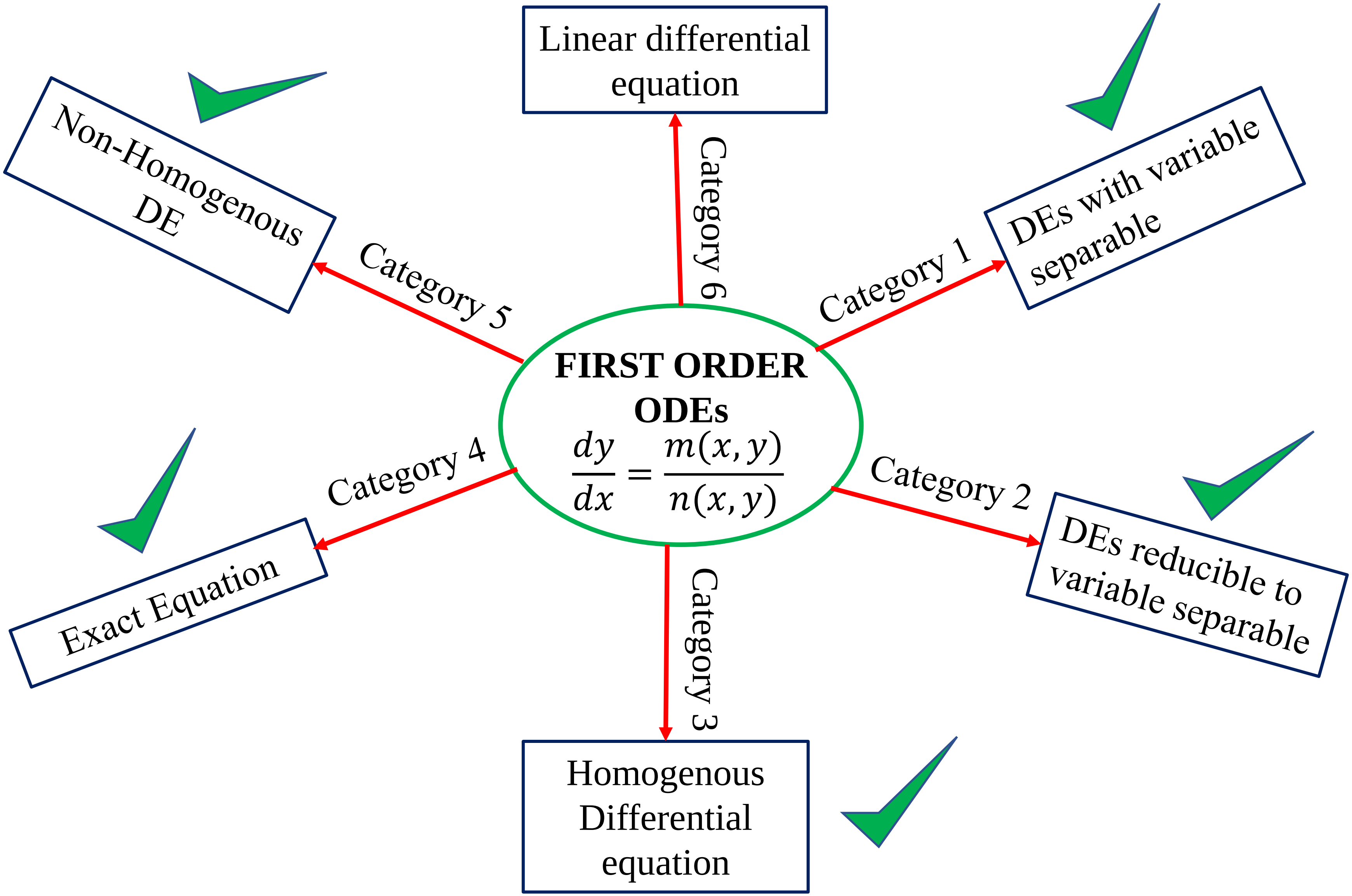
Separating the variables, $$\frac{(a_2 + b_2t)\,dt}{a_1 + (b_1 - a_2)t - b_2t^2} = \frac{du}{u}$$

Integrating, $$\int \frac{(a_2 + b_2t)\,dt}{a_1 + (b_1 - a_2)t - b_2t^2} = \int \frac{\mathrm{d}u}{u} + c$$ Integrating constant

$$-\frac{1}{2}\log[a_1 + (b_1 - a_2)t - b_2t^2] + \frac{a_2tb_1}{2}\int \frac{\mathrm{d}t}{a_1 + (b_1 - a_2)t + b_2t^2} \quad = \log u + c$$

Substituting $t = \frac{v}{u}$ and then *u*= *x*-*h*, *v*=*y*-*k* we will obtain the general solution

SOLUTION OF FIRST ORDER AND FIRST DEGREE ODEs



Category 6: First order Linear Differential Equation (Leibnitz's Linear DE)

A 1st order 1st degree DE has the form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

Independent variable only

Power of $\frac{dy}{dx} = 1$

Power of $y = 1$

Hence the given equation is a linear equation

Sub-case $\boxed{Q(x) = 0}$ $\boxed{\frac{dy}{dx} + P(x)y = 0} \longrightarrow$ Separable form

Proof:

$$\frac{dy}{y} = -P(x)dx \Rightarrow \int \frac{dy}{y} = -\int P(x)\,dx + \log c \Rightarrow \log y = -\int P(x)\,dx + \log c$$

$$\log y - \log c = -\int P(x)\,dx \Rightarrow \log\left(\frac{y}{c}\right) = -\int P(x)\,dx \Rightarrow e^{\log(y/c)} = e^{-\int P(x)\,dx}$$

$$\frac{y}{c} = e^{-\int P(x)\,dx} \Rightarrow y = c\,e^{-\int P(x)\,dx} \longrightarrow (1)$$

Cross-check:

Differentiating (1) with respect to x

$$\frac{dy}{dx} = ce^{-\int P(x)\,dx}(-P(x)) \Rightarrow \frac{dy}{dx} = -\boxed{ce^{-\int P(x)\,dx}}(P(x))$$

$$\frac{dy}{dx} = -y\,P(x) \Rightarrow \frac{dy}{dx} + P(x)y = 0 \quad \text{Given Equation}$$

Recall Given equation $\frac{dy}{dx} + P(x)y = 0$

Multiplying the given equation by $e^{\int P\,dx}$

$$e^{\int P\,dx}\left(\frac{dy}{dx} + P(x)y\right) = 0$$

$$\frac{dy}{dx}\,e^{\int P\,dx} + P(x)y\,e^{\int P\,dx} = 0$$

The above equation can be written as, $\frac{d}{dx}\left[ye^{\int P\,dx}\right] = 0$

Multiplying by dx $\cancel{dx}\,\frac{d}{\cancel{dx}}\left[ye^{\int P\,dx}\right] = 0\,dx \Rightarrow d\left[ye^{\int P\,dx}\right] = 0\,dx$

Upon integrating, $\cancel{\int}\cancel{d}\left(y\,e^{\int P\,dx}\right) = c$

$$\boxed{y\,e^{\int P\,dx} = c} \Rightarrow \text{General Solution}$$

Given equation $\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = 0$

Multiplying the given equation by $\mathrm{e}^{\int P \, \mathrm{d}x}$, we can rewrite

$$\underbrace{\frac{\mathrm{d}}{\mathrm{d}x}\left[y\mathrm{e}^{\int P \, \mathrm{d}x}\right] =}_{\text{Perfect differential}} \quad 0$$

So, integration becomes trivial

The function which we multiply to get a perfect differential is called an **Integrating factor**. In this case $\mathrm{e}^{\int P \, \mathrm{d}x}$ is the Integrating factor.

<u>Integrating Factor</u>

An expression $\mathrm{F}(x, y)$ of the variables x, y is called an Integrating factor of the differential equation $M(x,y)dx + N(x,y)dy = 0$ if $\mathrm{F}(x,y)[\,M(x,y)dx + N(x,y)dy] = d\big(u(x,y)\big)$, where $u(x,y)$ is some expression of x, y.

Category 6: $$\frac{dy}{dx} + P(x)y = Q(x)$$

Multiplying both sides by the integrating factor $e^{\int P\,dx}$, we can rewrite

$$\underbrace{e^{\int P\,dx}\frac{dy}{dx} + P(x)ye^{\int P\,dx}} = \underbrace{Q(x)e^{\int P\,dx}}_{\text{Function of } x \text{ only}}$$

$$\frac{d}{dx}\left(ye^{\int P\,dx}\right) = Q(x)e^{\int P\,dx}$$

Multiplying by dx on both sides

$$\cancel{dx}\frac{d}{\cancel{dx}}\left(ye^{\int P\,dx}\right) = Q(x)e^{\int P\,dx}\,dx$$

$$d\left(ye^{\int P\,dx}\right) = Q(x)e^{\int P\,dx}\,dx$$

Integrating

$$\int d\left(ye^{\int P\,dx}\right) = \int Q(x)e^{\int P\,dx}\,dx + c$$

Integration constant

$$ye^{\int P\,dx} = \int Q(x)e^{\int P\,dx}\,dx + c$$

$$\boxed{y = e^{-\int P\,dx}\left[\int Q(x)e^{\int P\,dx}\,dx + c\right]}$$ General Solution

Example 1: Solve the differential equation $\dfrac{dy}{dx} + \dfrac{y}{x} = x^2$

Comparing with $\dfrac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = \mathrm{Q}(x)$

We find $P(x) = \dfrac{1}{x}$ $\quad Q(x) = x^2$

Then the integrating factor is $e^{\int P(x)\,dx} = e^{\int (1/x)\,dx} = e^{\log x} = x$

Let us multiply the given equation by integration factor x,

$$x\frac{\mathrm{d}y}{\mathrm{d}x} + y = x^3 \Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}[xy] = x^3$$

Multiplying by dx

$$\cancel{\mathrm{d}x}\,\frac{\mathrm{d}}{\cancel{\mathrm{d}x}}[xy] = x^3\,\mathrm{d}x \Rightarrow \int \mathrm{d}(xy) = \int x^3\,\mathrm{d}x + c \xrightarrow{\text{Integrating}} xy = \frac{x^4}{4} + c$$

Integration constant

$$\boxed{y = \frac{x^3}{4} + \frac{c}{x}}$$

Cross-check:

Differentiating ,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3x^2}{4} - \frac{c}{x^2} \quad = \frac{3x^2}{4} - \frac{1}{x^2}\left[xy - \frac{x^4}{4}\right] = \frac{3x^2}{4} - \frac{y}{x} + \frac{x^2}{4}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x^2 - \frac{y}{x} \qquad \frac{dy}{dx} + \frac{y}{x} = x^2$$

Verified

Initial value problem

To determine the solution of a differential equation subject to some initial conditions is known as initial value problem.

The conditions that are prescribed along with the differential equation are referred to as the Initial Conditions.

For a differential equation of 1st order and 1st degree only one initial condition is required since the general solution contains only one arbitrary constant.

Example :

Solve $\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{y}{x} = x^2$ given that $\boxed{y = \frac{5}{4} \text{ when } x = 1}$ → Initial condition

Solution : $y = \frac{x^3}{4} + \frac{c}{x}$

Substituting the Initial Condition, $y = \frac{5}{4}$ at $x = 1$

$$\frac{5}{4} = \frac{1}{4} + c \quad \Rightarrow \quad c = 1$$

The required particular solution is,

$$y = \frac{x^3}{4} + \frac{1}{x}$$

Arbitrary constant is fixed through initial condition

Example 2:

Solve the differential equation

$(x^2 - y^2)\,\mathrm{d}x + 2xy\,\mathrm{d}y = 0$ given that $\boxed{y = 1 \text{ when } x = 1}$ → Initial condition

General Solution $x^2 + y^2 = cx$ **(HW)**

Substituting tinitial conditions, $1 + 1 = c$ ⇨ $c = 2$

Particular Solution $x^2 + y^2 = 2x$

More on Integrating factors

Consider a differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{x}$. Find out the integrating factors.

Note: The equation is of separable type. We can derive the solution easily.

$$\frac{\mathrm{d}y}{y} = \frac{dx}{x}$$

$$\log y = \log x + \log c$$

$$y = c\, x$$

Integrating factors

The aim is to find a function which upon multiplication on the given equation we should able to write the given equation as a perfect differential.

Given equation $$\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{y}{x} = 0 \longrightarrow (1)$$

Multiplying (1) by $\frac{1}{x}$ $$\frac{1}{x}y' - \frac{y}{x^2} = 0$$

Integrating factor

$$\frac{xy' - y}{x^2} = 0$$

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{x}\right) = 0} \longrightarrow \text{Perfect derivative}$$

Given equation $$\frac{dy}{dx} - \frac{y}{x} = 0 \longrightarrow (1)$$

Multiplying (1) by $\frac{1}{x^2+y^2}$ $$\frac{y'}{x^2+y^2} - \frac{y}{x(x^2+y^2)} = 0$$

Second
Integrating factor

$$\frac{xy'-y}{x(x^2+y^2)} = 0$$

$$\underbrace{\frac{d}{dx}\left[\tan^{-1}\left(\frac{y}{x}\right)\right]}_{\text{Perfect derivative}} = 0 \Rightarrow \frac{1}{\left(1+\left(\frac{y}{x}\right)^2\right)} \times \left(\frac{-y}{x^2} + \frac{y'}{x}\right) = 0 \Rightarrow \frac{xy'-y}{\left(1+\left(\frac{y}{x}\right)^2\right)x^2} = 0$$

$$\Rightarrow \frac{\frac{xy'-y}{(x^2+y^2)\cancel{x^2}}}{\cancel{x^2}} = 0 \Rightarrow \frac{xy'-y}{(x^2+y^2)} = 0$$

Multiplying (1) by $\frac{1}{y}$ $$\frac{1}{y}y' - \frac{1}{x} = 0$$

Third
Integrating factor

$$\frac{d}{dx}(\log y) - \frac{d}{dx}(\log x) = 0 \Rightarrow \frac{d}{dx}(\log y - \log x) = 0$$

$$\frac{d}{dx}\left(\log\frac{y}{x}\right) = 0 \Rightarrow \log\frac{y}{x} = c \Rightarrow \boxed{\frac{y}{x} = c}$$

Lesson:

1. Integrating factors (I.F.) are many for the given equation.
2. Finding integrating factors is as difficult as solving the given differential equation.

Method of finding Integrating Factors for first order ODEs

Given equation

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

CASE 1

$$xM(x,y) + yN(x,y) \neq 0$$

$$\text{I.F} = \frac{1}{xM(x,y)+yN(x,y)}$$

CASE 2

$$xM(x,y) - yN(x,y) \neq 0$$

$$\text{I.F} = \frac{1}{xM(x,y)-yN(x,y)}$$

CASE 3

$$\frac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N} = f(x)$$

$$\text{IF} = e^{\int f(x)\,dx}$$

CASE 4

$$\frac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}{N} = f(y)$$

$$\text{IF} = e^{\int f(y)\,dy}$$

CASE 1: $\dfrac{dy}{dx} = \dfrac{y^2}{x(y-x)} \Longleftrightarrow y^2\,\mathrm{d}x + x(x-y)\,\mathrm{d}y = 0 \longrightarrow (1)$

$$M(x,y) = y^2 \quad ; \quad N(x,y) = x(x-y)$$

$$xM(x,y) + yN(x,y) = xy^2 + xy(x-y) = x^2y \neq 0$$

$$\text{I.F} = \frac{1}{xM+yN} = \frac{1}{x^2y}$$

Multiplying (1) by $\frac{1}{x^2y}$

$$\frac{1}{x^2y}\left[y^2\,\mathrm{d}x + x(x-y)\,\mathrm{d}y\right] = 0 \Longrightarrow \frac{y\,\mathrm{d}x - x\,\mathrm{d}y}{x^2} + \frac{\mathrm{d}y}{y} = 0$$

$$-\,\mathrm{d}\left(\frac{y}{x}\right) + \mathrm{d}(\log y) = 0$$

On integrating

$$-\frac{y}{x} + log\ y = \mathrm{c} \Rightarrow \mathrm{e}^{-y/x}.\mathrm{e}^{\log y} = c'$$

$$\mathrm{e}^{-y/x}.\mathrm{e}^{\log y} = c'$$

$$y.e^{-y/x} = c'$$

$$\boxed{y = c'\mathrm{e}^{y/x}}$$

Implicit solution

Method of finding Integrating Factors for first order ODEs

Given equation

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

CASE 1

$$xM(x,y) + yN(x,y) \neq 0$$

$$\text{I.F} = \frac{1}{xM(x,y)+yN(x,y)}$$

CASE 2

$$xM(x,y) - yN(x,y) \neq 0$$

$$\text{I.F} = \frac{1}{xM(x,y)-yN(x,y)}$$

CASE 3

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)$$

$$\text{IF} = \mathrm{e}^{\int f(x)\,\mathrm{d}x}$$

CASE 4

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{N} = f(y)$$

$$\text{IF} = \mathrm{e}^{\int f(y)\,\mathrm{d}y}$$

CASE 2: Solve : $(x^3y^2 + x)\,\mathrm{d}y + (x^2y^3 - y)\,\mathrm{d}x = 0$

$$M(x,y) = x^3y^2 - y; \qquad N(x,y) = x^2y^3 + x$$

$$xM - yN = -2xy \neq 0$$

$$\text{I.F} = \frac{1}{xM(x,y) - yN(x,y)} = \frac{1}{2xy}$$

Multiplying (1) by - $\frac{1}{2xy}$,

$$-\frac{1}{2xy}\left[y(x^2y^2 - 1)\,\mathrm{d}x + x(x^2y^2 + 1)\,\mathrm{d}y\right] = 0$$

$$-\frac{1}{2}\left(\overset{\boxed{1}}{x^2y} - \overset{\boxed{2}}{\frac{1}{x}}\right)\mathrm{d}x - \frac{1}{2}\left(\overset{\boxed{1}}{x^2y} + \overset{\boxed{2}}{\frac{1}{y}}\right)\mathrm{d}y = 0$$

$$\underbrace{-\frac{1}{2}xy(ydx + x\,dy)}_{\boxed{1}} + \underbrace{\frac{1}{2}\left(\frac{\mathrm{d}x}{x} - \frac{dy}{y}\right)}_{\boxed{2}} = 0 \Rightarrow -\frac{1}{4}\frac{d}{dx}(xy)^2 + \frac{1}{2}\frac{d}{dx}(\log x - \log y) = 0$$

On integrating,

$$-\frac{1}{4}(xy)^2 + \frac{1}{2}(\log x - \log y) = c$$

$$\boxed{-\frac{1}{4}(xy)^2 + \frac{1}{2}\log\frac{x}{y} = c}$$

Method of finding Integrating Factors for first order ODEs

Given equation
$$M(x,y)\,dx + N(x,y)\,dy = 0$$

CASE 1

$$xM(x,y) + yN(x,y) \neq 0$$

$$\text{I.F} = \frac{1}{xM(x,y)+yN(x,y)}$$

CASE 2

$$xM(x,y) - yN(x,y) \neq 0$$

$$\text{I.F} = \frac{1}{xM(x,y)-yN(x,y)}$$

CASE 3

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)$$

$$\text{IF} = e^{\int f(x)\,dx}$$

CASE 4

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{N} = f(y)$$

$$\text{IF} = e^{\int f(y)\,dy}$$

CASE 3: Solve : $(x^2 + y^2 + x)\,\mathrm{d}x + xy\,\mathrm{d}y = 0$

H.W

Find out the IF and the solution.

CASE 4: Solve : $y\,\mathrm{d}x + (y^2 - x)\,\mathrm{d}y = 0$

H.W

Find out the IF and the solution.

Method of finding Integrating Factors for first order ODEs

CASE 1

$$xM(x,y) + yN(x,y) \neq 0$$

$$\text{I.F} = \frac{1}{xM(x,y)+yN(x,y)}$$

CASE 3

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)$$

$$\text{IF} = e^{\int f(x)\,dx}$$

Given equation

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

CASE 2

$$xM(x,y) - yN(x,y) \neq 0$$

$$\text{I.F} = \frac{1}{xM(x,y)-yN(x,y)}$$

CASE 4

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{N} = f(y)$$

$$\text{IF} = e^{\int f(y)\,dy}$$

APPLICATIONS

Application 1:

Bacteria Culture

In a laboratory, it is observed that the rate of increase of a bacteria in a certain culture is proportional to the number of bacteria present. If the number doubles in t_d hours, how many bacterial be expected at the end of nt_d hours?

Answer: Let x = number of bacterial at any time.

Rate of increase of bacteria w.r.t. 't'

According to the experimental result $\frac{dx}{dt}$

Information given in the 1st line in the problem

$$\boxed{\frac{dx}{dt} = kx}$$

1st order ODE

Proportionality constant

The other information given in the second line

$x \rightarrow 2x$ in t_d

Unknown variable

So we need to find x in terms of t

1st order ODE

$$\boxed{\frac{dx}{dt} = kx}$$

(separable type)

System parameters we know this value

$$\boxed{\frac{dx}{dt} = k\,x}$$

$$\frac{dx}{x} = k\,dt \Rightarrow \log x = kt + \log c \Rightarrow \log\frac{x}{c} = kt$$

$$\Rightarrow e^{\log(x/c)} = e^{kt}$$

$$\frac{x}{c} = e^{kt}$$

There are two arbitrary constants present in the solution. We have to fix them from the given information

$$\boxed{x(t) = ce^{kt}}$$

General solution with one arbitrary constant

Step 1: Fixing the constant *c*

In the second information x has been written in terms of $t_{d.}$

Let us express x in terms of $t_{d.}$

Let N_0 be the number of bacteria originally present in the culture.

At $t = 0$ $\quad x = N_0$ $\qquad$ In that case $N_0 = ce^{k(0)}$ $\qquad$ $N_0 = c$

$$\therefore\ x = N_0 e^{kt}$$

Step 2: Fixing the constant k

As per the second information $x = 2N_0$ at $t = t_d$

$$\therefore \; 2N_0 = N_0 e^{k t_d} \qquad 2 = e^{k t_d}$$

$$\log 2 = \log . e^{k t_d} \qquad \log 2 = k t_d$$

$$k = \frac{1}{t_d} \log 2$$

$$\boxed{x(t) = N_0 \, e^{\frac{t}{t_d}(\log 2)}}$$

(we have fixed these parameters)

Value of c fixed from initial value

Proportionality constant

Now let us answer the question

How many bacteria be expected at the end of $n\, t_d$ hours?

Answer: We have to find x at $t = n\, t_d$ $\quad x(t) = N_0 \, e^{\, n\, t_d (\log 2)/t_d}$

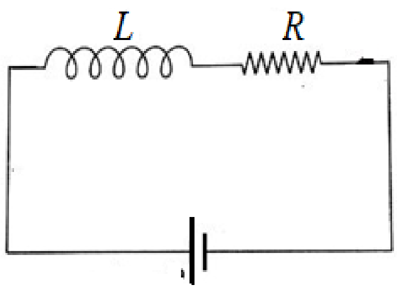
$$\boxed{x(t) = N_0 \, e^{n \log 2}}$$

Known

Known

Application 2:

Electronic circuit



$E = E_0(t)$

A resistance R and an inductance L are connected in series with a voltage supply $E(t)$. Find current in the circuit when $E = E_0 \sin \omega t$ is a E_0 is a constant.

Step 1: Set up the DE with i as an unknown variable.

Step 2: Integrate and find the expression for i

Step 1: The desired equation can be obtained by applying Kirchoff's voltage law to the circuit

Voltage drop by the inductor + Voltage drop by the resistance

$$L\frac{di}{dt} + Ri = E(t)$$

Dependent variable

$$\frac{di}{dt} + \frac{R}{L}i = \frac{1}{L}E\ (t)$$

Quantity which involves only independent variables

Independent variable

$$\frac{di}{dt} + \frac{R}{L} i = \frac{1}{L} E\ (t)$$

It is of the form $\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = \mathrm{Q}(x)$ $\quad P(x) = \frac{R}{L}$ = constant

$Q(x) = \frac{1}{L} E\ (t)$

Answer :

$$y = \mathrm{e}^{-\int P(x)\,\mathrm{d}x}\left[\int Q(x)\mathrm{e}^{\int P(x)}\,\mathrm{d}x + c\right]$$

$$= \mathrm{e}^{-(R/L)x}\left[\frac{1}{L}\int E(x)\mathrm{e}^{(R/L)x}\,\mathrm{d}x + c\right]$$

Current in the circuit $\boxed{i = \mathrm{e}^{-(R/L)x}\left[\frac{1}{L}\int E(t)\mathrm{e}^{(R/L)x}\,\mathrm{d}x + c\right]}$ In terms of original variable

Substitute $E = E_0 \sin \omega t$

$$i = \mathrm{e}^{-(R/L)x}\left[\frac{1}{L}\int E_0 \sin \omega t\ \mathrm{e}^{(R/L)t}\,\mathrm{d}x + c\right]$$

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