22PH102 CLASSICAL MECHANICS

I Semester

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Contents

2. Lagrangian Formalism

2.1 Introduction

Classical mechanics remains a cornerstone in solving many practical problems. Modern theories may not be necessary when describing simple phenomena, such as how a ball rolls down a ramp or how a rocket moves through space. While classical mechanics cannot explain the fundamental building blocks of nature, it is still the most effective theory for understanding macroscopic objects. This makes classical mechanics an invaluable tool in the physicist's toolbox. It also serves as an ideal platform for learning important concepts that are essential to modern physics. For example, Lagrangian mechanics is a powerful framework for exploring numerous physical models, offering intuitive insights into the mathematical tools that are indispensable in many areas of physics.

2.2 Newton's Laws

2.2.1 Newton's first law - The law of inertia

An object at rest remains at rest, and an object in motion continues to move at a constant velocity unless acted upon by an external force.

- The first law provides a fundamental criterion for identifying an inertial frame of reference.
- An inertial frame of reference is characterized as one in which equilibrium – whether in a state of rest or of constant momentum – is self-sustaining and determined entirely by the initial conditions.

A key implication of this law is the complete equivalence between the mechanical state of ***rest*** and that of ***constant momentum***.

2.2.2 Newton's second Law

An object or particle responds to an external force with a change in its momentum $\vec{p}$, such that the temporal rate of change of momentum, $\frac{d\vec{p}}{dt}$, is exactly equal to the external force $\vec{F}$.

The linear momentum $\vec{p}$ is defined as the product of an object's mass and its velocity:

$$\vec{p} = m\vec{v}. \tag{2.1}$$

The mechanics of the object is contained in this law. Specifically, it states that ***there exist frames of reference in which the motion of the object is described by the following differential equation***:

$$\vec{F} = \frac{d\vec{p}}{dt} \equiv \dot{\vec{p}}. \tag{2.2}$$

A frame of reference in which the above equation (2.2) is valid is called an ***inertial frame*** or a ***Galilean frame***.

2.2.3 Newton's third Law

The third law is commonly stated as *action and reaction are equal and opposite*. In its ***original form***, it states that the forces two objects (or particles) exert on each other are equal and opposite. This statement is sometimes referred to as the ***weak law of action and reaction***.

This law can be succinctly expressed as a mathematical equation:

$$\vec{F}_{ji} = -\vec{F}_{ij}, \tag{2.3}$$

where $\vec{F}_{ji}$ is the force exerted on particle i by particle j, and $\vec{F}_{ij}$ is the force exerted on particle j by particle i.

2.3 Mechanics of a particle

Let $\vec{r}$ denote the position vector of a particle relative to the origin O, and let $\vec{v}$ represent its velocity vector (see Figure 2.1). The linear momentum of a particle is defined as the product of its mass and

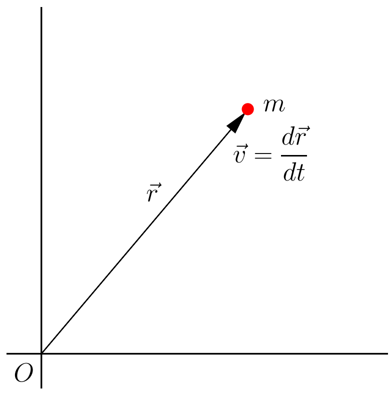

Figure 2.1: Mechanics of a single particle.

velocity:

$$\vec{p} = m\vec{v}. \tag{2.4}$$

As a result of interactions with external objects and fields, the particle may experience forces of various types. Let $\vec{F}$ be the sum of all forces (the net force) exerted on the particle. The mechanics of the particle is *contained* in Newton's second law of motion, which is given by the following differential equation:

$$\vec{F} = \frac{d\vec{p}}{dt} \equiv \dot{\vec{p}}. \tag{2.5}$$

or

$$\vec{F} = \frac{d}{dt}(m\vec{v}). \tag{2.6}$$

In most circumstances, the mass of the particle is constant and equation (2.6) reduces to

$$\vec{F} = m\frac{dv}{dt} \equiv m\vec{a}, \tag{2.7}$$

where $\vec{a}$ is the vector acceleration of the particle defined by

$$\vec{a} = \frac{d^2\vec{r}}{dt^2}. \tag{2.8}$$

The equation of motion, in general, is a second order ordinary differential equation, assuming $\vec{F}$ does not depend on higher derivatives.

2.3.1 Conservation Theorems

Many important conclusions in mechanics can be expressed in terms of conservation theorems, which indicate the conditions under which mechanical quantities remain constant over time. For instance, equation (2.5) directly provides the first conservation theorem.

Conservation Theorem for the Linear Momentum

If the total force $\vec{F}$ is zero, then the linear momentum $\vec{p}$ is conserved.

This means that when the net force exerted on the particle is zero, the linear momentum remains constant. A solution to the above problem can be directly obtained through integration.

Conservation Theorem for the Angular Momentum

The angular momentum of the particle about the point O, denoted by $\vec{L}$ is defined as

$$\vec{L} = \vec{r} \times \vec{p}. \tag{2.9}$$

Now define the *moment of force* or *torque* about the point O as

$$\vec{N} = \vec{r} \times \vec{F}. \tag{2.10}$$

An analogous equation of motion for the torque $\vec{N}$ is obtained by taking the cross product of $\vec{r}$ with equation (2.6). That is

$$\vec{r} \times \vec{F} = \vec{r} \times \frac{d}{dt}(m\vec{v}) \tag{2.11}$$

Consider the differentiation of the term $\vec{r} \times m\vec{v}$ with respect to t:

$$\begin{aligned} \frac{d}{dt}(\vec{r} \times m\vec{v}) &= \left(\frac{d\vec{r}}{dt} \times m\vec{v}\right) + \left[\vec{r} \times \frac{d}{dt}(m\vec{v})\right] \\ &= (\vec{v} \times m\vec{v}) + \left[\vec{r} \times \frac{d}{dt}(m\vec{v})\right], \end{aligned}$$

$$\Longrightarrow \vec{r} \times \frac{d}{dt}(m\vec{v}) = \frac{d}{dt}(\vec{r} \times m\vec{v}) \equiv \frac{d\vec{L}}{dt}, \quad [\because \ \vec{v} \times m\vec{v} = 0], \tag{2.12}$$

Thus

$$\vec{N} = \frac{d\vec{L}}{dt} \equiv \dot{\vec{L}}. \tag{2.13}$$

Note that both $\vec{N}$ and $\vec{L}$ depend on the point O about which the moments are taken.

The torque equation above (2.13) also leads to an immediate conservation theorem: ***If the total torque, $\vec{N}$, is zero, then $\dot{\vec{L}} = 0$, and angular momentum is conserved.***

Energy Conservation Theorem for a particle

Next, we consider the work done, W_{12}, by the external force $\vec{F}$ on the particle as it moves from point 1 to point 2 (see Figure 2.2). By definition, the work is

$$W_{12} = \int_1^2 \vec{F} \cdot d\vec{s}. \tag{2.14}$$

For constant mass, the integral in equation (2.14) reduces to

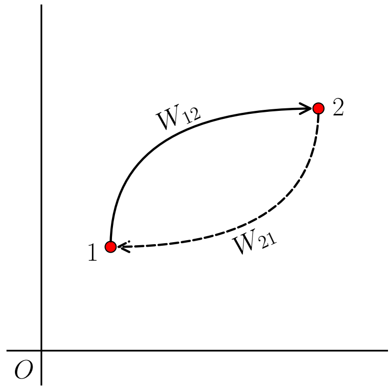

Figure 2.2: Work done by the external force upon the particle in going from point 1 to point 2.

$$\begin{aligned} \int \vec{F} \cdot d\vec{s} &= m \int \frac{d\vec{v}}{dt} \cdot \vec{v}\, dt \qquad \left[\because \ d\vec{s} = \frac{d\vec{s}}{dt} dt \equiv \vec{v}\, dt \right] \\ &= \frac{m}{2} \int \frac{d}{dt} \left(v^2\right) dt, \end{aligned}$$

and therefore

$$W_{12} = \frac{m}{2} v^2 \Big|_1^2 = \frac{m}{2} \left(v_2^2 - v_1^2\right). \tag{2.15}$$

The scalar quantity $\dfrac{mv^2}{2}$ is called the kinetic energy of the particle and is denoted by T, such that the work done is equal to the change in kinetic energy:

$$W_{12} = T_2 - T_1. \tag{2.16}$$

If the force field is such that the work W_{12} is the same for any physically possible path between points 1 and 2, then the force is said to be ***conservative***. The system is called a ***conservative system***.

Alternatively, let us suppose the particle is taken from point 1 to point 2 along one possible path (the solid line in Figure 2.2) and then returned to point 1 along another path (the dashed line in Figure 2.2). The independence of W_{12} from the particular path implies that $W_{12} = W_{21}$. This means that the work done around a closed loop is zero, that is,

$$\oint \vec{F} \cdot d\vec{s} = 0. \tag{2.17}$$

A necessary and sufficient condition for the work, W_{12}, to be independent of the physical path taken by the particle is that $\vec{F}$ is the gradient of some scalar function of position:

$$\vec{F} = -\vec{\nabla} V(\vec{r}), \tag{2.18}$$

where V is called the ***potential*** or ***potential energy***. This implies, assuming a Cartesian coordinate system,

$$\begin{aligned}\vec{F}\cdot d\vec{s} &= -\vec{\nabla}V\cdot d\vec{s},\\ &= -\left(\hat{i}\frac{\partial V}{\partial x}+\hat{j}\frac{\partial V}{\partial y}+\hat{k}\frac{\partial V}{\partial z}\right)\cdot\left(\hat{i}dx+\hat{j}dy+\hat{k}dz\right),\\ &= -\left(\frac{\partial V}{\partial x}dx+\frac{\partial V}{\partial y}dy+\frac{\partial V}{\partial z}dz\right),\end{aligned}$$

or

$$\vec{F}\cdot d\vec{s} = -dV. \tag{2.19}$$

Thus, if W_{12} is independent of the path of integration between the endpoints 1 and 2, it should be possible to express W_{12} as the change in a quantity that depends only on the positions of the endpoints. Substituting (2.19) in equation (2.14), we get

$$W_{12} = -\int_1^2 dV \equiv -V\Big|_1^2,$$

or

$$W_{12} = V_1 - V_2. \tag{2.20}$$

Equating W_{12} in equations (2.16) and (2.20), we get

$$T_2 - T_1 = V_1 - V_2 \quad\Longrightarrow\quad T_1 + V_1 = T_2 + V_2. \tag{2.21}$$

Thus, the energy conservation theorem for a particle reads as ***if the forces acting on a particle are conservative, then the total energy of the particle, $T+V$, is conserved***.

2.4 Mechanics of a system of particles

Next, we are generalizing the ideas studied in the previous section to systems of many particles. However, first, we must distinguish between the ***external forces*** acting on the particles due to sources outside the system, and ***internal forces*** on, say, some particle i due to all other particles in the system. The equation of motion for the i-th particle is written as

$$\sum_j \vec{F}_{ji} + \vec{F}_i^{(e)} = \dot{\vec{p}}_i, \tag{2.22}$$

where $\vec{F}_i^{(e)}$ stands for an external force, and $\vec{F}_{ji}$ is the internal force on the i-th particle due to the j-th particle. We shall assume that the $\vec{F}_{ji}$ obey Newton's third law of motion in its original form: that ***the forces two particles exert on each other are equal and opposite***.

Summed over all the particles, equation (2.22) takes the form

$$\frac{d^2}{dt^2}\sum_i m_i\vec{r}_i = \sum_i \vec{F}_i^{(e)} + \sum_{\substack{i,j\\ i\neq j}} \vec{F}_{ji}. \tag{2.23}$$

The first sum on the R.H.S. is simply the total external force $\vec{F}^{(e)}$, while the second term vanishes, since the law of action and reaction states that each pair $\vec{F}_{ji}+\vec{F}_{ij}$ is zero $\left(\because \vec{F}_{ji} = -\vec{F}_{ij}\right)$. To further

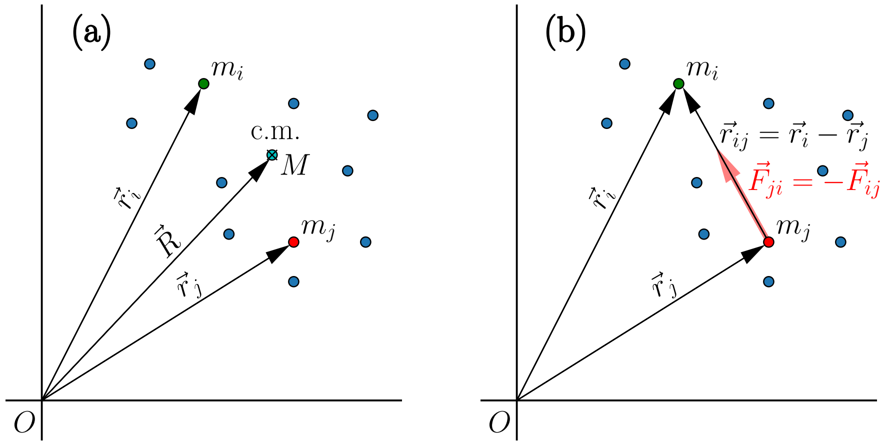

Figure 2.3: Mechanics of a system of particles: (a) The center of mass of a system of particles, and (b) an illustration of internal forces obeying the strong law of action and reaction.

simplify the left-hand side (L.H.S.), we define a vector $\vec{R}$ as the mass-weighted average of the radii vectors of the particles as [see Figure 2.3(a)]:

$$\vec{R} = \frac{\sum m_i \vec{r}_i}{\sum m_i} = \frac{\sum m_i \vec{r}_i}{M}. \tag{2.24}$$

The vector $\vec{R}$ defines the ***center of mass*** of the system. With this definition, Equation (2.23) reduces to

$$M\frac{d^2\vec{R}}{dt^2} = \sum_i \vec{F}_i^{(e)} \equiv \vec{F}^{(e)}, \tag{2.25}$$

which states that the center of mass moves as if the total external force acts on the entire mass of the system concentrated at the center of mass. The internal forces, if they obey Newton's third law, have no effect on the motion of the center of mass. %item An example is the motion of an exploding shell–the center of mass of the fragments travels as if the shell were still a single piece (neglecting air resistance).

2.4.1 Conservation Theorem for the Total Linear Momentum

The total linear momentum of the system,

$$\vec{P} = \sum_i m_i \frac{d\vec{r}_i}{dt} \equiv M\frac{d\vec{R}}{dt}, \tag{2.26}$$

is the total mass of the system times the velocity of the center of mass.

The conservation theorem of total linear momentum for a system of particles states: ***If the total external force acting on the system is zero, then the total linear momentum of the system remains conserved.***

2.4.2 Conservation Theorem for the Total Angular Momentum

Next, we examine the angular momentum of a system of particles. It is obtained by forming the cross product $\vec{r}_i \times \vec{p}_i$ and summing over all particles i. Using the identity provided in Equation

(2.12), we write:

$$\sum_i \left(\vec{r}_i \times \dot{\vec{p}}_i\right) = \sum_i \frac{d}{dt}\left(\vec{r}_i \times \vec{p}_i\right) \equiv \dot{\vec{L}},$$

where $\vec{L}$ represents the total angular momentum. Thus

$$\frac{d\vec{L}}{dt} = \sum_i \left(\vec{r}_i \times \vec{F}_i^{(e)}\right) + \sum_{\substack{i,j\\ i\neq j}} \left(\vec{r}_i \times \vec{F}_{ji}\right). \tag{2.27}$$

The last term on the R.H.S. of equation (2.27) can be considered as a sum of pairs of the form

$$\begin{aligned}\left(\vec{r}_i \times \vec{F}_{ji}\right) + \left(\vec{r}_j \times \vec{F}_{ij}\right) &= \left(\vec{r}_i \times \vec{F}_{ji}\right) - \left(\vec{r}_j \times \vec{F}_{ji}\right),\\ &= \left(\vec{r}_i - \vec{r}_j\right) \times \vec{F}_{ji} \equiv \vec{r}_{ij} \times \vec{F}_{ji}.\end{aligned} \tag{2.28}$$

Then equation (2.27) can be rewritten as

$$\frac{d\vec{L}}{dt} = \sum_i \left(\vec{r}_i \times \vec{F}_i^{(e)}\right) + \frac{1}{2}\sum_{\substack{i,j\\ i\neq j}} \left(\vec{r}_{ij} \times \vec{F}_{ji}\right). \tag{2.29}$$

To establish the equation of motion for the total torque, it is evident that the term $\vec{r}_{ij} \times \vec{F}_{ji}$ must vanish. This implies that the internal forces between two particles, in addition to being equal and opposite, also ***lie along the line joining the particles***. Such a condition is referred to as the ***strong law of action and reaction***. Consequently, all the cross products $\vec{r}_{ij} \times \vec{F}_{ji}$ vanish. As illustrated in Figure 2.3(b), the force $\vec{F}_{ji}$ must lie along the vector $\vec{r}_{ij}$. The equation of motion is therefore given by

$$\frac{d\vec{L}}{dt} = \sum_i \vec{N}_i^{(e)} \equiv \vec{N}^{(e)}, \tag{2.30}$$

where $\vec{N}^{(e)}$ represents the total external torque.

The conservation theorem states: ***The total angular momentum, $\vec{L}$, is a constant of motion if the total external torque is zero, i.e.,*** $N^{(e)} = 0$.

Note that the conservation of total linear momentum for a system of particles, in the absence of applied forces, assumes the validity of the weak law of action and reaction for the internal forces. In contrast, the conservation of total angular momentum for the system, in the absence of applied torques, requires the validity of the strong law of action and reaction, which further necessitates that the internal forces are ***central***.

An example is the force of gravity, which satisfies the strong form of the action-reaction law. We will discuss central force problems in more detail later.

- However, it is possible to find forces for which action and reaction are equal, even though the forces are not central.
- For instance, in a system involving moving charges, the forces between the charges, as predicted by the Biot-Savart law, violate both forms of the action-reaction law.
- The conservation theorems derived above are not applicable in such cases, at least in the form discussed here.
- In an isolated system of moving charges, the sum of the mechanical angular momentum and the electromagnetic angular momentum of the field is conserved.

Equation (2.26) states that the total linear momentum of the system is the same as if the entire mass were concentrated at the center of mass and moving with it. An analogous theorem for the angular momentum involving the center of mass can be derived as follows. Considering the origin O as the reference point, the total angular momentum of the system is given by

$$\vec{L} = \sum_i (\vec{r}_i \times \vec{p}_i) \tag{2.31}$$

Let $\vec{R}$ be the radius vector from O to the center of mass, and let $\vec{r}_i'$ be the radius vector from the center of mass to the i-th particle. We define

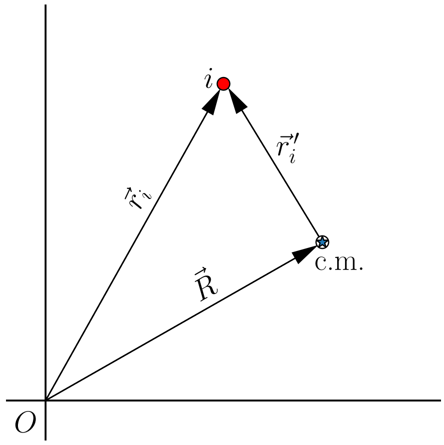

Figure 2.4: Illustration of vectors representing the shift in the reference point for angular momentum.

$$\vec{r}_i = \vec{r}_i' + \vec{R}, \quad \text{and} \quad \vec{v}_i = \vec{v}_i' + \vec{v}, \tag{2.32}$$

where $\vec{v} = \dfrac{d\vec{R}}{dt}$ is the velocity of the center of mass relative to O, and

$$\vec{v}_i' = \frac{d\vec{r}_i'}{dt}, \tag{2.33}$$

is the velocity of the i-th particle relative to the center of mass of the system. Using equation (2.31), we rewrite the total angular momentum as

$$\vec{L} = \sum_i \left(\vec{R} \times m_i \vec{v}\right) + \sum_i \left(\vec{r}_i' \times m_i \vec{v}_i'\right) + \left[\left(\frac{d}{dt}\sum_i m_i \vec{r}_i'\right) \times \vec{v}\right] + \left(\vec{R} \times \sum_i m_i \vec{r}_i'\right). \tag{2.34}$$

The last two terms in the above expression vanishes as both contain the factor $\sum_i m_i \vec{r}_i'$. Actually, $\sum_i m_i \vec{r}_i'$ defines the radius vector of the center of mass in the very coordinate system whose origin is the center of mass itself, which is a null vector. Then the total angular momentum about O is

$$\vec{L} = \left(\vec{R} \times M\vec{v}\right) + \sum_i \left(\vec{r}_i' \times \vec{p}_i'\right). \tag{2.35}$$

Thus, the total angular momentum about a point O is the angular momentum of the motion concentrated at the center of mass, plus the angular momentum of the motion ***about*** the center of mass. The form above emphasizes that, in general, $\vec{L}$ depends on the origin O, through the vector $\vec{R}$ only. If the center of mass is at rest with respect to O, the angular momentum is independent of the point of reference, and $\vec{L}$ reduces to the angular momentum about the center of mass.

2.4.3 Conservation of Total Energy

Finally, let us consider the energy equation. We calculate the work done by all forces in moving the system from the initial ***configuration*** 1 to the final ***configuration*** 2:

$$W_{12} = \sum_i \int_1^2 \vec{F}_i \cdot d\vec{s}_i = \sum_i \int_1^2 \vec{F}_i^{(e)} \cdot d\vec{s}_i + \sum_{\substack{i,j \\ i \neq j}} \int_1^2 \vec{F}_{ji} \cdot d\vec{s}_i. \tag{2.36}$$

The equation of motion can be used to reduce the integrals to

$$\begin{aligned} \sum_i \int_1^2 \vec{F}_i \cdot d\vec{s}_i &= \sum_i \int_1^2 m_i \frac{d\vec{v}_i}{dt} \cdot \vec{v}_i dt, \quad \left[\because \vec{F}_i = m_i \vec{a}_i \equiv m_i \frac{d\vec{v}_i}{dt}, \;\; d\vec{s}_i = \frac{d\vec{s}_i}{dt} dt \equiv \vec{v}_i dt\right] \\ &= \sum_i \int_1^2 m_i \, d\left(\frac{1}{2}\vec{v}_i \cdot \vec{v}_i\right) = \sum_i \int_1^2 d\left(\frac{1}{2} m_i v_i^2\right), \\ &= \sum_i \frac{1}{2} m_i v_i^2 \Big|_1^2 = \sum_i \left(\frac{1}{2} m_i v_{i2}^2 - \frac{1}{2} m_i v_{i1}^2\right). \end{aligned} \tag{2.37}$$

Or

$$W_{12} = T_2 - T_1, \tag{2.38}$$

where $T = \sum_i \frac{1}{2} m_i v_i^2$ is the total kinetic energy of the system.

Transformation to center of mass coordinates $\rightarrow T$ can also be written as

$$\begin{aligned} T &= \frac{1}{2} \sum_i m_i \left(\vec{v} + \vec{v}'\right) \cdot \left(\vec{v} + \vec{v}'\right), \\ &= \frac{1}{2} \sum_i m_i v^2 + \frac{1}{2} \sum_i m_i \vec{v}'^2 + \vec{v} \cdot \frac{d}{dt} \left(\sum_i m_i \vec{r}'\right). \end{aligned}$$

By virtue, the last term in the above equation is zero. Thus

$$T = \frac{1}{2} M v^2 + \frac{1}{2} \sum_i m_i \vec{v}'^2. \tag{2.39}$$

Hence, the kinetic energy, like angular momentum, consists of two parts: the kinetic energy as if all the mass were concentrated at the center of mass, plus the kinetic energy of motion about the center of mass.

In the special case where the external forces can be expressed as the gradient of a scalar potential, the first term in Eq. (2.36) can be written as

$$\sum_i \int_1^2 \vec{F}_i^{(e)} \cdot d\vec{s}_i = -\sum_i \int_1^2 \vec{\nabla}_i V_i \cdot d\vec{s}_i = -\sum_i V_i \Big|_1^2, \tag{2.40}$$

where the subscript i on the $\vec{\nabla}$ operator indicates that the derivatives are taken with respect to the components of $\vec{r}_i$.

If the internal forces are also conservative, then the mutual forces between the i-th and j-th particles, $\vec{F}_{ij}$ and $\vec{F}_{ji}$, can be derived from a potential function, say V_{ij}. To satisfy the strong law of action and reaction, V_{ij} must be a function of the distance between the particles only:

$$V_{ij} = V_{ij}\left(\left|\vec{r}_i - \vec{r}_j\right|\right). \tag{2.41}$$

The two forces are the automatically equal and opposite,

$$\vec{F}_{ji} = -\vec{\nabla}_i V_{ij} = +\vec{\nabla}_j V_{ji} = -\vec{F}_{ij}, \tag{2.42}$$

and lie along the line joining the two particles,

$$\vec{\nabla}_{ij} V_{ij} \left(\left|\vec{r}_i - \vec{r}_j\right|\right) = \left(\left|\vec{r}_i - \vec{r}_j\right|\right) f, \tag{2.43}$$

where f is some scalar function.

Note: If V_{ij} were also a function of the difference between some other pair of vectors associated with the particles, such as their velocities or their intrinsic "spin" angular momenta, then the forces would still be equal and opposite, but would not necessarily lie along the direction between the particles.

When the forces are all conservative, the second term in Eq. (2.36) can be rewritten as a sum over pairs of particles, with the terms for each pair taking the form

$$-\int_1^2 \left(\vec{\nabla}_i V_{ij} \cdot d\vec{s}_i + \vec{\nabla}_j V_{ij} \cdot d\vec{s}_j\right). \tag{2.44}$$

If we denote the difference vector $\vec{r}_i - \vec{r}_j$ by $\vec{r}_{ij}$ and if $\vec{\nabla}_{ij}$ stands for the gradient with respect to r_{ij}, then

$$\vec{\nabla}_i V_{ij} = \vec{\nabla}_{ij} V_{ij} = -\vec{\nabla}_j V_{ij},$$

and

$$d\vec{s}_i - d\vec{s}_j = d\vec{r}_i - d\vec{r}_j = d\vec{r}_{ij},$$

so that the term for the ij pair has the form

$$-\int \vec{\nabla}_{ij} V_{ij} \cdot d\vec{r}_{ij}.$$

The total work arising from internal forces then reduces to

$$-\frac{1}{2} \sum_{\substack{i,j \\ i \neq j}} \int_1^2 \vec{\nabla}_i V_{ij} \cdot d\vec{r}_{ij} = -\frac{1}{2} \sum_{\substack{i,j \\ i \neq j}} V_{ij} \Big|_1^2. \tag{2.45}$$

The factor $\frac{1}{2}$ appears in Eq. (2.45) because, in summing over both i and j, each member of a given pair is included twice.

Thus, if the external forces and internal forces are both derivable from potentials, it is possible to define a *total potential energy*, V, of the system,

$$V = \sum_i V_i + \frac{1}{2} \sum_{\substack{i,j \\ i \neq j}} V_{ij}. \tag{2.46}$$

so that the total energy $T + V$ is conserved.

- The second term in Eq. (2.46) is called the internal potential energy of the system.
- In general, the internal potential energy need not be zero, and it may vary as the system evolves with time.
- Only for the particular class of systems known as *rigid bodies* will the internal potential energy always be constant.
- A rigid body can be defined as a system of particles in which the distances $\vec{r}_{ij}$ are fixed and cannot vary with time. In such a case, the vectors $d\vec{r}_{ij}$ can only be perpendicular to the corresponding $\vec{r}_{ij}$, and therefore to the F_{ij}.
- Therefore, in a rigid body, the *internal forces do not do work*, and the internal potential energy must remain constant.

2.5 Constraints

We all obtain the impression that all problems in mechanics have been reduced to solving the following set of differential equations:

$$m_i\ddot{\vec{r}}_i = \vec{F}_i^{(e)} - \sum_i \vec{F}_{ji}. \tag{2.47}$$

Substituting the various forces acting upon the particles of the system, it becomes a mathematical problem of solving a set of ordinary differential equations. However, solving these equations is a real challenge, which we will not focus on for the time being.

In most cases, it becomes necessary to take into account the ***constraints*** that limit or restrict the motion of the system. For instance, in rigid bodies, the constraints on the motion of the particles keep the distances r_{ij} unchanged. Other examples are:

- The beads of an abacus are constrained to one-dimensional motion by the supporting wires.
- Gas molecules within a container are constrained by the walls of the vessel to move only inside the container.
- A particle placed on the surface of a solid sphere is subject to the constraint that it can move only on the surface or in the region exterior to the sphere.

2.5.1 Classification of constraints

Constraints may be classified in various ways. However, we restrict ourselves to the following classification.

If the conditions of constraints are expressed as *equations connecting the coordinates of the particles* of the form

$$f(\vec{r}_1, \vec{r}_2, \vec{r}_3, \ldots, t) = 0, \tag{2.48}$$

then the constraints are said to be ***holonomic***. The simplest example of holonomic constraints is the rigid body, where the constraints are expressed by equations of the form

$$(\vec{r}_i - \vec{r}_j)^2 - c_{ij}^2 = 0, \tag{2.49}$$

with c_{ij}'s constants.

A particle constrained to move along a curve or on a given surface is another example of a holonomic constraint. In this case, the equations defining the curve or surface are the equations of the constraint.

Constraints that cannot be expressed in the above manner are called ***nonholonomic***. The walls of a gas container constitute a nonholonomic constraint. The constraint involved in the example of a particle placed on the surface of a sphere is also nonholonomic. It can be expressed as an inequality of the form

$$r^2 - a^2 \geq 0, \tag{2.50}$$

where a is the radius of the sphere.

If the equations of constraint contain time as an explicit variable, then it is said to be ***rheonomous***. On the other hand, if time does not appear explicitly in the equations of constraint, then it is known as ***scleronomous***.

A bead sliding on a rigid curved wire fixed in space is an example of a scleronomous constraint. If the wire is moving in a prescribed manner, the constraint becomes rheonomous.

Constraints pose two types of difficulties in the solution of mechanical problems:

(1) The coordinates $\vec{r}_i$ are no longer all independent since they are connected by the equations of constraint. Hence, the equations of motion are not all independent.

(2) The force of constraint is not specified. For instance, the force that the wire exerts on the bead or the force that the wall exerts on the gas particle, as seen in the examples above, is not known.

Imposing constraints on the system is simply another method of stating that there are forces present in the problem that cannot be specified directly. However, these constraint forces are known in terms of their effects on the system.

Exercise 2.1 Classify the following systems according to the type of constraints:

(a) A sphere rolling downward without friction on a fixed sphere.
(b) A cylinder rolling down an inclined plane with inclination angle α.
(c) A particle gliding on the rough inner surface of a rotating paraboloid.
(d) A particle moving without friction along a very long bar. The bar rotates with angular velocity ω in the vertical plane about the horizontal axis.

2.5.2 Generelized coordinates

In the case of holonomic constraints, the first difficulty is solved by the introduction of ***generalized coordinates***. So far, we have been implicitly thinking in terms of Cartesian coordinates. A system of N particles, free from constraints, has $3N$ ***independent coordinates*** or ***degrees of freedom***.

If there are holonomic constraints, expressed in k equations of the form (2.48), these k equations can be used to eliminate k of the $3N$ coordinates. This leaves us with $3N-k$ independent coordinates, and the system is said to have $3N-k$ degrees of freedom. The elimination of the dependent coordinates can be achieved by introducing new, independent variables $q_1, q_2, q_3, \ldots, q_{3N-k}$, where the old coordinates $\vec{r}_1, \vec{r}_2, \vec{r}_3, \ldots, \vec{r}_N$ are expressed in terms of equations of the form

$$\vec{r}_1 = \vec{r}_1(q_1, q_2, q_3, \ldots, q_{3N-k}, t), \tag{2.51a}$$
$$\vec{r}_2 = \vec{r}_2(q_1, q_2, q_3, \ldots, q_{3N-k}, t), \tag{2.51b}$$
$$\vdots$$
$$\vec{r}_N = \vec{r}_N(q_1, q_2, q_3, \ldots, q_{3N-k}, t), \tag{2.51c}$$

which implicitly contain the constraints. These are ***transformation*** equations from $\{\vec{r}_i,\ i = 1, 2, \ldots, N\}$ to $\{q_j,\ \ j = 1, 2, \ldots, 3N-k\}$. The above equations can also be considered as a parametric representation of the variables $\vec{r}_i$. Furthermore, it is assumed that one can transform back from q_j to the $\vec{r}_i$, i.e., Eqs. (2.51), combined with the k constraint equations, can be inverted to obtain any q_j as a function of $\vec{r}_i$ and time, t. Or simply,

$$q_1 = q_1(\vec{r}_1, \vec{r}_2, \vec{r}_3, \ldots, \vec{r}_N, t), \tag{2.52a}$$
$$q_2 = q_2(\vec{r}_1, \vec{r}_2, \vec{r}_3, \ldots, \vec{r}_N, t), \tag{2.52b}$$
$$\vdots$$
$$q_N = q_{3N-k}(\vec{r}_1, \vec{r}_2, \vec{r}_3, \ldots, \vec{r}_N, t), \tag{2.52c}$$

In the case of a particle constrained to move on the surface of a sphere, the two angles expressing its position on the sphere (latitude and longitude) are the possible generalized coordinates. In the example of a double pendulum moving in a plane, the generalized coordinates are the two angles θ_1 and θ_2 (see Figure 2.5). In a double pendulum, two particles are connected by an inextensible, light rod and suspended by a similar rod fastened to one of the particles, as illustrated in Figure 2.5.

Exercise 2.2 A particle moving on the ellipse as shown in Fig. 2.6. Identify suitable generalized coordinate(s) and write down the transformation equations.

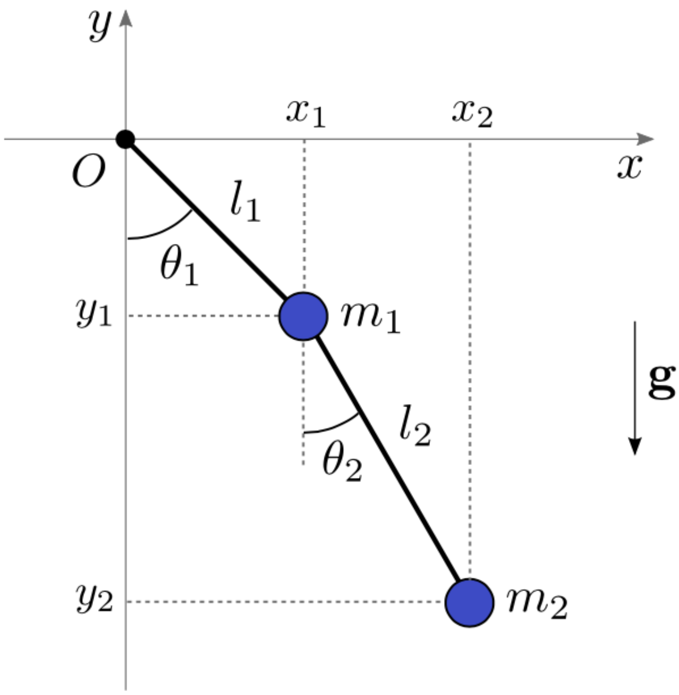

Figure 2.5: Illustration of a double pendulum.

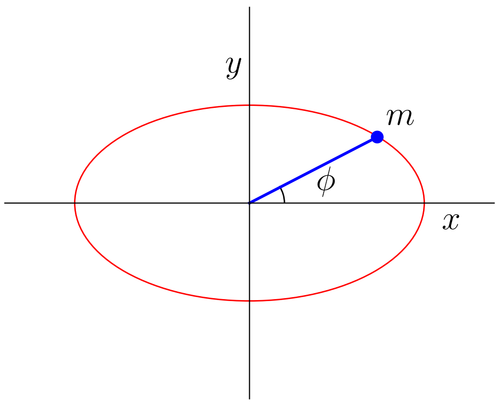

Figure 2.6: Illustration of a particle moving in an ellipse.

2.6 D'Alembert's principle and Lagrange equations

2.6.1 Principle of virtual work and D'Alembert's principle

Consider a change in the configuration of a system as a result of an arbitrary infinitesimal change in the coordinates, $\delta\vec{r}_i$, ***consistent with the forces and constraints imposed on the system at the given instant*** t. Such a change leads to a virtual displacement, as there is no passage of time. We refer to this infinitesimal displacement as "virtual" to distinguish it from an actual displacement of the system occurring over a time interval dt. The virtual displacement must be consistent with the forces and constraints imposed on the system, meaning it should not violate the laws of force or the constraints acting on the system.

Suppose the system is in equilibrium, meaning that the total force on each particle vanishes, i.e., $\vec{F}_i = 0$. The dot product $\vec{F}_i \cdot \delta\vec{r}_i$, which represents the ***virtual work*** of the force $\vec{F}_i$ during the displacement $\delta\vec{r}_i$, also vanishes. The sum of these vanishing products over all particles must therefore be zero:

$$\sum_i \vec{F}_i \cdot \delta\vec{r}_i = 0. \tag{2.53}$$

$\vec{F}_i$ may be decomposed into the applied force, $\vec{F}_i^{(a)}$, and the force of constraint, $\vec{f}_i$, that is,

$$\vec{F}_i = \vec{F}_i^{(a)} + \vec{f}_i. \tag{2.54}$$

Then Eq. (2.53) becomes

$$\sum_i \vec{F}_i^{(a)} \cdot \delta\vec{r}_i + \sum_i \vec{f}_i \cdot \delta\vec{r}_i = 0. \tag{2.55}$$

Now, we restrict ourselves to systems for which the *net virtual work of the constraint forces is zero*. This condition holds for rigid bodies and is also valid for a large class of other constraints.

For instance, if a particle is restricted to move on the surface of a table, the force of constraint is perpendicular to the surface, while the virtual displacement must be tangent to it, and hence the virtual work vanishes. However, this is no longer true if sliding friction forces are present, in which case we exclude such systems from consideration.

The condition for the equilibrium of a system is that the virtual work of the applied forces vanishes:

$$\sum_i \vec{F}_i^{(a)} \cdot \delta\vec{r}_i = 0. \tag{2.56}$$

Equation (2.56) is called the ***principle of virtual work***.

Note that the coefficients of $\delta\vec{r}_i$ cannot be set equal to zero. That is, in general, $\vec{F}_i^{(a)} \neq 0$, since the $\delta\vec{r}_i$ are not completely independent but are connected by the constraints. In order to equate the coefficients to zero, one must transform the principle into a form involving the virtual displacement of the generalized coordinates q_i, which are independent.

Let us write the equation of motion, $\vec{F}_i = \dot{\vec{p}}_i$, as

$$\vec{F}_i - \dot{\vec{p}}_i = 0. \tag{2.57}$$

Eq. (2.57) states that the particles in the system will be in equilibrium under a force equal to the actual force plus a ***reverse effective force*** $-\dot{\vec{p}}_i$. Then, we can rewrite Eq. (2.53) as

$$\sum_i \left(\vec{F}_i - \dot{\vec{p}}_i\right) \cdot \delta\vec{r}_i = 0, \tag{2.58}$$

and, making the same resolution into the applied forces and forces of constraint results

$$\sum_i \left(\vec{F}_i^{(a)} - \dot{\vec{p}}_i\right) \cdot \delta\vec{r}_i + \sum_i \vec{f}_i \cdot \delta\vec{r}_i = 0, \tag{2.59}$$

By restricting ourselves to systems for which the virtual work of the forces of constraint vanishes, we obtain

$$\sum_i \left(\vec{F}_i^{(a)} - \dot{\vec{p}}_i\right) \cdot \delta\vec{r}_i = 0. \tag{2.60}$$

The above equation (2.60) is often called ***D'Alembert's principle***. Note that the forces of constraint no longer appear in the above expression.

2.6.2 Derivation of Euler-Lagrange equations from D'Alembert's principle

Now, we need to transform the principle into an expression involving virtual displacements of the generalized coordinates, which are then independent of each other so that the coefficients of the δq_i can be set separately equal to zero. The translation $\vec{r}_i$ to q_j starts from the transformation equations (2.51),

$$\vec{r}_i = \vec{r}_i\left(q_1, q_2, q_3, \ldots, q_n, t\right), \tag{2.61}$$

assuming n independent coordinates (or n degrees of freedom). Then the velocity, $\vec{v}_i$ can be expressed as

$$\begin{aligned}\vec{v}_i = \frac{d\vec{r}_i}{dt} &= \frac{\partial \vec{r}_i}{\partial q_1}\frac{dq_1}{dt} + \frac{\partial \vec{r}_i}{\partial q_2}\frac{dq_2}{dt} + \ldots + \frac{\partial \vec{r}_i}{\partial q_n}\frac{dq_n}{dt} + \frac{\partial \vec{r}_i}{\partial t}, \\ &= \frac{\partial \vec{r}_i}{\partial q_1}\dot{q}_1 + \frac{\partial \vec{r}_i}{\partial q_2}\dot{q}_2 + \ldots + \frac{\partial \vec{r}_i}{\partial q_n}\dot{q}_n + \frac{\partial \vec{r}_i}{\partial t}.\end{aligned}$$

Or

$$\vec{v}_i = \sum_{k=1}^{n} \frac{\partial \vec{r}_i}{\partial q_k}\dot{q}_k + \frac{\partial \vec{r}_i}{\partial t}. \tag{2.62}$$

Similarly, the arbitrary virtual displacement $\delta \vec{r}_i$ can be connected with the virtual displacements δq_i by

$$\delta \vec{r}_i = \sum_j \frac{\partial \vec{r}_i}{\partial q_j}\delta q_j. \tag{2.63}$$

Note that no variation of time, that is $\delta t = 0$ (since virtual displacement by definition considers only displacements of the coordinates). Since we are dealing with the applied forces only, hereafter we shall drop the superscript $^{(a)}$ without ambiguity.

In terms of generalized coordinates, the virtual work of the force $\vec{F}_i$ becomes

$$\sum_i \vec{F}_i \cdot \delta \vec{r}_i = \sum_{i,j} \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}\delta q_j = \sum_j Q_j \delta q_j, \tag{2.64}$$

where Q_j are the components of the *generalized force*, defined as

$$Q_j = \sum_i \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}. \tag{2.65}$$

Note that q's need not have the dimension of length, so the Q's do not necessarily have the dimensions of force, but $Q_j \delta q_j$ must always have the dimensions of work. For instance, Q_j might be a torque N_J and dq_j a differential angle $d\theta_j$, which makes $N_j d\theta_j$ a differential of work. The other term in Eq. (2.6) becomes

$$\sum_i \dot{\vec{p}}_i \cdot \delta \vec{r}_i = \sum_i m_i \ddot{\vec{r}}_i \cdot \delta \vec{r}_i.$$

Expressing $\delta \vec{r}_i$ by (2.63), the above expression becomes

$$\sum_i \dot{\vec{p}}_i \cdot \delta \vec{r}_i = \sum_{i,j} m_i \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}\delta q_j. \tag{2.66}$$

Consider now the relation

$$\sum_i m_i \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} = \sum_i \left[\frac{d}{dt}\left(m_i \dot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} \right) - m_i \dot{\vec{r}}_i \cdot \frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_j} \right) \right]. \tag{2.67}$$

In the last term of the above expression, the differentiation with respect to t and q_j can be interchanged, that is,

$$\frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_j} \right) = \frac{\partial \dot{\vec{r}}_i}{\partial q_j} = \sum_k \frac{\partial^2 \vec{r}_i}{\partial q_j \partial q_k} \dot{q}_k + \frac{\partial^2 \vec{r}_i}{\partial q_j \partial t} \equiv \frac{\partial \vec{v}_i}{\partial q_j}, \tag{2.68}$$

by Eq. (2.62). We can also see from Eq. (2.62) that

$$\frac{\partial \vec{v}_i}{\partial \dot{q}_j} = \frac{\partial \vec{r}_i}{\partial q_j}. \tag{2.69}$$

Substituting these changes in Eq. (2.67)

$$\sum_i m_i \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} = \sum_i \left[\frac{d}{dt}\left(m_i \vec{v}_i \cdot \frac{\partial \vec{v}_i}{\partial \dot{q}_j} \right) - m_i \vec{v}_i \cdot \frac{\partial \vec{v}_i}{\partial q_j} \right],$$

Then putting all the expressions together in expression (2.60), we get

$$\sum_j \left\{ Q_j - \sum_i \left[\frac{d}{dt}\left(m_i \vec{v}_i \cdot \frac{\partial \vec{v}_i}{\partial \dot{q}_j} \right) - m_i \vec{v}_i \cdot \frac{\partial \vec{v}_i}{\partial q_j} \right] \right\} \delta q_j = 0,$$

or

$$\sum_j \left\{ \frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_j}\left(\sum_i \frac{1}{2} m_i v_i^2 \right) \right] - \frac{\partial}{\partial q_j}\left(\sum_i \frac{1}{2} m_i v_i^2 \right) - Q_j \right\} \delta q_j = 0. \quad \left[\because \; v_i^2 = \vec{v}_i \cdot \vec{v}_i\right] \tag{2.70}$$

With $T = \sum_i \frac{1}{2} m_i v_i^2$ being the system kinetic energy, the above equation can be written as

$$\sum_j \left\{ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} - Q_j \right\} \delta q_j = 0. \tag{2.71}$$

Since the generalized coordinates q_j's are all independent and so the virtual displacement δq_j, the above equation is valid only if each of the coefficients of δq_j identically vanishes. That is,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} - Q_j = 0, \tag{2.72}$$

where $j = 1, 2, \ldots, n$.

When the forces are derivable from a scalar potential function, V,

$$\vec{F}_i = -\vec{\nabla}_i V.$$

Then the generalized forces can be written as

$$Q_j = \sum_i \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} = -\sum_i \vec{\nabla}_i V \cdot \frac{\partial \vec{r}_i}{\partial q_j},$$

which is exactly the same expression for the partial derivative of a function $-V(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N, t)$ with respect to q_j:

$$Q_j = -\frac{\partial V}{\partial q_j}.$$

Equations (2.72) can then be rewritten as

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial (T-V)}{\partial q_j} = 0. \tag{2.73}$$

As the potential V is does not depend on the generalized velocities, we can include a term in V in the partial derivative with respect to $\dot{q}$:

$$\frac{d}{dt}\left(\frac{\partial (T-V)}{\partial \dot{q}_j}\right) - \frac{\partial (T-V)}{\partial q_j} = 0. \tag{2.74}$$

Defining a new function, the Lagrangian L, as

$$L = T - V, \tag{2.75}$$

the Eqs. (2.74) become

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0. \tag{2.76}$$

The above expressions referred to as ***Lagrange's equations*** or ***Euler-Lagrange's equations***.

Note that for a particular set of equations of motion there is no unique choice of Lagrangian such that Eqs. (2.76) lead to the equations of motion in the given generalized coordinates.

Exercise 2.3 If $L(\dot{q}, q, t)$ is a Lagrangian for a system satisfying Lagrange's equations, show by direct substitution that

$$L' = L(\dot{q}, q, t) + \frac{dF(q,t)}{dt},$$

also satisfies Lagrange's equations where F is arbitrary, but differentiable, function of q ant t.

Given

$$L' = L + \frac{dF}{dt} = L + \sum_k \frac{\partial F}{\partial q_k}\dot{q}_k + \frac{\partial F}{\partial t},$$

We need to prove

$$\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}_j}\right) - \frac{\partial L'}{\partial q_j} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0.$$

Calculate the required derivatives:

$$\frac{\partial L'}{\partial q_j} = \frac{\partial L}{\partial \dot{q}_j} + \sum_k \frac{\partial^2 F}{\partial q_j \partial q_k}\dot{q}_k + \frac{\partial^2 F}{\partial q_j \partial t},$$

$$\frac{\partial L'}{\partial \dot{q}_j} = \frac{\partial L}{\partial \dot{q}_j} + \frac{\partial F}{\partial q_j}; \quad \frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}_j}\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) + \sum_k \frac{\partial^2 F}{\partial q_k \partial q_j}\dot{q}_k + \frac{\partial^2 F}{\partial t \partial q_j}$$

Therefore

$$\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}_j}\right) - \frac{\partial L'}{\partial q_j} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) + \sum_k \frac{\partial^2 F}{\partial q_k \partial q_j}\dot{q}_k + \frac{\partial^2 F}{\partial t \partial q_j} - \left[\frac{\partial L}{\partial \dot{q}_j} + \sum_k \frac{\partial^2 F}{\partial q_j \partial q_k}\dot{q}_k + \frac{\partial^2 F}{\partial q_j \partial t}\right],$$

which implies

$$\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}_j}\right) - \frac{\partial L'}{\partial q_j} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial \dot{q}_j}.$$

Exercise 2.4 Construct the Lagrangian of a particle of m moving on a rotating parabolic surface and deduce the equations of motion.

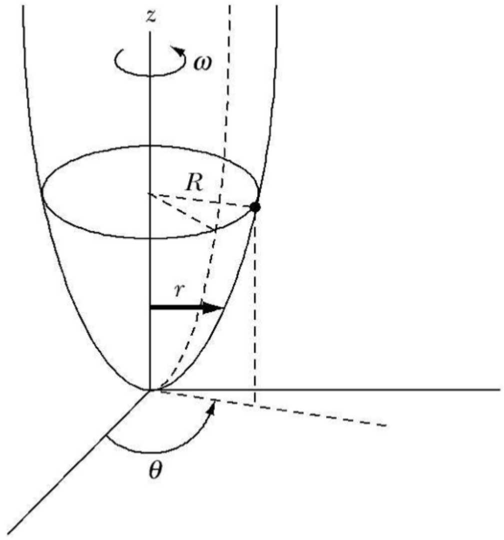

Figure 2.7: Illustration of a moving particle on the surface of a rotating parabola.

Exercise 2.5 A pendulum is attached to a massless rim of radius a and rotates at a constant velocity ω. The bob is of mass m and the length of the inextensible thread is l as illustrated in Figure 2.8. Obtain the Lagrangian and deduce the equations of motion.

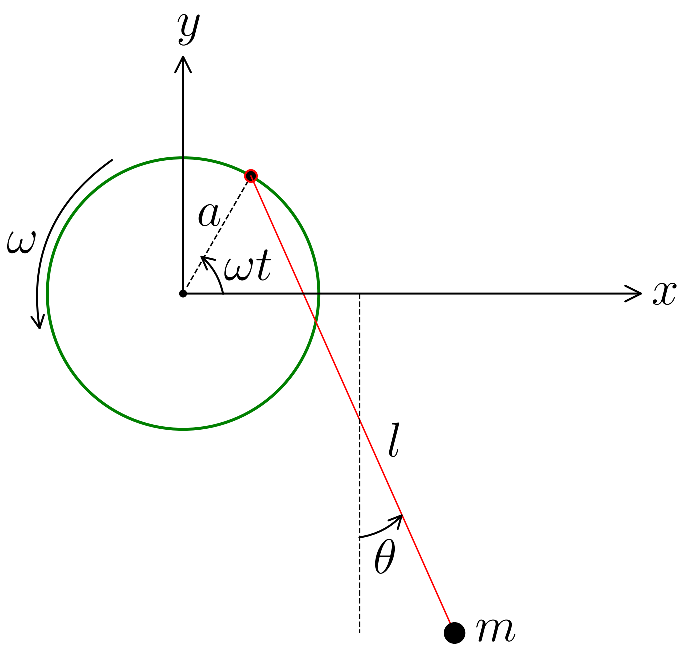

Figure 2.8: Illustration of a pendulum on a rim.

2.7 Velocity dependent potentials

Lagrange equations can be described the the form (2.76) even if there is no potential function, V, in the usual sense, providing the generalized are obtained from a function $U(q,\dot{q})$ by the prescription

$$Q_j = -\frac{\partial U}{\partial q_j} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_j}\right). \tag{2.77}$$

In such case, Eqs. (2.76) still follow from (2.72) with the Lagrangian given by

$$L = T - U. \tag{2.78}$$

Here U may be called a ***generalized potential*** or ***velocity-dependent potential***.

2.7.1 Moving charge in both an electric and a magnetic field

Consider a particle of mass m, and an electric charge, q, moving at a velocity, $\vec{v}$, in an otherwise charge-free region containing both an electric field, $\vec{E}$, and a magnetic field, $\vec{B}$, which may depend upon time and position.

The charge experience a force, called the Lorentz force, given by

$$\vec{F} = q\left[\vec{E} + \left(\vec{v}\times\vec{B}\right)\right]. \tag{2.79}$$

Both $\vec{E} = \vec{E}(t,x,y,z)$ and $\vec{B} = \vec{B}(t,x,y,z)$ are continuous functions of time and position derivable from a scalar potential $\phi(t,x,y,z)$ and a vector potential $\vec{A}(t,x,y,z)$ by

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}, \tag{2.80a}$$

and

$$\vec{B} = -\vec{\nabla}\times\vec{A}. \tag{2.80b}$$

The force can be derived from the following velocity-dependent potential energy

$$U = q\phi - q\vec{A}\cdot\vec{v}. \tag{2.81}$$

so the Lagrangian, $L = T - U$, is

$$L = \frac{1}{2}mv^2 - q\phi + q\vec{A}\cdot\vec{v}. \tag{2.82}$$

The Lagrange's equations are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0, \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = 0, \quad \text{and} \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) - \frac{\partial L}{\partial z} = 0. \tag{2.83}$$

The x-component of Lagrange's equations becomes,

$$m\ddot{x} = q\left(v_x\frac{\partial A_x}{\partial x} + v_y\frac{\partial A_y}{\partial x} + v_z\frac{\partial A_z}{\partial x}\right) - q\left(\frac{\partial \phi}{\partial x} + \frac{dA_x}{dt}\right) \tag{2.84}$$

The total time derivative of A_x can be expressed as

$$\begin{aligned}\frac{dA_x}{dt} &= \frac{\partial A_x}{\partial t} + \frac{\partial A_x}{\partial x}\frac{dx}{dt} + \frac{\partial A_x}{\partial y}\frac{dy}{dt} + \frac{\partial A_x}{\partial z}\frac{dz}{dt} \equiv \frac{\partial A_x}{\partial t} + \frac{\partial A_x}{\partial x}\dot{x} + \frac{\partial A_x}{\partial y}\dot{y} + \frac{\partial A_x}{\partial z}\dot{z} \\ &= \frac{\partial A_x}{\partial t} + \left(\hat{\imath}\frac{\partial A_x}{\partial x} + \hat{\jmath}\frac{\partial A_x}{\partial y} + \hat{k}\frac{\partial A_x}{\partial z}\right)\cdot\left(\hat{\imath}v_x + \hat{\jmath}v_y + \hat{k}v_z\right) \quad [\because \dot{x} = v_x,\ \dot{y} = v_y,\ \dot{z} = v_z]\end{aligned} \tag{2.85}$$

or

$$\frac{dA_x}{dt} = \frac{\partial A_x}{\partial t} + \vec{v}\cdot\vec{\nabla}A_x. \tag{2.86}$$

The x component of $\left(\vec{v}\times\vec{B}\right)$, using (2.80b), is given as

$$\left(\vec{v}\times\vec{B}\right)_x = v_y\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) + v_z\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right). \tag{2.87}$$

Combining these expressions gives the equation of motion in the x-direction

$$m\ddot{x} = q\left[E_x + \left(\vec{v}\times\vec{B}\right)_x\right]. \tag{2.88}$$

On a component-by-component comparison, equations (2.88) and (2.79) are identical. This shows that the Lorentz force equation is derivable from a velocity dependent potential of the form (2.81).

2.7.2 Rayleigh's dissipation function

If not all the forces acting on the system are derivable from a potential, then Lagrange's equations can always be written in the form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = Q_j, \tag{2.89}$$

where L contains the potential of the conservative forces as before, and Q_j represents the forces *not* arising from a potential. Such a situation occurs when frictional forces are present. In most cases, the frictional force is proportional to the velocity of the particle, so that its x-component has the form

$$F_{f_x} = -k_x v_x. \tag{2.90}$$

Frictional forces of this type may be derived in terms of a function $\mathscr{F}$, known as ***Rayleigh's dissipation function***, and defined as

$$\mathscr{F} = \frac{1}{2}\sum_i\left(k_x v_{ix}^2 + k_y v_{iy}^2 + k_z v_{iz}^2\right), \tag{2.91}$$

where the summation is over the particle of the system. From this definition we can write

$$F_{f_x} = -\frac{\partial \mathscr{F}}{\partial v_x}, \tag{2.92}$$

or

$$F_{f_x} = -\vec{\nabla}_v \mathscr{F}. \tag{2.93}$$

A physical interpretation may be given to the dissipation function by evaluating the work done by the system against friction as

$$dw_f = -\vec{F}_f\cdot d\vec{r} = -\vec{F}_f\cdot\vec{v}dt = \left(k_x v_x^2 + k_y v_y^2 + k_z v_z^2\right)dt \equiv 2\mathscr{F}. \tag{2.94}$$

Hence, $2\mathscr{F}$ is the rate of energy dissipation due to friction. The component of the force resulting from the force of friction is then given by

$$\begin{aligned} Q_j = \sum_i \vec{F}_{f_i}\cdot\frac{\partial \vec{r}_i}{\partial q_j} &= -\sum_i \vec{\nabla}_v\mathscr{F}\cdot\frac{\partial \vec{r}_i}{\partial q_j} \\ &= -\sum_i \vec{\nabla}_v\mathscr{F}\cdot\frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_j}. \end{aligned} \tag{2.95}$$

The Lagrange equations with dissipation becomes

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} + \frac{\partial \mathscr{F}}{\partial \dot{q}_j} = 0, \tag{2.96}$$

so that two scalar functions, L and $\mathscr{F}$, must be specified to obtain the equations of motion.

■ **Example 2.1** Stoke's law: A sphere of radius a moving at a speed v, in a medium of viscosity η experiences the frictional drag force $F_f = 6\pi\eta av$. ■

2.8 Simple Applications

■ **Example 2.2 Motion of a particle is space using Cartesian coordinates:** The generalized forces needed are F_x, F_y and F_z (the components of the force vector). ■

The kinetic energy is given by

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right). \tag{2.97}$$

$$\frac{\partial T}{\partial x} = \frac{\partial T}{\partial y} = \frac{\partial T}{\partial z} = 0, \text{ and } \frac{\partial T}{\partial \dot{x}} = m\dot{x}, \quad \frac{\partial T}{\partial \dot{y}} = m\dot{y}, \quad \frac{\partial T}{\partial \dot{z}} = m\dot{z}.$$

and the equations of motion are [using Eq. (2.72)]

$$\frac{d}{dt}(m\dot{x}) = F_x, \quad \frac{d}{dt}(m\dot{y}) = F_y, \quad \frac{d}{dt}(m\dot{z}) = F_z \tag{2.98}$$

■ **Example 2.3 Atwood's machine: Figure 2.9(a)** An example of a conservative system with holonomic, scleronomous constraint (the pulley is assumed to be frictionless and massless). ■

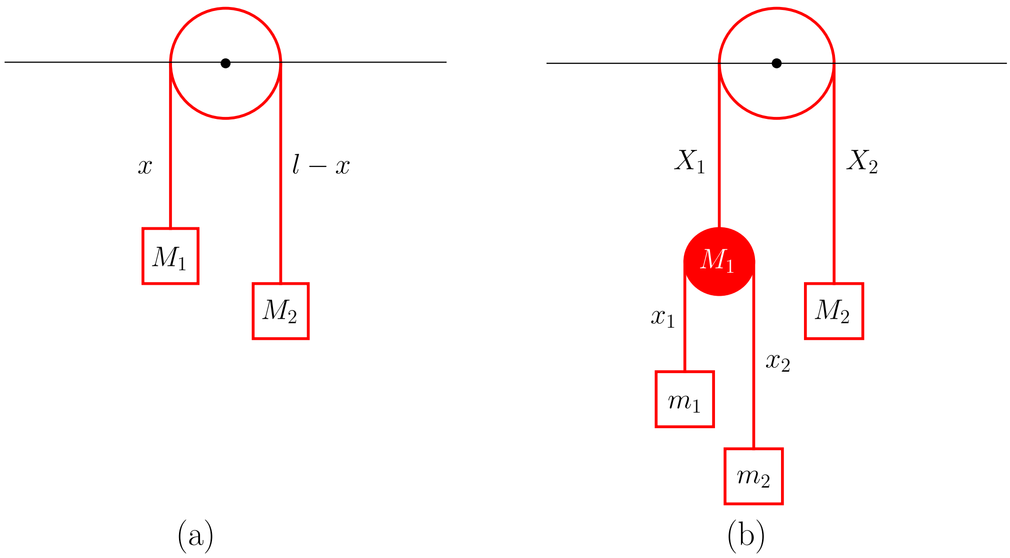

Figure 2.9: (a) Atwood's machine (b) Atwood's machine with two pulleys.

There is only one independent coordinate x, the position of the other weight being determined by the constraint that the length of the rope is l. The potential energy is

$$V = -M_1 g x - M_2 g(l - x), \tag{2.99}$$

while the kinetic energy is

$$T = \frac{1}{2}(M_1 + M_2)\dot{x}^2. \tag{2.100}$$

The Lagrangian can be written as

$$L = T - V = \frac{1}{2}(M_1 + M_2)\dot{x}^2 + M_1 g x + M_2 g(l - x). \tag{2.101}$$

The required derivatives are

$$\frac{\partial L}{\partial x} = (M_1 - M_2)\, g, \text{ and } \frac{\partial L}{\partial \dot{x}} = (M_1 + M_2)\,\dot{x}.$$

The Euler-Lagrange equation for the Atwood's machine can be written as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0, \tag{2.102}$$

implies

$$(M_1 + M_2)\,\ddot{x} = (M_1 - M_2)\, g, \tag{2.103}$$

or

$$\ddot{x} = \frac{(M_1 - M_2)}{(M_1 + M_2)} g, \tag{2.104}$$

which is the familiar result obtained by more elementary means. This trivial problem emphasizes that the forces of constraint (here the tension in the rope) appear nowhere in the Lagrangian formulation. The tension in the rope can not be found directly by the Lagrangian method.

■ **Example 2.4 A bead (or ring) sliding on a uniformly rotating wire in a force-free space:** The wire is straight, and is rotated uniformly about some fixed axis perpendicular to the wire (see Figure 2.10). This example is a simple illustration of a constraint being time dependent, with the

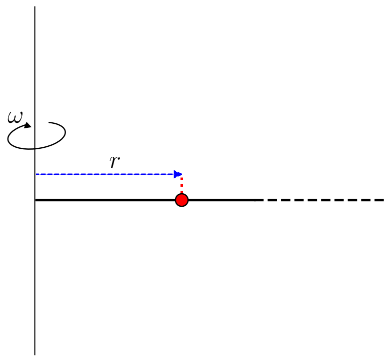

Figure 2.10: A bead sliding on a uniformly rotating wire in a force-free space

rotation axis along z and the wire in the xy plane. ■

The transformation equations explicitly contain the time.

$$x = r\cos\omega t.$$
$$y = r\sin\omega t.$$

where r is the distance along wire from the rotation axis and ω is the angular velocity of rotation.

In this example, the constraint can be expressed by the relation $\dot{\theta} = \omega$. Hence, the kinetic energy turns out to be

$$T = \frac{1}{2}m\left(\dot{r}^2 + r^2\omega^2\right).$$

The equation of motion is

$$m\ddot{r} - mr\omega^2 = 0,$$

which is the familiar simple harmonic oscillator equation with a change of sign. The solution $r = e^{\omega t}$ shows that the bead moves exponentially outward because of the centripetal acceleration. Again, the method cannot furnish the force of constraint that keep the bead on the wire.

2.9 Review questions

Q. 2.1 The validity of *strong law of action and reaction* is required for the conservation of ________.

Q. 2.2 The space spanned by the generalised coordinates is known as __________ space.

Q. 2.3 What are constraints?

Q. 2.4 Derive Lagrange's equations using D'Alembert's principle.

Q. 2.5 Set up the Lagrangian for a simple pendulum and obtain an equation of motion.

Q. 2.6 A mass M_2 hangs at one end of a string that passes over a fixed, frictionless, non-rotating pulley. At the other end of this string, there is another non-rotating pulley of mass M_1, over which a second string carries masses m_1 and m_2, as illustrated in Figure 2.9(b). Set up the Lagrangian for this system and determine the acceleration of the mass M_2.

Q. 2.7 Two identical masses m are connected by springs having equal spring constants, as shown in figure below (Figure 2.11), so that the masses are free to slide on a frictionless AB. The

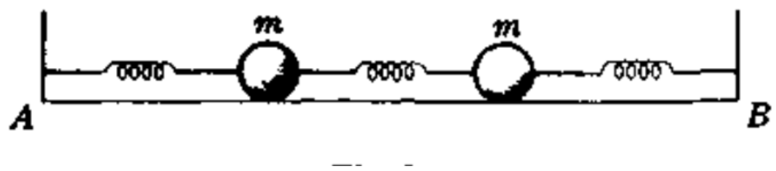

Figure 2.11: Two identical masses m are connected by springs having equal spring constants.

walls at A and B to which the ends of the springs are attached are fixed. Construct the Lagrangian and deduce the equations of motion.

Q. 2.8 Construct a Lagrangian and deduce the equation of motion for the compound pendulum illustrated in Figure 2.12, which oscillates in a vertical plane about a fixed perpendicular axis passing through O.

Q. 2.9 A bead slides without friction under the influence of gravity on a frictionless wire in the shape of a cycloid as illustrated in Figure 2.13 with equations. $x = a(\theta - \sin\theta)$ and $y = a(1 + \cos\theta)$, where $0 \le \theta \le 2\pi$. Find the Lagrangian and obtain the equation of

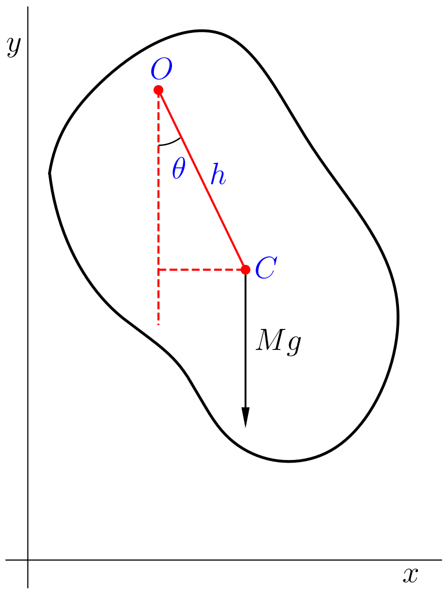

Figure 2.12: Illustration of a compound pendulum.

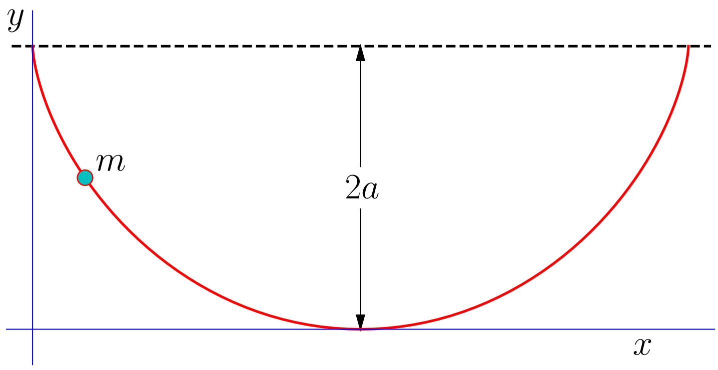

Figure 2.13: Illustration of a bead sliding without friction under the influence of gravity on a frictionless wire shaped like a cycloid.

motion. Also, show that the equation of motion can be represented as $\frac{d^2u}{dt^2} + \frac{g}{4a}u = 0$ using the transformation $u = \cos\frac{\theta}{2}$.

Q. 2.10 The Lagrangian of a point mass m falling freely from rest by gravity is given by

$$L = \frac{1}{2}m\dot{y}^2 + mgy,$$

with g being the acceleration due to gravity. Deduce the equation of motion.

Q. 2.11 A particle of mass m moves under the influence of gravity on the frictionless inner surface of the paraboloid of revolution $\rho^2 = az$, where $\rho^2 = x^2 + y^2$. Obtain the equations of motion. (i) Identify the generalized coordinates and construct the Lagrangian and (ii) obtain the equations of motion.

Q. 2.12 State whether the force force $F = -\frac{3a}{r^4}$, where a is constant, is a central force or not. Find the potential.

Q. 2.13 Show that if not all the forces are derivable from a potential, then

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = Q_j, \tag{2.105}$$

where L contains the potential of the conservative forces, and Q_j represents the forces that are *not* arising from a potential. Prove that the above Lagrange equation (2.105) is equivalent to

$$\frac{\partial \dot{L}}{\partial \dot{q}_j} - 2\frac{\partial L}{\partial q_j} = Q_j. \tag{2.106}$$

Q. 2.14 The Lagrangian $L = \frac{1}{2}\dot{q}^2 + q\dot{q} - \frac{1}{2}q^2$ corresponds to ________ system.

Q. 2.15 A bead is constrained to slide on a frictionless rod that is fixed at an angle θ with a vertical

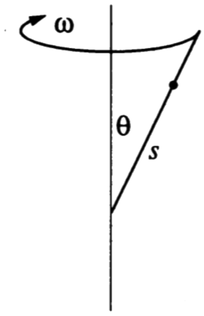

Figure 2.14: Illustration of a bead constrained to slide on a frictionless rod.

axis and is rotating with angular frequency ω, about the axis as show in Figure 2.14. Taking the distance s along the rod as variable, construct the Lagrangian of the bead.

2.10 Reference

[1] H. Goldstein, J. L. Safko, and C. P. Poole. *Classical Mechanics, 3e*. Pearson Education, 2011.