# Bharathidasan University

Tiruchirappalli – 620 024, Tamil Nadu, India

## Programme: M. Sc., Physics

**Course Title : Electromagnetic Theory**

**Course Code : 22PH301**

## Unit V

## Electromagnetism


**Dr. T.C. Sabari Girisun**

Assistant Professor

Department of Physics

# Electrodynamics After Maxwell

$\nabla . \boldsymbol{E} = \frac{1}{\varepsilon_0} \rho$ Electrostatics (Gauss's law)

$\nabla . \boldsymbol{B} = 0$ Magnetostatics (no name)

$\nabla \; X \; \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$ Electromagnetism (Faraday's law)

$\nabla \; X \; \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t}$ Electromagnetism (Modified Ampere's law)

# Poynting's Theorem

The work necessary to assemble a static charge distribution

$$W_e = \frac{\varepsilon_0}{2} \int E^2 d\tau$$

The work required to get current going is

$$W_m = \frac{1}{2\mu_0} \int B^2 d\tau$$

The total energy stored in electromagnetic fields is

$$W_{em} = \frac{1}{2} \int \left( \varepsilon_0 E^2 + \frac{B^2}{\mu_0} \right) d\tau = \frac{dU_{em}}{dt}$$

According to Lorentz law, the work done on a charge is

$$\boldsymbol{F}.\, d\boldsymbol{l} = q(\boldsymbol{E} + \boldsymbol{v}\; X\; \boldsymbol{B}).\, \boldsymbol{v}dt = q\boldsymbol{E}.\, \boldsymbol{v}dt$$

The rate at which work is done on all charges in a volume is

$$\frac{dW}{dt} = F.\, v = qE.\, v = \sum_i n_i q_i v_i.\, \boldsymbol{E}_i = \sum_i \boldsymbol{J}_i.\, \boldsymbol{E}_i = \int_V \; (\boldsymbol{E}.\boldsymbol{J}) d\tau$$

# Poynting's Theorem

From the equation of continuity

$$\nabla.\boldsymbol{J} + \frac{\partial \rho}{\partial t} = 0$$

From Gauss Law

$$\nabla.\boldsymbol{E} = 4\pi\rho \qquad \rho = \frac{1}{4\pi}(\nabla.\boldsymbol{E}) \qquad \frac{\partial \rho}{\partial t} = \frac{1}{4\pi}\left(\nabla.\frac{\partial \boldsymbol{E}}{\partial t}\right)$$

$$\nabla.\boldsymbol{J} + \frac{1}{4\pi}\left(\nabla.\frac{\partial \boldsymbol{E}}{\partial t}\right) = 0 \qquad \nabla.\left(\boldsymbol{J} + \frac{1}{4\pi}\frac{\partial \boldsymbol{E}}{\partial t}\right) = 0$$

From corrected Ampere's law

$$\nabla \times \mathbf{B} = \frac{4\pi}{c}\left[\boldsymbol{J} + \frac{1}{4\pi}\frac{\partial \boldsymbol{E}}{\partial t}\right]$$

$$\nabla \times \mathbf{B} = \frac{4\pi}{c}\boldsymbol{J} + \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t}$$

$$\boldsymbol{J} = \frac{c}{4\pi}\left(\nabla \times \mathbf{B} - \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t}\right) = \frac{1}{4\pi}\left(c\,\nabla \times \mathbf{B} - \frac{\partial \boldsymbol{E}}{\partial t}\right)$$

# Poynting's Theorem

The total work done by the field is

$$\frac{dW}{dt} = \int_V (\boldsymbol{J}.\boldsymbol{E}) d^3x = \int_V \left(\frac{1}{4\pi}\left(c\,\nabla \text{ X } \mathbf{B} - \frac{\partial \boldsymbol{E}}{\partial t}\right)\right).\boldsymbol{E}\; d^3x$$

$$\frac{dW}{dt} = \frac{1}{4\pi}\int \left(c\,\boldsymbol{E}.(\nabla \text{ X } \mathbf{B}) - \boldsymbol{E}.\frac{\partial \boldsymbol{E}}{\partial t}\right) d^3x$$

$Since, \boldsymbol{E}.(\nabla \text{ X } \mathbf{B}) = \boldsymbol{B}.(\nabla \text{ X } \mathbf{E}) - \nabla.(\mathbf{E} \text{ X } \mathbf{B})$

$$\frac{dW}{dt} = \frac{1}{4\pi}\int \left(c\,\boldsymbol{B}.(\nabla \text{ X } \mathbf{E}) - c\,\nabla.(\mathbf{E} \text{ X } \mathbf{B}) - \boldsymbol{E}.\frac{\partial \boldsymbol{E}}{\partial t}\right) d^3x$$

$$\frac{dW}{dt} = \frac{1}{4\pi}\int \left(-\boldsymbol{B}.\frac{\partial \boldsymbol{B}}{\partial t} - c\,\nabla.(\mathbf{E} \text{ X } \mathbf{B}) - \boldsymbol{E}.\frac{\partial \boldsymbol{E}}{\partial t}\right) d^3x$$

# Poynting's Theorem

$$Mean\ while,\ \boldsymbol{B}.\frac{\partial \boldsymbol{B}}{\partial t} = \frac{1}{2}.\frac{\partial}{\partial t}(\boldsymbol{B}^2)\ and\ \boldsymbol{E}.\frac{\partial \boldsymbol{E}}{\partial t} = \frac{1}{2}.\frac{\partial}{\partial t}(\boldsymbol{E}^2)$$

$$\frac{dW}{dt} = \frac{1}{4\pi}\int\left(-\frac{1}{2}.\frac{\partial}{\partial t}(\boldsymbol{B}^2) - \frac{1}{2}.\frac{\partial}{\partial t}(\boldsymbol{E}^2)\right) d^3x - \int\big(c\ \nabla.(\mathbf{E}\ \mathrm{X}\ \mathbf{B})\big)\, d^3x$$

Hence the rate at which work is done on all charges in a volume is

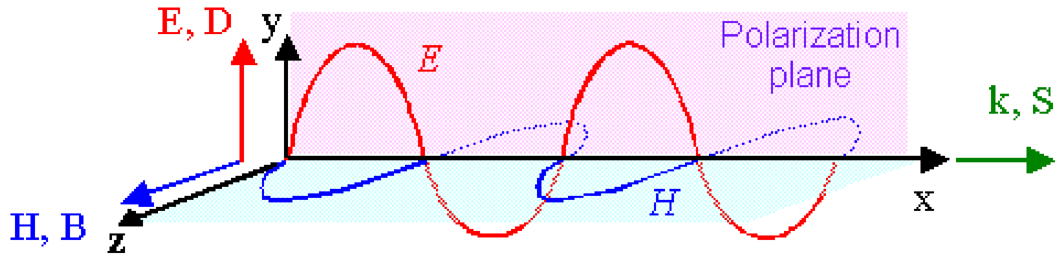
$$\frac{dW}{dt} = -\frac{d}{dt}\int_V \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{B^2}{\mu_0}\right) d\tau - \frac{1}{\mu_0}\oint_S (\mathbf{E}\ \mathrm{X}\ \mathbf{B}).\,d\boldsymbol{a}$$

The first integral is the total energy stored in the field, $U_{em}$ and the second term is the rate at which energy is carried out of ϒ across its boundary surface by the em fields. This is Poynting's theorem or 'work-energy theorem of electrodynamics'

# Poynting's Theorem

The energy per unit time per unit area transported by the fields is called Poynting's vector,

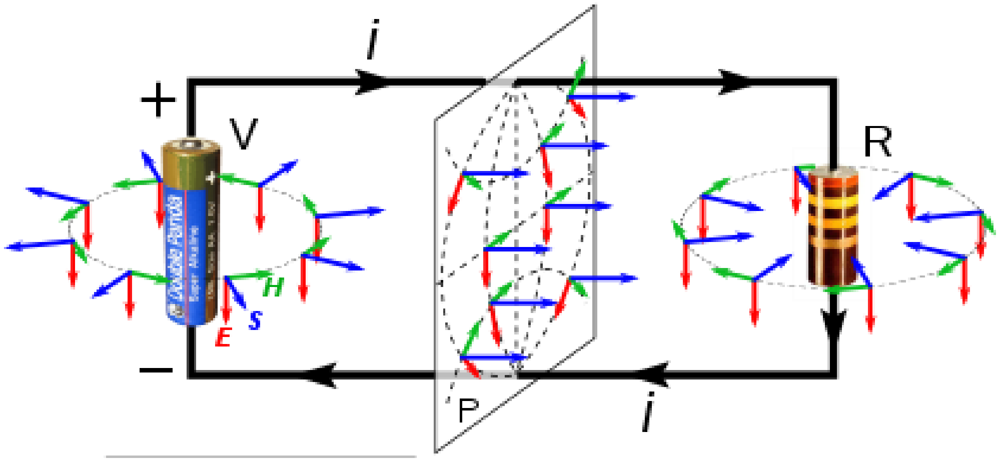
$$\boldsymbol{S} = \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B})$$




Poynting's theorem states that the work done on the charges by the electromagnetic force is equal to the decrease in energy stored in the field, less the energy that flowed through the surface

$$\frac{dW}{dt} = -\frac{dU_{em}}{dt} - \oint_S \boldsymbol{S}.d\boldsymbol{a}$$

$$\frac{\partial U}{\partial t} + \nabla.\boldsymbol{S} = -\boldsymbol{J}.\boldsymbol{E}$$



# Poynting Vector

It is the power flux – amount of energy crossing unit area placed perpendicular to the vector per unit time

$$\mathbf{S} = \frac{P}{A}$$

Calculate the magnitude of Poynting vector at the surface of the sun having radius 7 x $10^8$ m which radiate a power of 3.8 x $10^{26}$ W

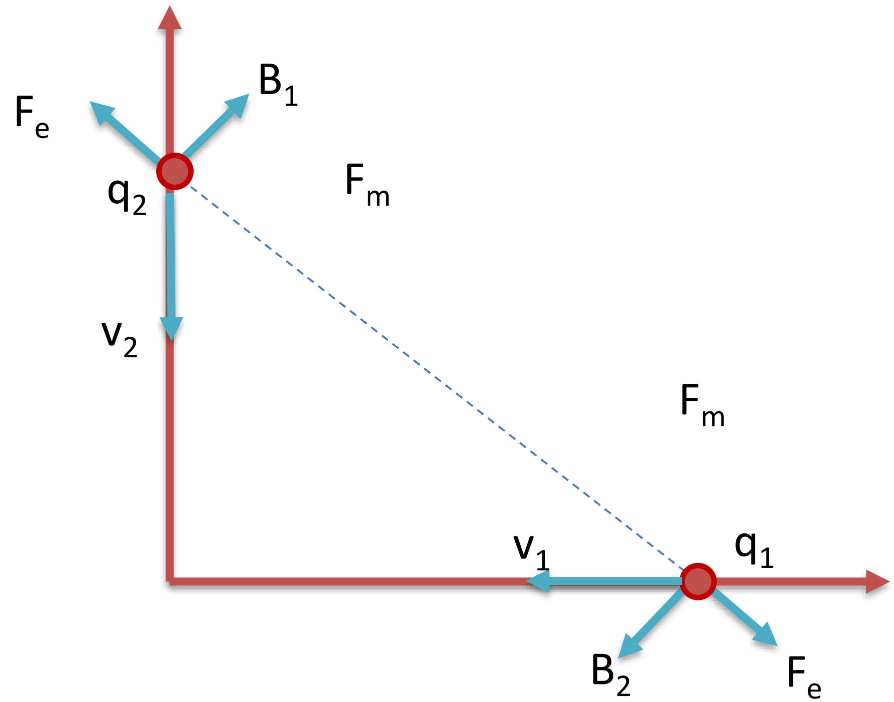
$$\mathbf{S} = \frac{P}{4\pi r^2} = 6.175\ x\ 107\ W/m^2$$

Prove Joule's law of heating using Poynting vector. Given a current flows down a wire with uniform electric field E=V/L

# Newton's III Law in Electrodynamics

Imagine a point charge is moving along x-axis at constant speed, v. As it is moving, its electric field is not given by Coulomb's law, but **E** still points radially outward. Also as moving point charge, do not give a steady current, its magnetic field is not given by Biot-Savart's law but **B** still circles around the axis.

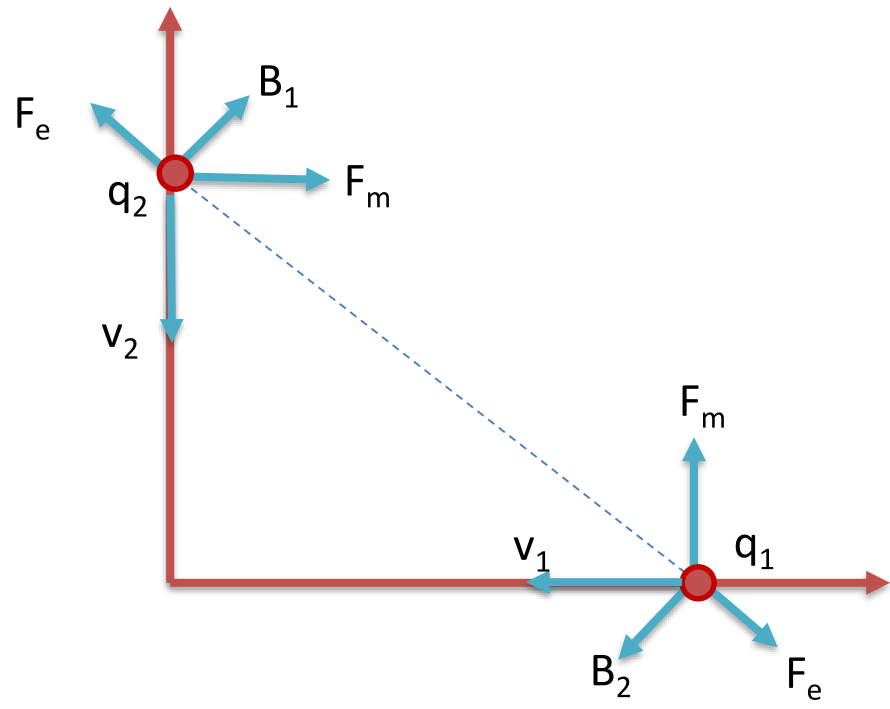
If a charge moving along x-axis encounters an identical charge along y-axis, the emf between them would tend to drive them off the axes. But some means it is placed on the axes. The electric force between them is now repulsive, but magnetic force is quiet different.



# Newton’s III Law in Electrodynamics

The magnetic field of $q_1$ points into the page at the position of $q_2$, whereas the magnetic field of $q_2$ is out of page at $q_1$ and magnetic force on $q_1$ is upward.

The emf of $q_1$ on $q_2$ is equal but not opposite to the force of $q_2$ on $q_1$, which violates Newton’s third law of motion. But this is not the case in electrostatics and magnetostatics.




The paradox can be explained by the proof of conservation of momentum that rests on the cancellation of internal forces. It can be realized that the fields themselves carry momentum. In the above case, whatever momentum is lost to the particles is gained by the fields. Only when the field momentum is added to the mechanical momentum of the charges, conservation of momentum is restored.

# Conservation of Momentum

The total electromagnetic force on the charges in volume V is

$$\boldsymbol{F} = \int_V (\boldsymbol{E} + \boldsymbol{v}\, X\, \boldsymbol{B})\rho d\tau = \int_V (\rho\boldsymbol{E} + \boldsymbol{J}\, X\, \boldsymbol{B})d\tau$$

Force per unit volume V is

$$f = \rho\boldsymbol{E} + \frac{1}{c}\boldsymbol{J}\, X\, \boldsymbol{B}$$

Rewriting the above equation in terms of fields alone by eliminating sources ρ and **J** using Maxwell's equations is

$$\nabla . \boldsymbol{E} = 4\pi\rho \qquad \rho = \frac{1}{4\pi}\nabla . \boldsymbol{E}$$

$$\nabla\, \mathrm{X}\, \mathbf{B} = \frac{4\pi}{c}\boldsymbol{J} + \frac{1}{c}\frac{\partial\boldsymbol{E}}{\partial t} \qquad \boldsymbol{J} = \frac{c}{4\pi}\left(\nabla\, \mathrm{X}\, \mathbf{B} - \frac{1}{c}\frac{\partial\boldsymbol{E}}{\partial t}\right)$$

$$\rho\boldsymbol{E} + \frac{1}{c}\boldsymbol{J}\, X\, \boldsymbol{B} = \frac{1}{4\pi}(\nabla . \boldsymbol{E})\boldsymbol{E} + \frac{1}{4\pi}\left(\nabla\, \mathrm{X}\, \mathbf{B} - \frac{1}{c}\frac{\partial\boldsymbol{E}}{\partial t}\right) X\, \boldsymbol{B}$$

$$\rho\boldsymbol{E} + \frac{1}{c}\boldsymbol{J}\, X\, \boldsymbol{B} = \frac{1}{4\pi}(\nabla . \boldsymbol{E})\boldsymbol{E} - \frac{1}{4\pi}\left(\boldsymbol{B}\, X\, (\nabla\, \mathrm{X}\, \mathbf{B}) + \frac{1}{c}\boldsymbol{B}\, X\, \frac{\partial\boldsymbol{E}}{\partial t}\right)$$

# Conservation of Momentum

But

$$\frac{1}{c}\frac{\partial}{\partial t}(\boldsymbol{E}\;X\;\boldsymbol{B}) = \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t}\;X\;\boldsymbol{B} + \frac{1}{c}\boldsymbol{E}\;X\;\frac{\partial \boldsymbol{B}}{\partial t} = -\frac{1}{c}\boldsymbol{B}\;X\;\frac{\partial \boldsymbol{E}}{\partial t} + \frac{1}{c}\boldsymbol{E}\;X\;\frac{\partial \boldsymbol{B}}{\partial t}$$

$$\boldsymbol{B}\;X\;\frac{\partial \boldsymbol{E}}{\partial t} = \boldsymbol{E}\;X\;\frac{\partial \boldsymbol{B}}{\partial t} - \frac{\partial}{\partial t}(\boldsymbol{E}\;X\;\boldsymbol{B})$$

Thus

$$f = \frac{1}{4\pi}(\nabla.\boldsymbol{E})\boldsymbol{E} - \frac{1}{4\pi}\big(\boldsymbol{B}\;X\;(\nabla\;\mathrm{X}\;\mathbf{B})\big) + \frac{1}{4\pi c}\left(\boldsymbol{E}\;X\;\frac{\partial \boldsymbol{B}}{\partial t}\right) - \frac{1}{4\pi c}\frac{\partial}{\partial t}(\boldsymbol{E}\;X\;\boldsymbol{B})$$

$$f = \frac{1}{4\pi}[(\nabla.\boldsymbol{E})\boldsymbol{E} - \boldsymbol{E}\;X\;(\nabla\;\mathrm{X}\;\mathbf{E})] - \frac{1}{4\pi}\big(\boldsymbol{B}\;X\;(\nabla\;\mathrm{X}\;\mathbf{B})\big) - \frac{1}{4\pi c}\frac{\partial}{\partial t}(\boldsymbol{E}\;X\;\boldsymbol{B})$$

$$f = \frac{1}{4\pi}[(\nabla.\boldsymbol{E})\boldsymbol{E} - \boldsymbol{E}\;X\;(\nabla\;\mathrm{X}\;\mathbf{E})] + \frac{1}{4\pi}[(\nabla.\boldsymbol{B})\boldsymbol{B} - \boldsymbol{B}\;X\;(\nabla\;\mathrm{X}\mathbf{B})] - \frac{1}{4\pi c}\frac{\partial}{\partial t}(\boldsymbol{E}\;X\;\boldsymbol{B})$$

$$f = \frac{1}{4\pi}[(\nabla.\boldsymbol{E})\boldsymbol{E} + (\nabla.\mathbf{B})\boldsymbol{B} - \boldsymbol{E}\;X\;(\nabla\;\mathrm{X}\;\mathbf{E}) - \boldsymbol{B}\;X\;(\nabla\;\mathrm{X}\;\mathbf{B})] - \frac{1}{4\pi c}\frac{\partial}{\partial t}(\boldsymbol{E}\;X\;\boldsymbol{B})$$

# Conservation of Momentum

By Newton's law, $\boldsymbol{F} = \dfrac{\partial \boldsymbol{P}_{\boldsymbol{momentum}}}{\partial t}$

$$\frac{d\boldsymbol{P}_{\boldsymbol{momentum}}}{dt} + \frac{d}{dt}\int_V \frac{1}{4\pi c}(\boldsymbol{E}\ X\ \boldsymbol{B})d^3x$$

$$= \frac{1}{4\pi}\int_V [(\nabla.\boldsymbol{E})\boldsymbol{E} + (\nabla.\mathbf{B})\boldsymbol{B} - \boldsymbol{E}\ X\ (\nabla\ \mathrm{X}\ \mathbf{E}) - \boldsymbol{B}\ X\ (\nabla\ \mathrm{X}\ \mathbf{B})]\, d^3x$$

$$\frac{d\boldsymbol{P}_{\boldsymbol{momentum}}}{dt} + \frac{d\boldsymbol{P}_{\boldsymbol{field}}}{dt} = \frac{1}{4\pi}\int_V [(\nabla.\boldsymbol{E})\boldsymbol{E} + (\nabla.\mathbf{B})\boldsymbol{B} - \boldsymbol{E}\ X\ (\nabla\ \mathrm{X}\ \mathbf{E}) - \boldsymbol{B}\ X\ (\nabla\ \mathrm{X}\ \mathbf{B})]\, d^3x$$

The above equation can be simplified by introducing **Maxwell Stress Tensor**

$$T_{\alpha\beta} = \left[E_\alpha E_\beta + B_\alpha B_\beta - \frac{1}{2}\{(\boldsymbol{E}.\boldsymbol{E}) + (\boldsymbol{B}.\boldsymbol{B})\}\delta_{\alpha\beta}\right]$$

The indices refers to the co-ordinates of x, y and z. So the stress tensor has nine components. Here is Kronecker delta

$$\delta_{\alpha\beta} = \begin{cases} 1, & \delta_{xx} = \delta_{yy} = \delta_{zz} \\ 0, & \delta_{xy} = \delta_{yz} = \delta_{zx} \end{cases}$$

# Conservation of Momentum

$$(\nabla.\boldsymbol{E})\boldsymbol{E} + (\nabla.\mathbf{B})\boldsymbol{B} = \left(\frac{\partial E_1}{\partial x_1} + \frac{\partial E_2}{\partial x_2} + \frac{\partial E_3}{\partial x_3}\right)E_1 + \left(\frac{\partial B_1}{\partial x_1} + \frac{\partial B_2}{\partial x_2} + \frac{\partial B_3}{\partial x_3}\right)B_1$$

$$\boldsymbol{E}\ X\ (\nabla\ \mathrm{X}\ \mathbf{E}) + \boldsymbol{B}\ X\ (\nabla\ \mathrm{X}\ \mathbf{B})$$
$$= \left[\left(\frac{\partial E_2}{\partial x_1} - \frac{\partial E_1}{\partial x_2}\right)E_2 + \left(\frac{\partial E_1}{\partial x_3} - \frac{\partial E_3}{\partial x_1}\right)E_3 - \left(\frac{\partial B_2}{\partial x_1} - \frac{\partial B_1}{\partial x_2}\right)B_2 + \left(\frac{\partial B_1}{\partial x_3} - \frac{\partial B_3}{\partial x_1}\right)B_3\right]$$

The force per unit volume with $\overleftrightarrow{T}$ as Maxwell Stress Tensor and **S** is Poynting vector

$$f = \nabla.\overleftrightarrow{T} - \varepsilon_0\mu_0\frac{\partial S}{\partial t}$$

Total force on the charge is $$F = \oint \overleftrightarrow{T}.da - \varepsilon_0\mu_0\frac{d}{dt}\int_V \boldsymbol{S}.d\tau$$

The momentum due to the field is $$\boldsymbol{P}_{\boldsymbol{em}} = \varepsilon_0\mu_0\int_V \boldsymbol{S}.d\tau$$

$$\frac{d\boldsymbol{P}_{\boldsymbol{momentum}}}{dt} + \frac{d\boldsymbol{P}_{\boldsymbol{field}}}{dt} = -\varepsilon_0\mu_0\int_V \boldsymbol{S}.d\tau + \oint \overleftrightarrow{T}.da + \varepsilon_0\mu_0\int_V \boldsymbol{S}.d\tau$$

$$\frac{\partial}{\partial t}\left(\boldsymbol{P}_{\boldsymbol{momentum}} + \boldsymbol{P}_{\boldsymbol{field}}\right) = \nabla.\overleftrightarrow{T}$$

$\overleftrightarrow{T}$ is momentum flux density. It represents em stress and flow of momentum transported by fields

# Books for Reference

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3. **R. P. Feynman, R. B. Leighton and M. Sands**, *The Feynman Lectures on Physics: Vol. II (*Narosa Book Distributors, New Delhi, 1989)
4. **Satya Prakash**, *Electromagnetic Theory and Electrodynamics* (Kedar Nath Ram Nath, Meerut, 2015)