Bharathidasan University
Tiruchirappalli – 620 024, Tamil Nadu, India

Programme: M. Sc., Physics

Course Title : Electromagnetic Theory
Course Code : 22PH301

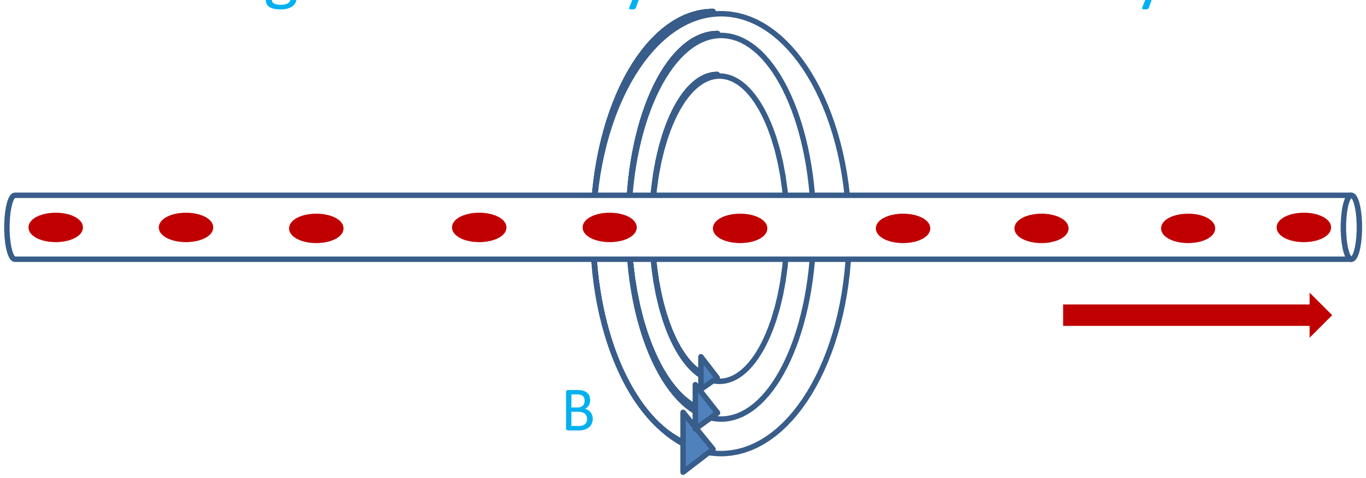
Unit III
Magnetostatics

Dr. T.C. Sabari Girisun
Assistant Professor
Department of Physics

What is Magnetostatics?

Stationary charges – Creates constant electric field

Charges in steady motion – Steady current

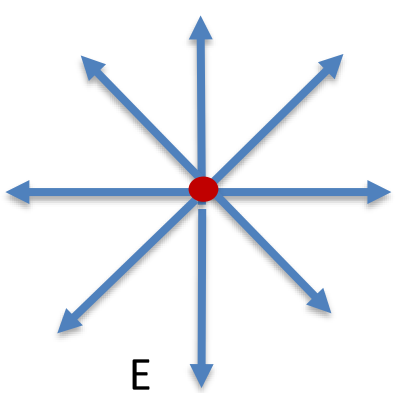


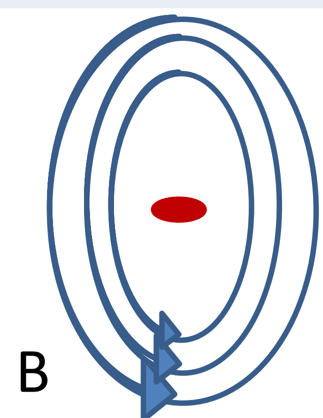
Steady current – Creates constant magnetic field

Magnetostatics means constant

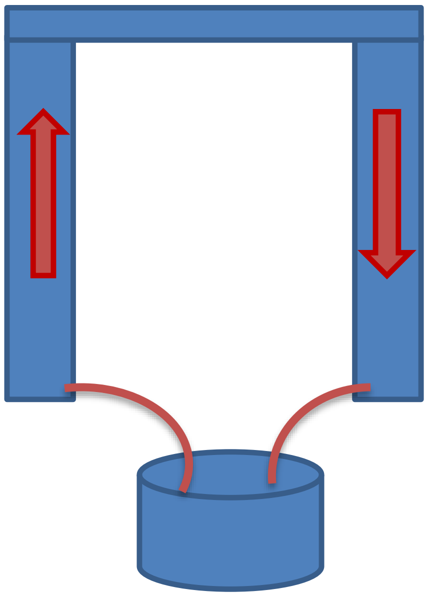
Moving from Electrostatics to Magnetostatics

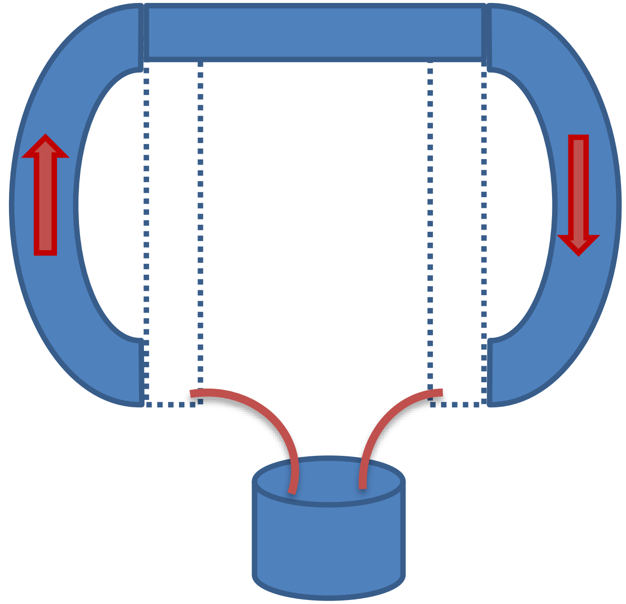
Particulars	Electrostatics	Magnetostatics
Force	Coulomb's Law (static charges)	Ampere's Law (current elements)
Electric Field	Gauss's Law	Biot-Savart's Law
Integral Statement	Gauss's law (Coulomb's law)	Ampere's circuital law (Ampere's law)
Divergence	div **E** = $-\rho/\epsilon_0$	div **B** = 0
Curl	curl **E** = 0	curl **B** = μ_0**J**
Potential	Scalar	Vector

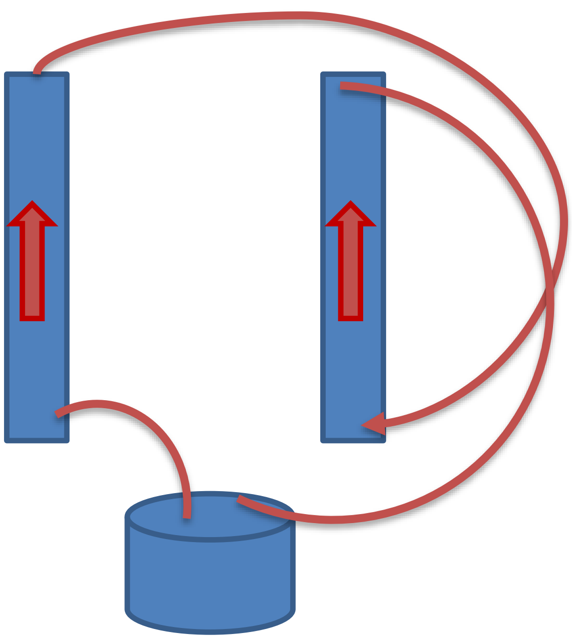


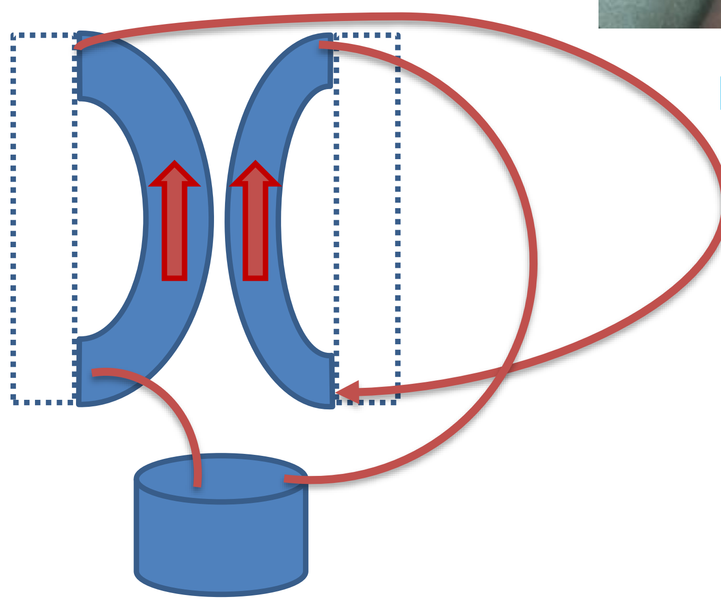


Lorentz Force Law

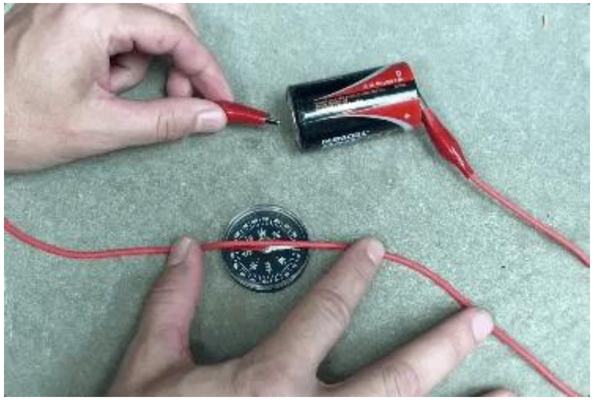




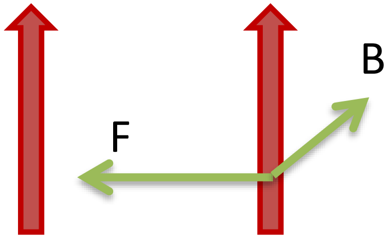




Anti-parallel current repels wire



Parallel current attracts wire



Lorentz Law: Force on Current Carrying Conductor

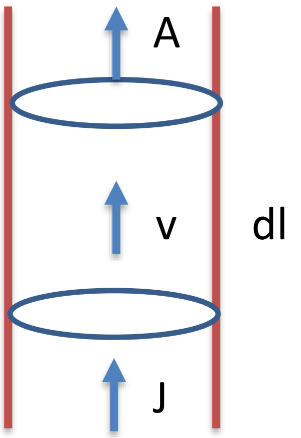
Force exerted by a current element **δl** carrying current I situated in magnetic field induction **B** is **δF$_m$**. For whole length each element is at right angels at B. The magnetic force on the conductor is

$$\boldsymbol{F_m} = \sum \boldsymbol{\delta F_m} = \sum I\delta\boldsymbol{l}\ X\ \boldsymbol{B} = IlB\sin\theta\ \hat{n} = IlB\ \hat{n}$$

Consider a conductor of cross-sectional area A carrying current I and current density **J**, with each carrier moving with velocity **v** placed in magnetic field **B**. If n is number of charges, total current flowing in conductor is

$$\boldsymbol{I} = \boldsymbol{J}.\boldsymbol{A} = nq\boldsymbol{v}.\boldsymbol{A}$$

$$\boldsymbol{\delta F_m} = (nq\boldsymbol{v}.\boldsymbol{A})\delta\boldsymbol{l}\ X\ \boldsymbol{B} = (nq\delta\boldsymbol{l}.\boldsymbol{A})\ \boldsymbol{v}\ X\ \boldsymbol{B}$$



Lorentz Law: Force on Current Carrying Conductor

The magnetic force known as Lorentz force by single carrier is

$$\boldsymbol{F_m} = \frac{\boldsymbol{\delta F_m}}{(n\delta \boldsymbol{l}.\boldsymbol{A})} = \frac{(nq\delta \boldsymbol{l}.\boldsymbol{A})\ \boldsymbol{v}\ X\ \boldsymbol{B}}{(n\delta \boldsymbol{l}.\boldsymbol{A})} \qquad \boldsymbol{F_m} = q(\boldsymbol{v}\ X\ \boldsymbol{B})$$

In the presence of both electric and magnetic field is

$$\boldsymbol{F} = \boldsymbol{F_e} + \boldsymbol{F_m} = q[\boldsymbol{E} + (\boldsymbol{v}\ X\ \boldsymbol{B})]$$

If Q moves through a distance dl and the work done by magnetic force is

$$\boldsymbol{dW_{mag}} = \boldsymbol{F_m} d\boldsymbol{l} = q(\boldsymbol{v}\ X\ \boldsymbol{B}).\boldsymbol{v}dt$$

As **v** x **B** is perpendicular to v, here (**v** x **B**)=0

$$\boldsymbol{dW_{mag}} = 0$$

Magnetic force do no work. It just alters the direction in which a particle moves, but never stop or speed them.

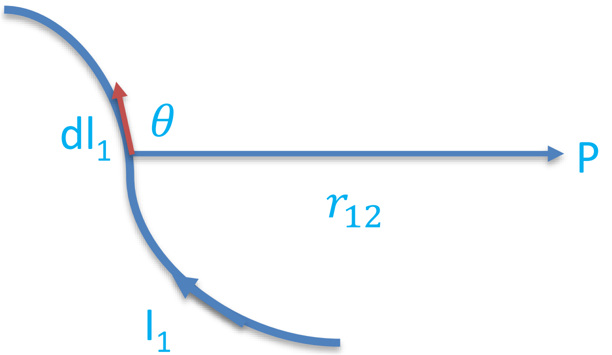
Biot-Savart Law

In reality as moving charges do not create steady current, in magnetostatics, it is assumed that ${}^{\partial\rho}/_{\partial t} = 0$, with equation of continuity as $\boldsymbol{\nabla}.\boldsymbol{J} = 0$

In electrostatics, the force exerted by a charge configuration upon charge Q is **F**=Q**E**. Similarly in magnetostatics, the force exerted is

$$d\boldsymbol{F}_{12} = I_2 d\boldsymbol{l}_2 X d\boldsymbol{B}_1 \qquad d\boldsymbol{B}_1 = \frac{\mu}{4\pi}\frac{I_1 d\boldsymbol{l}_1 X \boldsymbol{r}_{12}}{r_{12}^3}$$

d$\mathbf{B_1}$ is the magnetic induction or magnetic flux density which represents the magnetic field produced by current element d$\mathbf{l_1}$. This is Biot-Savart's law.



$$d\boldsymbol{B}_1 = \frac{\mu}{4\pi}\frac{I_1 d\boldsymbol{l}_1 \sin\theta}{r_{12}^2} \; W\,b/m^2$$

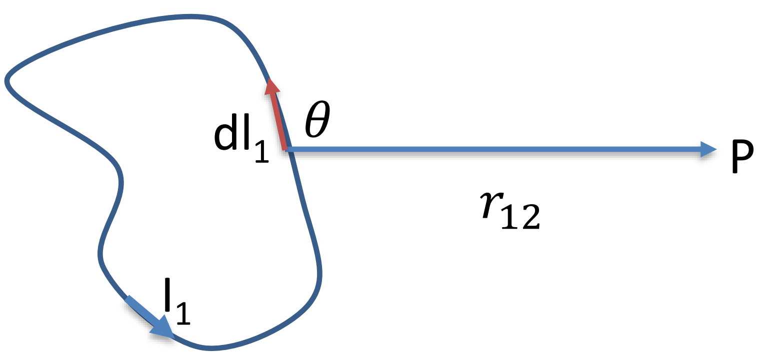
Biot-Savart Law

The resultant magnetic induction at P is

$$\boldsymbol{B}_1 = \int d\boldsymbol{B}_1 = \frac{\mu}{4\pi}\int \frac{I_1 d\boldsymbol{l}_1 X \boldsymbol{r}_{12}}{r_{12}^3}$$

For a closed loop, the resultant magnetic induction at P is

$$\boldsymbol{B}_1 = \frac{\mu}{4\pi}\oint \frac{I_1 d\boldsymbol{l}_1 X \boldsymbol{r}_{12}}{r_{12}^3}$$



If J_1 is current density,

$$I_1 d\boldsymbol{l}_1 = \boldsymbol{J}_1 dv$$

$$\boldsymbol{B}_1 = \frac{\mu}{4\pi}\oint \frac{\boldsymbol{J}_1 X \boldsymbol{r}_{12}}{r_{12}^3} dv$$

Thus Biot-Savart law is convenient to evaluate magnetic induction.

Ampere's Law of Force Between Current Elements

Ampere proposed a general statement to find the force between current elements by performing a series of experiment.

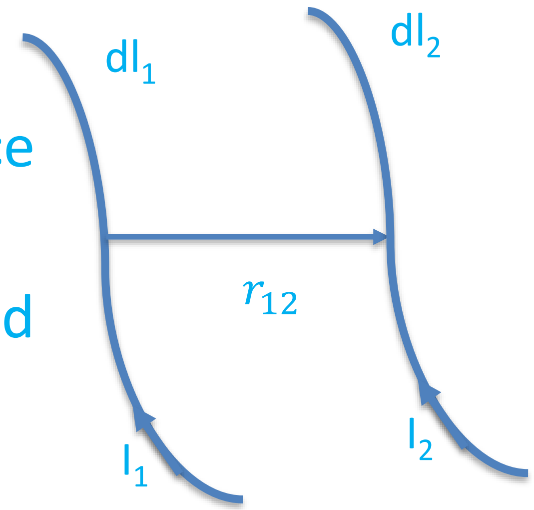
Consider two current elements dl_1 and dl_2 carrying steady currents I_1 and I_2. The force between two current elements is

i) directly proportional to magnitude of current

ii) inversely proportional to square of distance between current elements

iii) directly proportional to the length and orientation of two current elements

iv) nature of the medium



Force exerted by dl_1 upon dl_2 is

$$d\boldsymbol{F}_{12} = C(I_1 I_2)\left(\frac{1}{r_{12}^2}\right)\left[d\boldsymbol{l}_2 X d\boldsymbol{l}_1 X \frac{\boldsymbol{r}_{12}}{r_{12}}\right] \qquad d\boldsymbol{F}_{12} = \frac{\mu}{4\pi}\left(\frac{(I_2 d\boldsymbol{l}_2)X(I_1 d\boldsymbol{l}_1 X \boldsymbol{r}_{12})}{r_{12}^3}\right)$$

Here $\frac{\boldsymbol{r}_{12}}{r_{12}}$ represents unit vector along r_{12}

Ampere's Law of Force Between Current Elements

Force exerted by dl_2 upon dl_1 is

$$d\boldsymbol{F}_{21} = \frac{\mu}{4\pi}\left(\frac{(I_1 d\boldsymbol{l}_1)X(I_2 d\boldsymbol{l}_2 X \boldsymbol{r}_{21})}{r_{21}^3}\right)$$

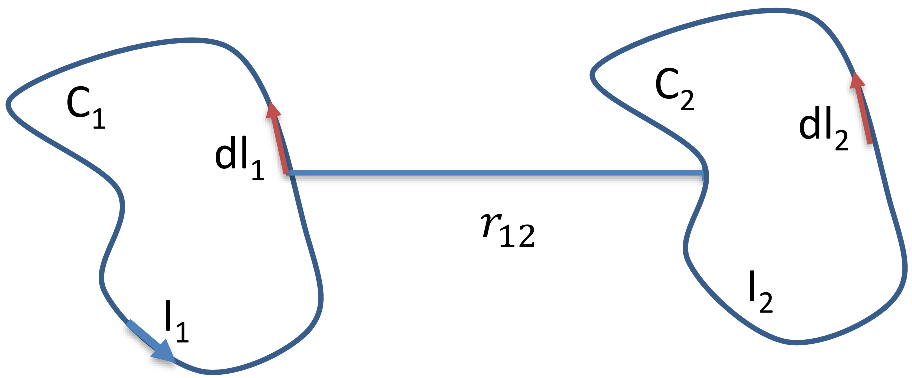
The above two equations are known as Ampere's law of force between two current elements. It resembles Coulomb's law

This equations appear to contradict Newton's III law, as I_1dl_1 and I_2dl_2 are not symmetric, $d\boldsymbol{F}_{21} \neq d\boldsymbol{F}_{21}$

To overcome the contradiction, consider closed loops C_1 and C_2 instead of current element. Force exerted by C_1 upon C_2 and C_2 upon C_1 is

$$\boldsymbol{F}_{12} = \frac{\mu}{4\pi}\oint_{C1}\oint_{C2}\frac{(I_2 d\boldsymbol{l}_2)X(I_1 d\boldsymbol{l}_1 X \boldsymbol{r}_{12})}{r_{12}^3}$$

$$\boldsymbol{F}_{21} = \frac{\mu}{4\pi}\oint_{C1}\oint_{C2}\frac{(I_1 d\boldsymbol{l}_1)X(I_2 d\boldsymbol{l}_2 X \boldsymbol{r}_{21})}{r_{21}^3}$$



Ampere's Law of Force Between Current Loops

Using vector triple product identity, $\boldsymbol{a}\ X\ (\boldsymbol{b}\ X\ \boldsymbol{c}) = (\boldsymbol{a}.\boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a}.\boldsymbol{b})\boldsymbol{c}$

$$\boldsymbol{F}_{12} = \frac{\mu}{4\pi}\oint_{C1}\oint_{C2}\frac{I_1 I_2}{r_{12}^3}[(d\boldsymbol{l}_2.\boldsymbol{r}_{12})d\boldsymbol{l}_1 - (d\boldsymbol{l}_2.d\boldsymbol{l}_1)\boldsymbol{r}_{12}]$$

$$\frac{\mu I_1 I_2}{4\pi}\oint_{C1}\oint_{C2}\frac{(d\boldsymbol{l}_2.\boldsymbol{r}_{12})d\boldsymbol{l}_1}{r_{12}^3} = \frac{-\mu I_1 I_2}{4\pi}\oint_{C1}d\boldsymbol{l}_1\oint_{C2}\nabla_2\left(\frac{1}{r_{12}}\right)d\boldsymbol{l}_2$$

$$\mathrm{d}_2\left(\frac{1}{r_{12}}\right) = \frac{\partial}{\partial x_2}\left(\frac{1}{r_{12}}\right)dx_2 + \frac{\partial}{\partial y_2}\left(\frac{1}{r_{12}}\right)dy_2 + \frac{\partial}{\partial z_2}\left(\frac{1}{r_{12}}\right)dz_2 = \nabla_2\left(\frac{1}{r_{12}}\right)d\boldsymbol{l}_2$$

$$\frac{\mu I_1 I_2}{4\pi}\oint_{C1}\oint_{C2}\frac{(d\boldsymbol{l}_2.\boldsymbol{r}_{12})d\boldsymbol{l}_1}{r_{12}^3} = \frac{-\mu I_1 I_2}{4\pi}\oint_{C1}d\boldsymbol{l}_1\oint_{C2}\mathrm{d}_2\left(\frac{1}{r_{12}}\right) = 0$$

$$\boldsymbol{F}_{12} = \frac{-\mu I_1 I_2}{4\pi}\oint_{C1}\oint_{C2}\frac{(d\boldsymbol{l}_2.d\boldsymbol{l}_1)\boldsymbol{r}_{12}}{r_{12}^3} \qquad \boldsymbol{F}_{21} = \frac{-\mu I_1 I_2}{4\pi}\oint_{C1}\oint_{C2}\frac{(d\boldsymbol{l}_2.d\boldsymbol{l}_1)\boldsymbol{r}_{21}}{r_{21}^3}$$

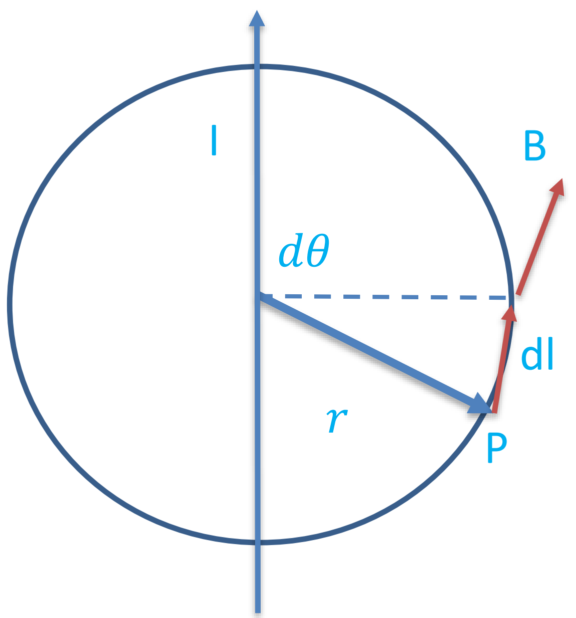
$$\boldsymbol{r}_{12} = -\boldsymbol{r}_{21} \qquad \boldsymbol{F}_{21} = -\boldsymbol{F}_{12}$$

Thus Ampere's law for current loop is considered instead of current elements

Ampere’s Law in Circuital Form

Consider a long wire carrying current I, produces a magnetic field induction **B**. The magnetic induction at any point P on the circular path of radius r is

$$\boldsymbol{B} = \frac{\mu I}{2\pi r}$$



At every point B is same and is parallel to the tangent of circular path. The line integral of the magnetic induction around the circular path is

$$\oint \boldsymbol{B}.d\boldsymbol{l} = \oint \frac{\mu I}{2\pi}\frac{r\,d\theta}{r} == \oint \frac{\mu I}{2\pi} d\theta = \frac{\mu I}{2\pi} 2\pi$$

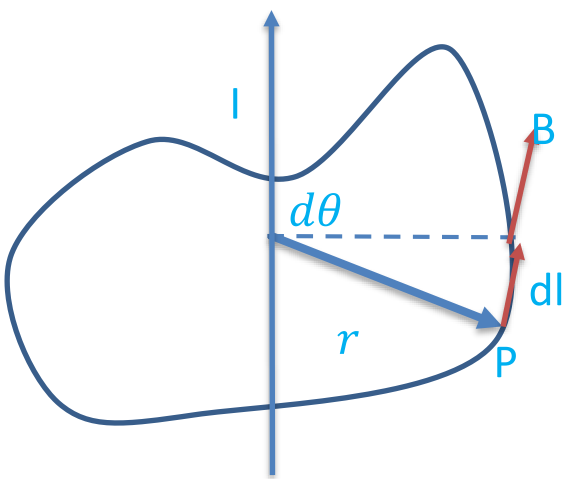
$$\oint \boldsymbol{B}.d\boldsymbol{l} = \mu I$$

The sign of the integral depends on the direction of circle encircled. The sign is positive, if path of line integral is parallel to **B** and is negative, if path of line integral is anti-parallel to **B**

Ampere's Law in Circuital Form

If the path enclosing current is not circular but irregular, then it can be divided into small elements of length dl, then for a small element,

$$\boldsymbol{B}.d\boldsymbol{l} = \frac{\mu_0 I}{2\pi}\frac{R\,d\theta}{R} = \frac{\mu_0 I}{2\pi}d\theta$$

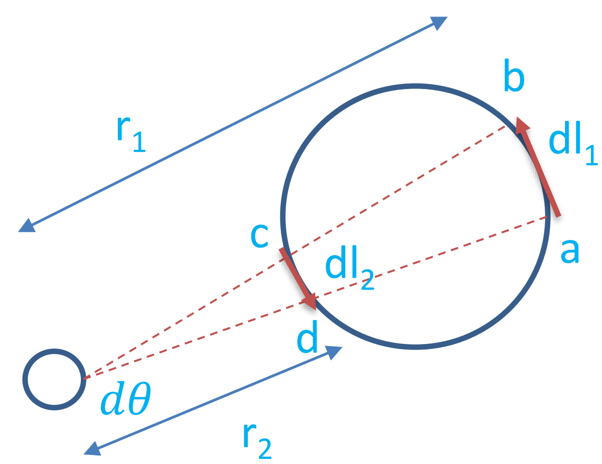


The line integral of the magnetic induction along the whole path is

$$\oint \boldsymbol{B}.d\boldsymbol{l} = \frac{\mu_0 I}{2\pi}\oint d\theta = \frac{\mu_0 I}{2\pi}2\pi = \mu_0 I$$

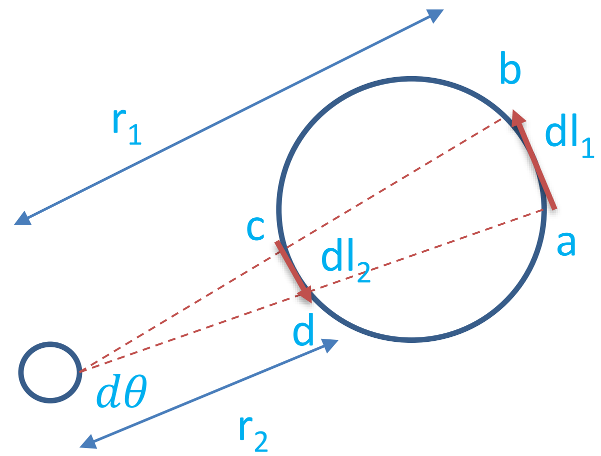
This shows that Ampere's law holds good for closed path of any shape.

When the closed path does not enclose current, consider a current I and neighbouring closed path which does not enclose any current. The path is divided into large number of small elements. Draw two lines OA, OB making an angle dθ cutting the arcs ab and cd of the path.



Ampere's Law in Circuital Form

The line integral of the magnetic induction along the path abcd is



$$\oint_{abcd} \boldsymbol{B}.d\boldsymbol{l} = \oint_{ab} \boldsymbol{B}.d\boldsymbol{l} + \oint_{bc} \boldsymbol{B}.d\boldsymbol{l} + \oint_{cd} \boldsymbol{B}.d\boldsymbol{l} + \oint_{da} \boldsymbol{B}.d\boldsymbol{l}$$

The angle between dl and B is 90°. So

$$\oint_{bc} \boldsymbol{B}.d\boldsymbol{l} = \oint_{da} \boldsymbol{B}.d\boldsymbol{l} = 0$$

$$\oint_{abcd} \boldsymbol{B}.d\boldsymbol{l} = \oint_{ab} \boldsymbol{B}.d\boldsymbol{l} + \oint_{cd} \boldsymbol{B}.d\boldsymbol{l} == \oint_{ab} \boldsymbol{B}_1.d\boldsymbol{l}_1 \cos\theta_1 + \oint_{cd} \boldsymbol{B}_2.d\boldsymbol{l}_2 \cos\theta_2$$

$$\oint_{abcd} \boldsymbol{B}.d\boldsymbol{l} = \oint_{ab} \frac{\mu_0 I}{2\pi r_1} d\boldsymbol{l}_1 - \oint_{cd} \frac{\mu_0 I}{2\pi r_2} d\boldsymbol{l}_2$$

Ampere's Law in Circuital Form

$$\oint_{abcd} \boldsymbol{B}.d\boldsymbol{l} = \frac{\mu_0 I}{2\pi}\left[\frac{ab}{r_1} - \frac{cd}{r_2}\right]$$

$$\text{But,} \frac{ab}{r_1} = \frac{cd}{r_2} = d\theta$$

$$\oint \boldsymbol{B}.d\boldsymbol{l} = 0$$

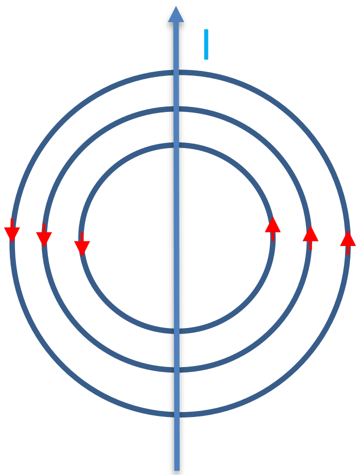
Thus Ampere's law in circuital form is summarized as the line integral of the magnetic induction around any closed path is equal to (i) permeability multiplied by the current enclosed by the path and (ii) zero, if the path does not enclose any current.

$$\oint \boldsymbol{B}.d\boldsymbol{l} = \begin{cases} \mu_0 I, if\ path\ encloses\ current\ I \\ 0, if\ path\ doesnot\ enclose\ current \end{cases}$$

Divergence of Magnetic Induction

Biot-Savart's law states that the line representing the direction of field d**B** are circles about the axis of current element and thus form closed curves. These lines do not start at(diverge from) any point nor do they stop (converge toward) any point. Thus the fixed d**B** is source free or solenoidal, that is divergence is zero.

$$d\boldsymbol{B} = \frac{\mu}{4\pi}\frac{Id\boldsymbol{l}\ X\ \boldsymbol{r}}{r^3}$$



$$\boldsymbol{B} = \frac{\mu}{4\pi}\oint\frac{Id\boldsymbol{l}\ X\ \boldsymbol{r}}{r^3}$$

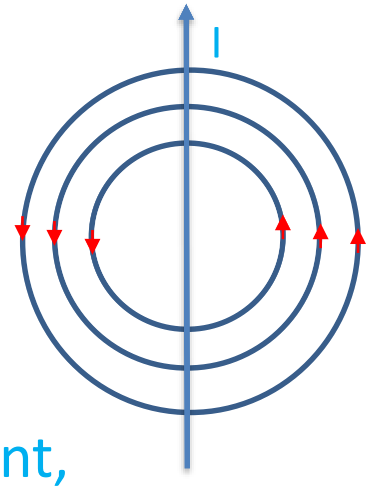
$$\nabla.\boldsymbol{B} = \frac{\mu I}{4\pi}\nabla.\oint\frac{d\boldsymbol{l}\ X\ \boldsymbol{r}}{r^3}$$

Since differentiation and integration are interchangeable,

$$\nabla.\boldsymbol{B} = \frac{\mu I}{4\pi}\oint\nabla.\left(\frac{d\boldsymbol{l}\ X\ \boldsymbol{r}}{r^3}\right)$$

Divergence of Magnetic Induction

Using the vector identity, $\nabla.(\boldsymbol{a}\ X\ \boldsymbol{b}) = \boldsymbol{b}.\nabla\ X\ \boldsymbol{a} - \boldsymbol{a}.\nabla\ X\ \boldsymbol{b}$



$$\nabla.\left(d\boldsymbol{l}\ X\frac{\boldsymbol{r}}{r^3}\right) = \frac{\boldsymbol{r}}{r^3}.\nabla\ X\ d\boldsymbol{l} - d\boldsymbol{l}.\nabla\ X\ \frac{\boldsymbol{r}}{r^3}$$

$$\nabla.\boldsymbol{B} = \frac{\mu I}{4\pi}\oint\left(\frac{\boldsymbol{r}}{r^3}.\nabla\ X\ d\boldsymbol{l} - d\boldsymbol{l}.\nabla\ X\ \frac{\boldsymbol{r}}{r^3}\right)$$

Since d**l** is not a function of co-ordinate (x,y,z) of field point,

$$\boldsymbol{\nabla}\ X\ d\boldsymbol{l} = 0$$

Here the curl of electric field due to a point charge,

$$\nabla\ \mathrm{X}\ \boldsymbol{E} = \nabla\ \mathrm{X}\left(\frac{1}{4\pi\varepsilon_0}\frac{q\boldsymbol{r}}{r^3}\right) = \frac{1}{4\pi\varepsilon_0}\nabla\ \mathrm{X}\left(\frac{q\boldsymbol{r}}{r^3}\right) = 0$$

$$\boldsymbol{\nabla}.\boldsymbol{B} = \boldsymbol{0}$$

Thus the divergence of magnetic induction is always zero.

Curl of Magnetic Induction

From Ampere's law in circuital form,

$$\oint \boldsymbol{B}.d\boldsymbol{l} = \mu(sum\ of\ currents\ enclosed\ by\ path) = \mu I$$

If **J** is current density, then the total current **I** through S is

$$I = \iint \boldsymbol{J}.d\boldsymbol{a} \qquad \oint \boldsymbol{B}.d\boldsymbol{l} = \mu \iint \boldsymbol{J}.d\boldsymbol{a}$$

From Stoke's theorem,

$$\oint \boldsymbol{B}.d\boldsymbol{l} = \iint \nabla \text{ X } \boldsymbol{B}.d\boldsymbol{a}$$

$$\iint \nabla \text{ X } \boldsymbol{B}.d\boldsymbol{a} = \mu \iint \boldsymbol{J}.d\boldsymbol{a} \qquad \iint (\nabla \text{ X } \boldsymbol{B} - \mu \boldsymbol{J}).d\boldsymbol{a} = 0$$

Since surface S arbitrary, $\nabla \text{ X } \boldsymbol{B} - \mu \boldsymbol{J} = 0$ $\qquad \nabla \text{ X } \boldsymbol{B} = \mu \boldsymbol{J}$

In free space, $\nabla \text{ X } \boldsymbol{B} = \mu_0 \boldsymbol{J}$

For steady flow of charge with charge density ρ having velocity **v** is

$$\nabla \text{ X } \boldsymbol{B} = \mu \rho \boldsymbol{v}$$

Magnetic Vector Potential

In electrostatics, $\nabla \times \boldsymbol{E} = 0$ permitted to introduce a scalar potential, $\mathbf{E} = -\nabla V$

In magnetostatics, $\nabla . \boldsymbol{B} = 0$ permitted to introduce a vector potential, $\mathbf{B} = \nabla \times \boldsymbol{A}$

If line integrals encircling any currents are considered, magnetic scalar potential cannot be used and hence B can not be derived from scalar magnetic potential. By Biot-Savart law, the magnetic induction due to a current element is

$$d\boldsymbol{B} = \frac{\mu I}{4\pi} \frac{d\boldsymbol{l} \, X \, \boldsymbol{r}}{r^3}$$

$$\nabla\left(\frac{1}{r}\right) = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left[\frac{1}{(x^2+y^2+z^2)^{1/2}}\right] \qquad \nabla\left(\frac{1}{r}\right) = \frac{-(x\hat{\imath} + y\hat{\jmath} + z\hat{k})}{r^3} = \frac{-\boldsymbol{r}}{r^3}$$

$$d\boldsymbol{B} = \frac{\mu I}{4\pi} d\boldsymbol{l} \, X \left[-\nabla\left(\frac{1}{r}\right)\right] = \frac{-\mu I}{4\pi}\left[\nabla\left(\frac{1}{r}\right) X \, d\boldsymbol{l}\right]$$

Magnetic Vector Potential

Using vector identity,

$$\nabla X(\emptyset \boldsymbol{A}) = \nabla\emptyset\ X\ \boldsymbol{A} + \emptyset\nabla\ X\ \boldsymbol{A}$$

$$\nabla\emptyset\ X\ \boldsymbol{A} = \nabla\ X(\emptyset\boldsymbol{A}) - \emptyset\nabla\ X\ \boldsymbol{A}$$

$$\nabla\left(\frac{1}{r}\right) X\ d\boldsymbol{l} = \nabla\ X\left(\left(\frac{1}{r}\right)d\boldsymbol{l}\right) - \left(\frac{1}{r}\right)\nabla\ X d\boldsymbol{l}$$

$$d\boldsymbol{B} = \frac{\mu I}{4\pi}\left[\nabla\ X\left(\frac{d\boldsymbol{l}}{r}\right) - \left(\frac{1}{r}\right)\nabla\ X d\boldsymbol{l}\right]$$

As $d\boldsymbol{l}$ is not a function of co-ordinate (x,y,z), $\nabla\ X d\boldsymbol{l} = 0$

$$d\boldsymbol{B} = \frac{\mu I}{4\pi}\left[\nabla\ X\left(\frac{d\boldsymbol{l}}{r}\right)\right]$$

Total magnetic induction at any given point by a closed current loop is

$$\boldsymbol{B} = \frac{\mu I}{4\pi}\oint\left[\nabla\ X\left(\frac{d\boldsymbol{l}}{r}\right)\right]$$

Magnetic Vector Potential

$$\boldsymbol{B} = \nabla \; X \left[\frac{\mu I}{4\pi} \oint \frac{d\boldsymbol{l}}{r} \right]$$

The quantity within the square bracket is a vector. By taking its curl, the magnetic induction produced at any point by a closed loop carrying current, the vector is obtained. This vector is known as magnetic vector potential (**A**).

$$\boldsymbol{B} = \text{curl}\;\mathbf{A} = \nabla \; X \; \boldsymbol{A}$$

$$\boldsymbol{A} = \frac{\mu I}{4\pi} \oint \frac{d\boldsymbol{l}}{r}$$

Using current density $\boldsymbol{J}\; d\boldsymbol{v} = \boldsymbol{I}\; d\boldsymbol{l}$, the magnetic vector potential is

$$\boldsymbol{A} = \frac{\mu}{4\pi} \oint \frac{\boldsymbol{J}(r)\; d\boldsymbol{v}}{r}$$

If the current element is at r’ instead of origin, then

$$\boldsymbol{A} = \frac{\mu}{4\pi} \oint \frac{\boldsymbol{J}(r')\; d\boldsymbol{v}'}{r - r'}$$

Characteristics of Magnetic Vector Potential

(i) It satisfies Poissson's equation

$$\nabla \, X \, \boldsymbol{B} = \mu_0 \boldsymbol{J}$$

$$\boldsymbol{B} = \nabla \, X \, \boldsymbol{A}$$

$$\nabla \, X \, (\nabla \, X \, \boldsymbol{A}) = \mu_0 \boldsymbol{J}$$

$$\nabla \, (\nabla . \boldsymbol{A}) - \nabla^2 \boldsymbol{A} \; = \mu_0 \boldsymbol{J}$$

$$\text{Since } \nabla . \boldsymbol{A} = 0, \nabla^2 \boldsymbol{A} \; = -\mu_0 \boldsymbol{J}$$

(ii) Scalar magnetic potential is

$$\emptyset_m \; = \oint \boldsymbol{A} . d\boldsymbol{l}$$

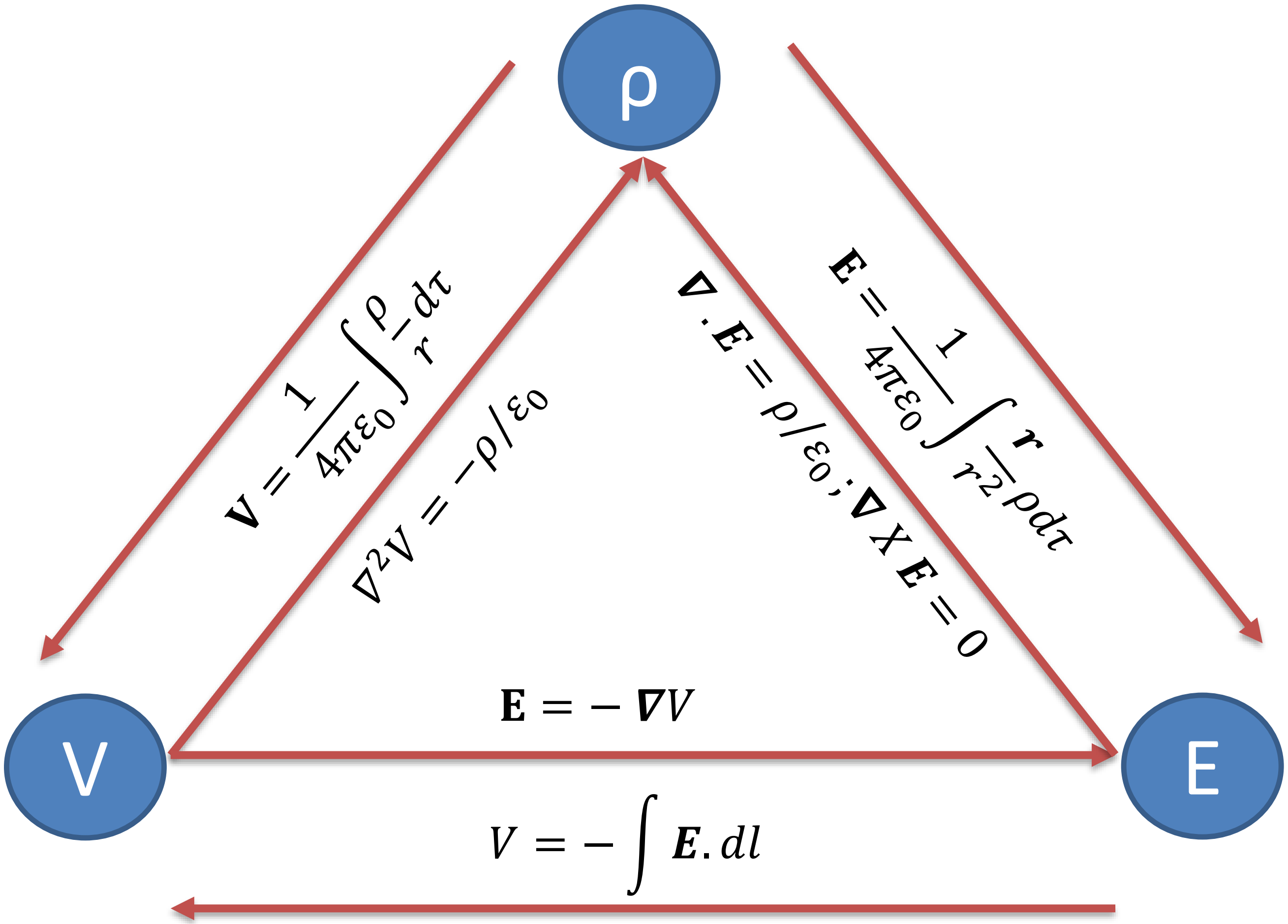
By definition,

$$\emptyset_m \; = \oint \boldsymbol{B} . d\boldsymbol{S}$$

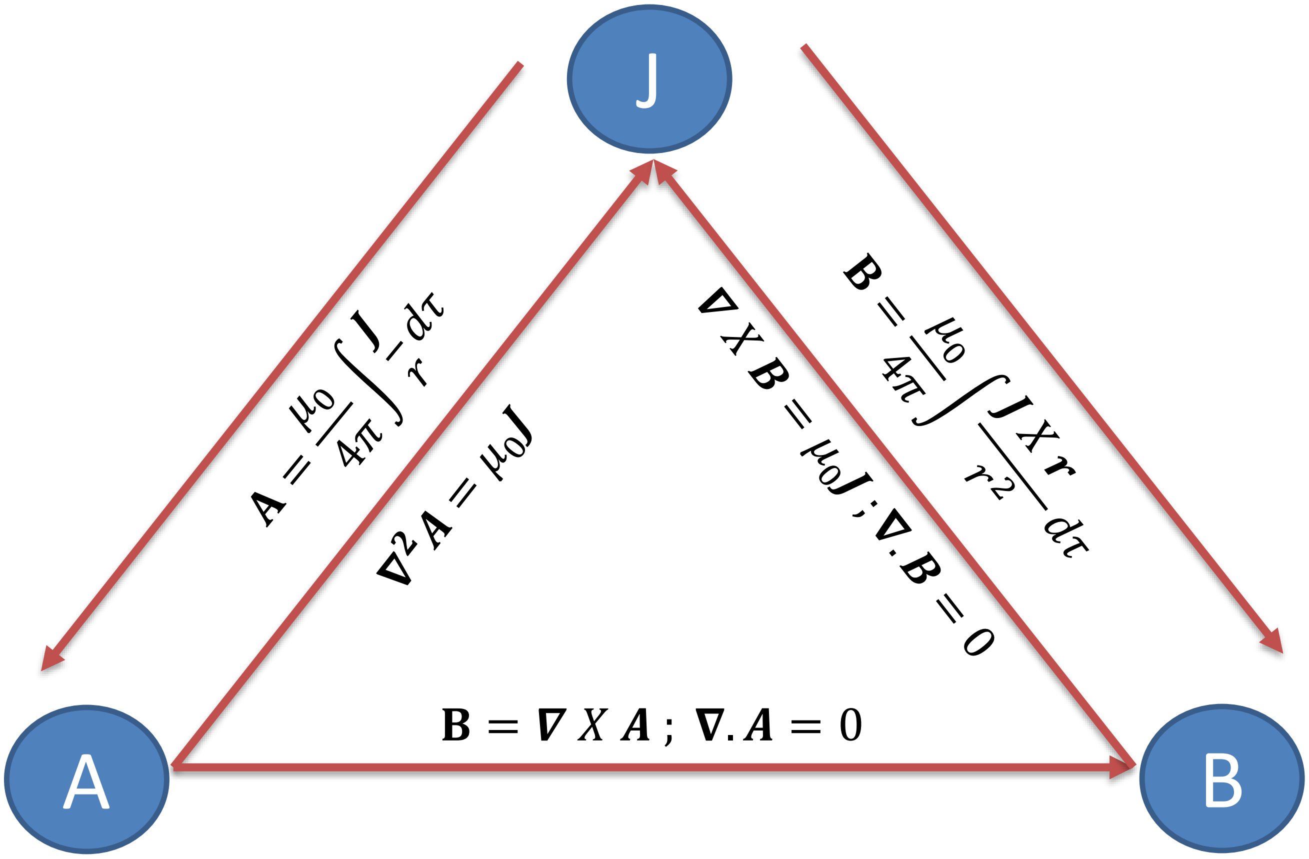
$$\emptyset_m \; = \oint (\nabla \; X \; \boldsymbol{A}) . d\boldsymbol{S}$$

Using Stoke's theorem, $\emptyset_m \; = \oint \boldsymbol{A} . d\boldsymbol{l}$

Electrostatics Boundary Conditions



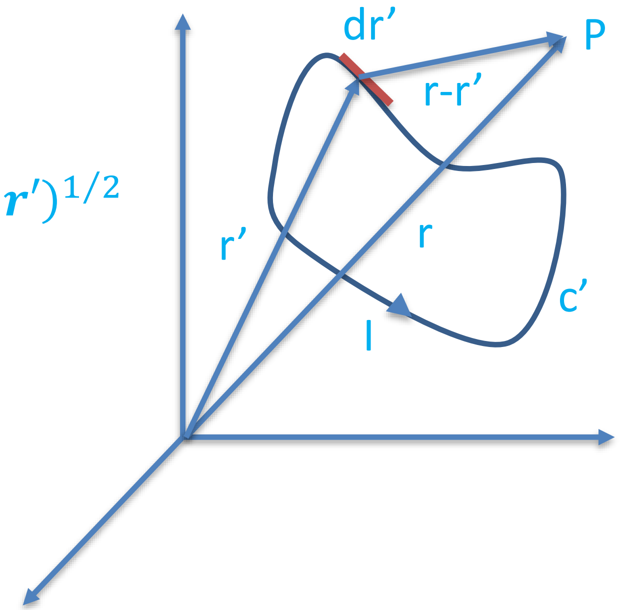
Magnetostatics Boundary Conditions



Magnetic Field of a Distant Current Loop

Consider a current loop carrying current **I**. Let **I.dr'** be the current element having position vector **r'** with respect to origin. Then the vector potential at a distance P having position vector **r** (r>>r')

$$\boldsymbol{A} = \frac{\mu_0}{4\pi}\oint \frac{\boldsymbol{I}\, d\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|}$$



$$|\boldsymbol{r}-\boldsymbol{r}'| = (r^2 + r'^2 - 2rr'\cos\theta)^{1/2} = (r^2 + r'^2 - 2\boldsymbol{r}.\boldsymbol{r}')^{1/2}$$

$$|\boldsymbol{r}-\boldsymbol{r}'| = r\left(1 + \frac{r'^2}{r^2} - \frac{2\boldsymbol{r}.\boldsymbol{r}'}{r^2}\right)^{1/2}$$

$$\boldsymbol{A} = \frac{\mu_0 \boldsymbol{I}}{4\pi r}\oint \frac{d\boldsymbol{r}'}{\left(1 + \frac{r'^2}{r^2} - \frac{2\boldsymbol{r}.\boldsymbol{r}'}{r^2}\right)^{1/2}}$$

Applying Binomial expression and keeping only first order terms,

$$\boldsymbol{A} = \frac{\mu_0 \boldsymbol{I}}{4\pi r}\oint d\boldsymbol{r}'\left(1 + \frac{\boldsymbol{r}.\boldsymbol{r}'}{r^2} + \cdots\right)$$

Magnetic Field of a Distant Current Loop

$$\boldsymbol{A} = \frac{\mu_0 \boldsymbol{I}}{4\pi r}\left[\oint d\boldsymbol{r}' + \oint \left(\frac{\boldsymbol{r}.\boldsymbol{r}'}{r^2}\right) d\boldsymbol{r}'\right]$$

For an arbitrary closed loop C', $\oint d\boldsymbol{r}' = 0$

$$\boldsymbol{A} = \frac{\mu_0 \boldsymbol{I}}{4\pi r}\left[\oint \left(\frac{\boldsymbol{r}.\boldsymbol{r}'}{r^2}\right) d\boldsymbol{r}'\right] = \frac{\mu_0 \boldsymbol{I}}{4\pi r^3}\left[\oint (\boldsymbol{r}.\boldsymbol{r}')d\boldsymbol{r}'\right]$$

Using vector triple product, $\boldsymbol{r}\ X\ (d\boldsymbol{r}'\ X\ \boldsymbol{r}') = (\boldsymbol{r}.\boldsymbol{r}')d\boldsymbol{r}' - (\boldsymbol{r}.d\boldsymbol{r}')\boldsymbol{r}'$

Differentiating, $d[\boldsymbol{r}'(\boldsymbol{r}.\boldsymbol{r}')] = d\boldsymbol{r}'(\boldsymbol{r}.\boldsymbol{r}') + \boldsymbol{r}'(\boldsymbol{r}.d\boldsymbol{r}')$

Adding above two equations, $\boldsymbol{r}\ X\ (d\boldsymbol{r}'\ X\ \boldsymbol{r}') + d[\boldsymbol{r}'(\boldsymbol{r}.\boldsymbol{r}')] = 2(\boldsymbol{r}.\boldsymbol{r}')d\boldsymbol{r}'$

$$(\boldsymbol{r}.\boldsymbol{r}')d\boldsymbol{r}' = \frac{1}{2}\boldsymbol{r}\ X\ (d\boldsymbol{r}'\ X\ \boldsymbol{r}') + \frac{1}{2}\ d[\boldsymbol{r}'(\boldsymbol{r}.\boldsymbol{r}')]$$

$$\boldsymbol{A} = \frac{\mu_0 \boldsymbol{I}}{4\pi r^3}\left[\oint \frac{1}{2}\boldsymbol{r}\ X\ (d\boldsymbol{r}'\ X\ \boldsymbol{r}') + \oint \frac{1}{2}\ d[\boldsymbol{r}'(\boldsymbol{r}.\boldsymbol{r}')]\right]$$

Magnetic Field of a Distant Current Loop

Since the integrand in the second integral is perfect differential, its integral around a closed loop is zero.

$$\boldsymbol{A} = \frac{\mu_0 \boldsymbol{I}}{4\pi r^3}\left[\oint \frac{1}{2}\boldsymbol{r}\ X\ (d\boldsymbol{r}'\ X\ \boldsymbol{r}')\right]$$

$$\boldsymbol{A} = \frac{\mu_0}{4\pi}\left[\frac{1}{2}\oint \boldsymbol{r}' X\ (\boldsymbol{I}\ d\boldsymbol{r}')X\ \frac{\boldsymbol{r}}{r^3}\right]$$

Introducing a new quantity known as magnetic moment **m**,

$$\boldsymbol{m} = \frac{1}{2}\oint [\boldsymbol{r}' X\ (\boldsymbol{I}\ d\boldsymbol{r}')]$$

$$\boldsymbol{A} = \frac{\mu_0}{4\pi}\left[\frac{\boldsymbol{m}\ X\ \boldsymbol{r}}{r^3}\right]$$

Thus the magnetic induction is,

$$\boldsymbol{B} = curl\ \boldsymbol{A}(r) = \nabla\ X\left\{\frac{\mu_0}{4\pi}\left[\frac{\boldsymbol{m}\ X\ \boldsymbol{r}}{r^3}\right]\right\}$$

Magnetic Field of a Distant Current Loop

$$\boldsymbol{B} = \frac{\mu_0}{4\pi}\left[-(\boldsymbol{m}.\nabla)\frac{\boldsymbol{r}}{r^3} + \boldsymbol{m}\,\nabla.\left(\frac{\boldsymbol{r}}{r^3}\right)\right]$$

$$\nabla.\left(\frac{\boldsymbol{r}}{r^3}\right) = 0 \qquad\qquad m_x\frac{\partial}{\partial x}\left(\frac{\boldsymbol{r}}{r^3}\right) = \frac{m_x\boldsymbol{i}}{r^3} - 3m_x x\frac{\boldsymbol{r}}{r^5}$$

$$\boldsymbol{m}\,\nabla.\left(\frac{\boldsymbol{r}}{r^3}\right) = \frac{\boldsymbol{m}}{r^3} - \frac{3(\boldsymbol{m}.\boldsymbol{r})\boldsymbol{r}}{r^5}$$

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\left[\frac{3(\boldsymbol{m}.\boldsymbol{r})\boldsymbol{r}}{r^5} - \frac{\boldsymbol{m}}{r^3}\right]$$

This equation indicates that the magnetic field of a distant current circuit does not depend upon its detailed geometry, but only on the magnetic moment of the circuit, **m**.

The field is same as the electric field of dipole and hence the field is called as magnetic dipole field.

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\boldsymbol{\pi}\boldsymbol{\varepsilon}_0}\left[\left(\frac{3(\boldsymbol{P}.\boldsymbol{r})\boldsymbol{r}}{r^5}\right) - \frac{\boldsymbol{P}}{r^3}\right]$$

Books for Reference

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3. **R. P. Feynman, R. B. Leighton and M. Sands**, *The Feynman Lectures on Physics: Vol. II (*Narosa Book Distributors, New Delhi, 1989)
4. **Satya Prakash**, *Electromagnetic Theory and Electrodynamics* (Kedar Nath Ram Nath, Meerut, 2015)