Bharathidasan University

Tiruchirappalli – 620 024, Tamil Nadu, India

Programme: M. Sc., Physics

Course Title : Electromagnetic Theory

Course Code : 22PH301

Unit I

Perspectives of Electrostatics

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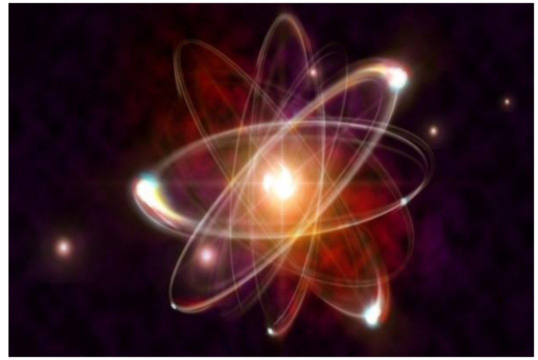
Roadmap of Electrodynamics

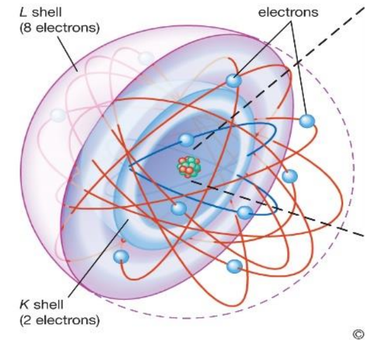
Newton Gravitational Law

Universe



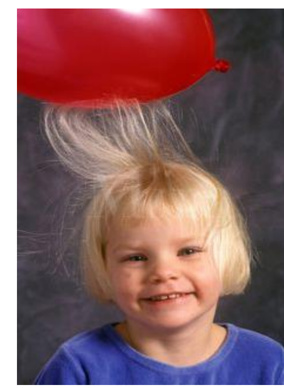
Atom

Coulombs Law

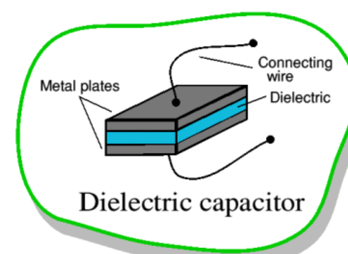


Electrons

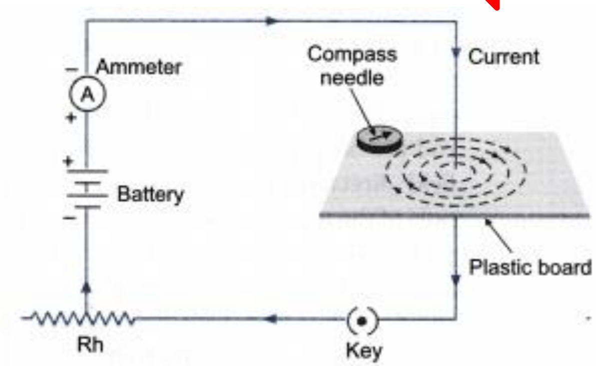
At Rest



Electrostatics



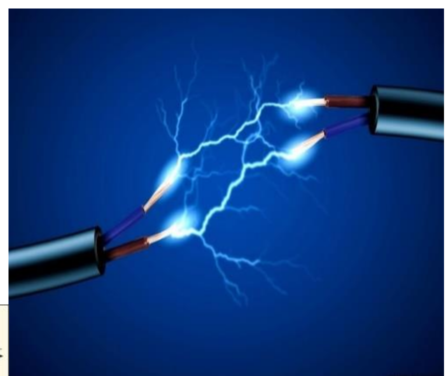
Steady Current



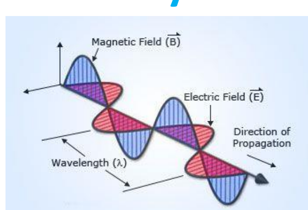
Magnetostatics

Biot Savart Law

Unsteady Current

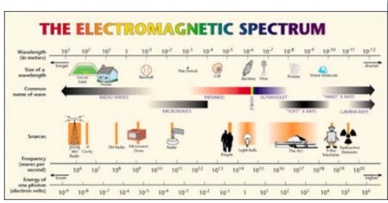


Electrodynamics



Maxwell's Equation

Optics



Relativistic Electrodynamics

What is Electromagnetic Theory?

Electrostatics

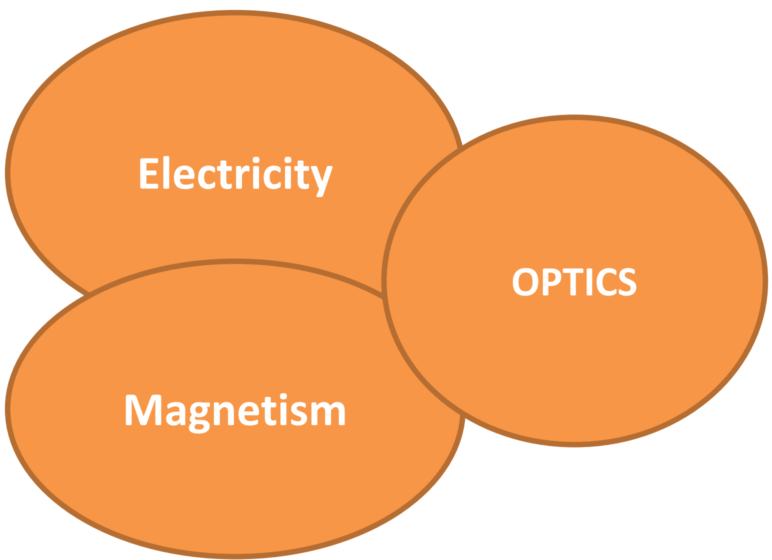
Charges at Rest

Magnetostatics

Charges in Steady Motion

Electromagnetism

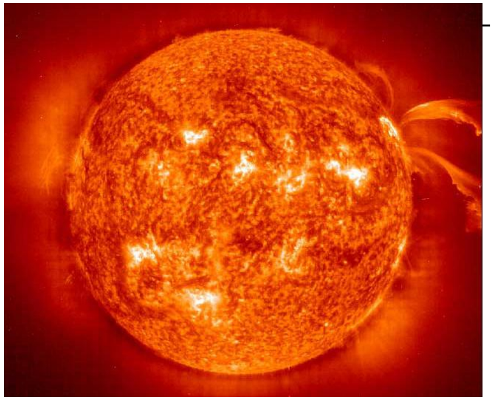
Charges in Unsteady Motion



Unification of Physical Theories - EMT

Universe to Atom

m_1



r_{12}

m_2

$$F_G = G\frac{m_1 m_2}{r_{12}^2}$$

Newton Gravitational Law

$G = 6.7 \; x \; 10^{-11} \; N.m^2/kg^2$

q_1

r_{12}

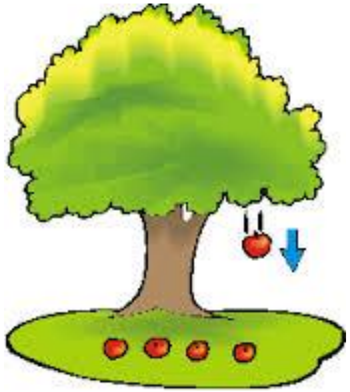
q_2

$$F_E = K\frac{q_1 q_2}{r_{12}^2}$$

Coulombs Law

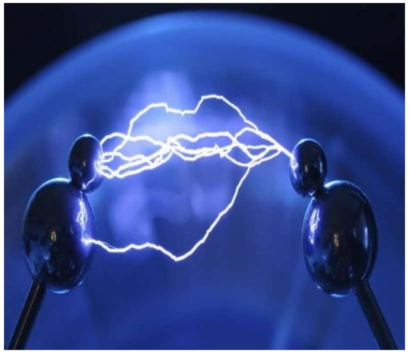
$K = \; 9 \; x \; 10^9 \; N.m^2/C^2$

Gravitational Vs Electrical Forces



- Gravitational force is attractive
- Gravitational force is weak
- Acceleration due to gravity points inward
- Unit of g is N/kg

$$\boldsymbol{F} = m_1\boldsymbol{g} = G\frac{m_1 m_2}{r_{12}^2} \qquad \boldsymbol{g} = \frac{Gm_2}{r_{12}^2}$$



- Electrical force is attractive/repulsive
- Electrical force is strong
- Electrical force points outward for positive charge
- Unit of E is N/C

$$\boldsymbol{F} = q_1\boldsymbol{E} = K\frac{q_1 q_2}{r_{12}^2} \qquad \boldsymbol{E} = \frac{Kq_2}{r_{12}^2}$$

Physics and Maths: Inseparables

Olympics Gold medal winner: Neeraj Chopra
Mathematical analysis

$R = \frac{U^2}{g} \sin 2\theta$

The maximum range for given initial velocity is R_{max} at $\theta = 45^0$.

At $\theta = 45^0$

$R_{max} = \frac{U^2}{g}$

Distance thrown, 87.58 m = $\frac{U^2}{9.81}$

$u^2 = 87.58 \times 9.8 = 859.1598$

$u = \sqrt{859.1598}$ = **29.311 m/sec**

Initial velocity with which **Neeraj Chopra** thrown the Javelin is **29.311 m/sec or 105. 52 KM /Hr.**

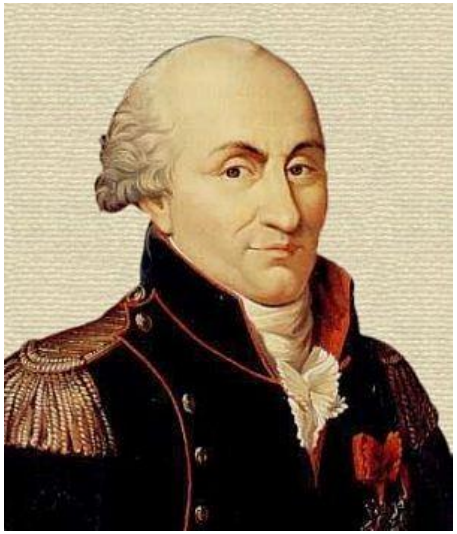
Science means "to know"
Physics means "Nature"
Maths is the Language of Science.

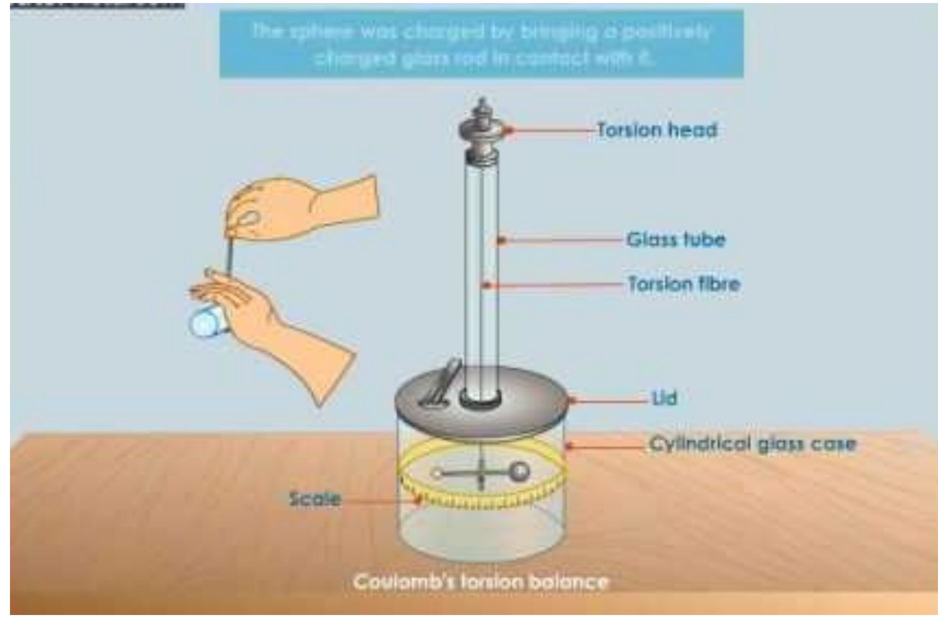
Master the language, else you can not communicate with nature!!!

Coulomb's Law

- State Coulomb's law and Hidden History
- Discuss magnitude and direction of electric force
- Draw diagram for two-body problem
- Frame equation – Force – Vector- Introduce constant
- Extend equation for n charges – Force of (n-1) charge on n^{th} charge
- Extend equation for charge distribution
- Nature examples : Lightning; Ionic bond; Coulomb's experiment

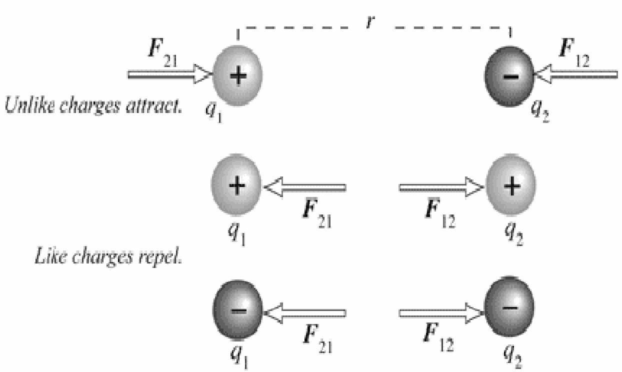
Coulomb’s Law

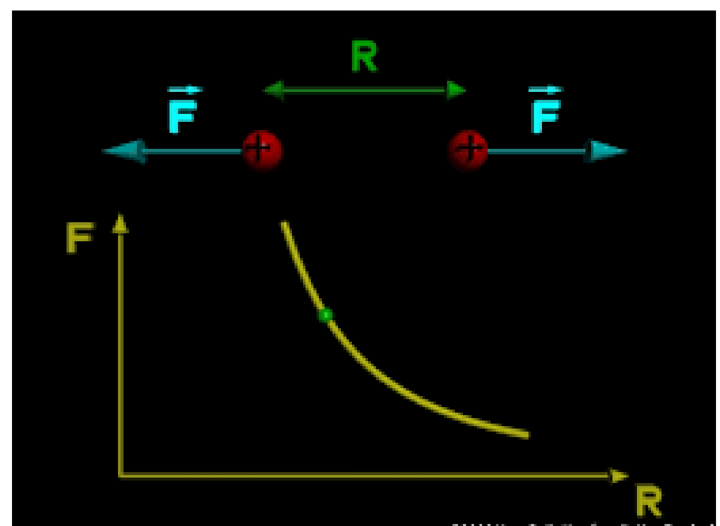




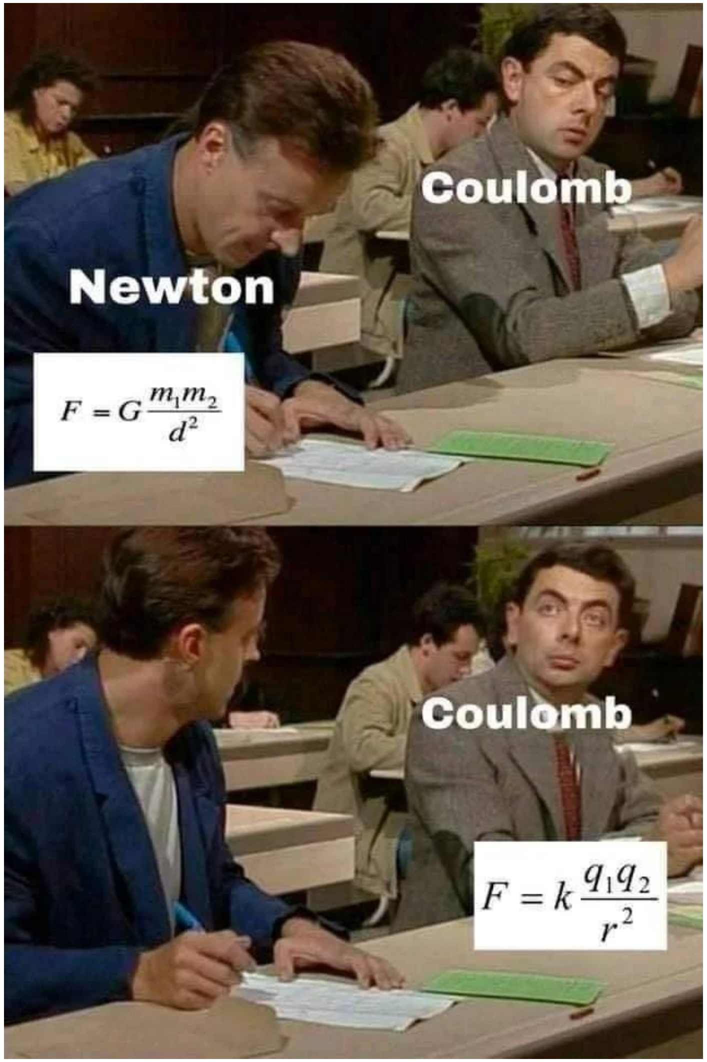
$$F = k\frac{q_1 q_2}{r^2}$$

F = electrostatic force
q = electric charge
r = distance between charge centers
k = Coulomb constant
9.0×10^9 N · m²/C²



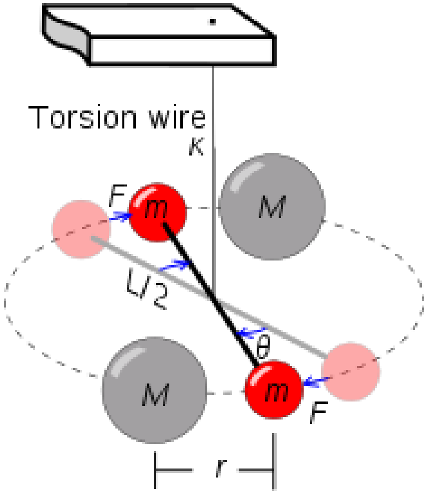


Is This True?



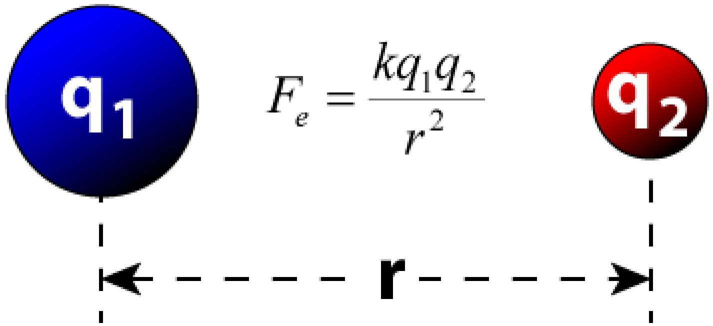
Newton (1643-1727)
Coulomb (1736-1806)

Coulomb constant – 1784
Gravitational constant - 1798



Henry Cavendish

Coulomb's Law : Statement



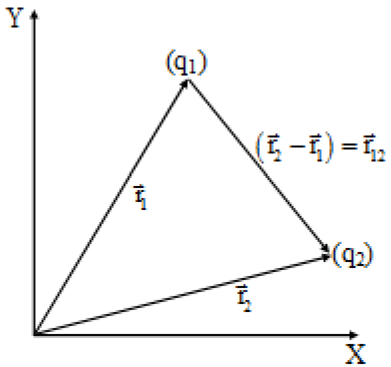
- ***Statement:*** *Force of attraction or repulsion between electric point charge at fixed distance apart is directly proportional to the product of magnitude of charges and inversely proportional to square of distance between them……* **????**

- Is force is a Scalar or Vector?
- Magnitude given…. Direction????

- *The direction of force is always along straight line joining the point charges.*

Coulomb's Law : Make it in Maths

- For two point charges separated by a finite distance



$$\overrightarrow{F_{ij}} = \lambda \frac{q_i q_j}{r_{ij}^2} \widehat{r_{ij}}$$

$$\overrightarrow{F_{ij}} = \frac{1}{4\pi\varepsilon} \frac{q_i q_j}{r_{ij}^2} \widehat{r_{ij}}; \frac{1}{4\pi\varepsilon} = 9 \, x \, 10^9 \, N.m^2/C^2$$

$$\overrightarrow{F_{ij}} = \frac{1}{4\pi\varepsilon_0} \frac{q_i q_j}{r_{ij}^3} \overrightarrow{r_{ij}} \qquad \varepsilon_0 = 8.854 \, x \, 10^{-12} \, C^2/N.m^2$$

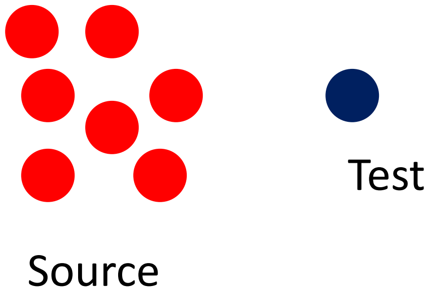
- Applicable for point charges like proton and electron with distance greater than 10^{-14} m.

Coulomb's Law: Generalization

- State Coulomb's law for n-point charges

- Source charges and Test Charges
- Force on j^{th} charge can be obtained by Principle of superposition

$$\overrightarrow{F_{ij}} = \sum F_{ij} = \frac{1}{4\pi\varepsilon_0} q_j \sum_{i \neq j}^{n} \frac{q_i}{r_{ij}^3} \overrightarrow{r_{ij}}$$

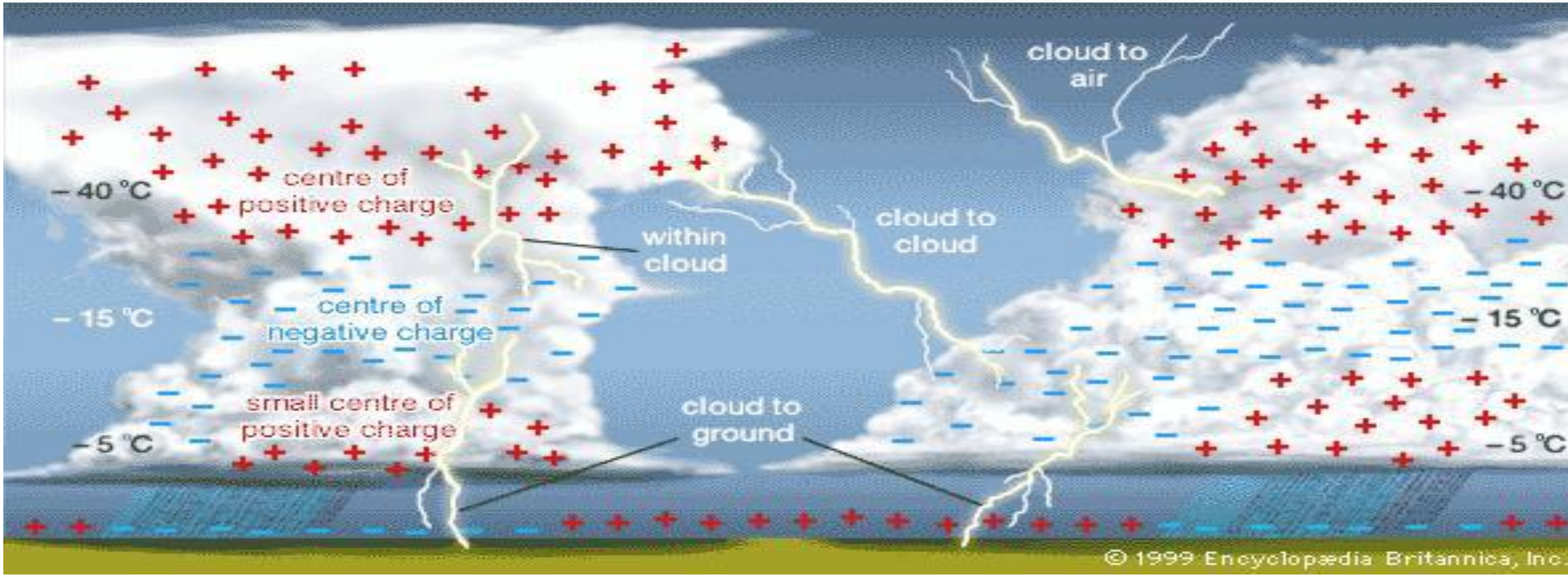


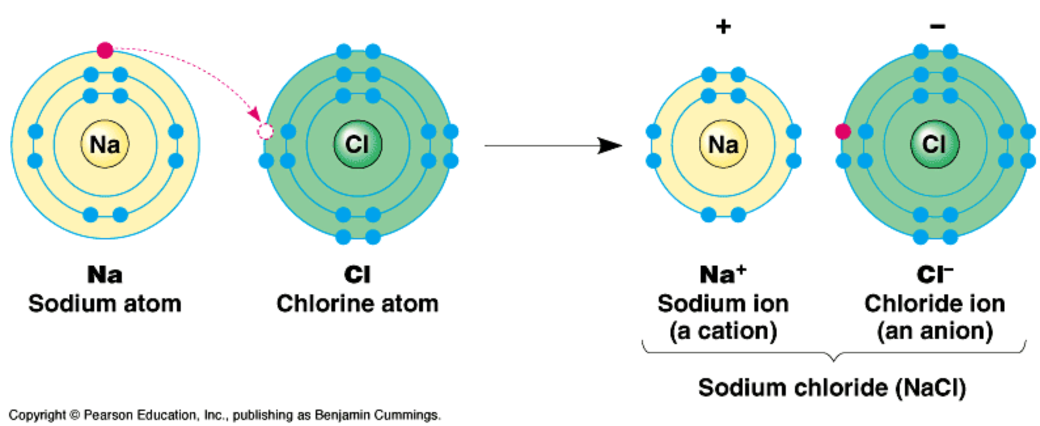
- Interaction of point charge (q) with continuous charge distribution (line, surface and volume)

$$F_q = \frac{q}{4\pi\varepsilon_0} \int \frac{dq}{r^3} \vec{r}$$

$$\rho = \lim_{\Delta v \to 0} \frac{\Delta q}{\Delta v} \; ; \; \sigma = \lim_{\Delta s \to 0} \frac{\Delta q}{\Delta s}$$

Coulomb's Law : Nature





Gauss Law

- State Gauss law
- Define electric field
- Draw diagram for Gauss law
- Electric field at internal and external point
- Divergence of electric field (differential form of Gauss law)
- Deduce Coulomb’s law from Gauss law
- Curl of electric field
- Nature examples

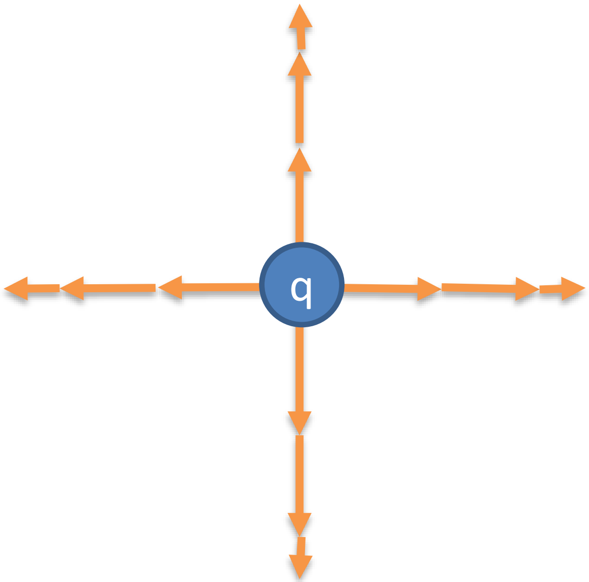
Electrical Field

- Force of point charges on q_1, q_2… q_i on test charge Q is,

$$\vec{F} = \overrightarrow{F_1} + \overrightarrow{F_2} + \cdots$$

$$\vec{F} = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1 Q}{r_1^3}\overrightarrow{r_1} + \frac{q_2 Q}{r_2^3}\overrightarrow{r_2} + \cdots.\right)$$

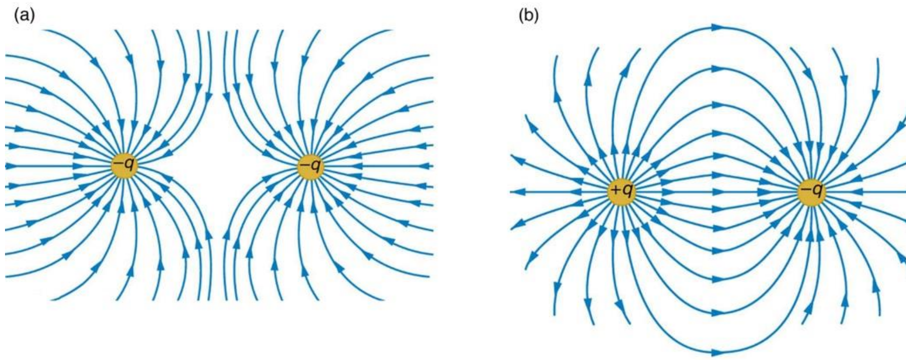
$$\vec{F} = Q\vec{E}$$



Electric field falls at $1/r^2$

- Electric field due to q_1, q_2… q_i on test charge Q at point P at distance r is,

$$\boldsymbol{E}(r) = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{q_i}{r_i^3}\overrightarrow{r_i}$$



Electrical Field

- Electric field due to continuous charge distribution on test charge Q at point P at distance r is,

$$\boldsymbol{E}(r) = \frac{1}{4\pi\varepsilon_0}\int \frac{dq}{r^3}\vec{r}$$

- Electric field due to continuous line charge distribution

$$\boldsymbol{E}(r)_{\boldsymbol{Line}} = \frac{1}{4\pi\varepsilon_0}\int \frac{\lambda(r')dl'}{r^3}\vec{r}$$

- Electric field due to continuous surface charge distribution

$$\boldsymbol{E}(r)_{\boldsymbol{Surface}} = \frac{1}{4\pi\varepsilon_0}\int \frac{\sigma(r')da'}{r^3}\vec{r}$$

- Electric field due to continuous volume charge distribution

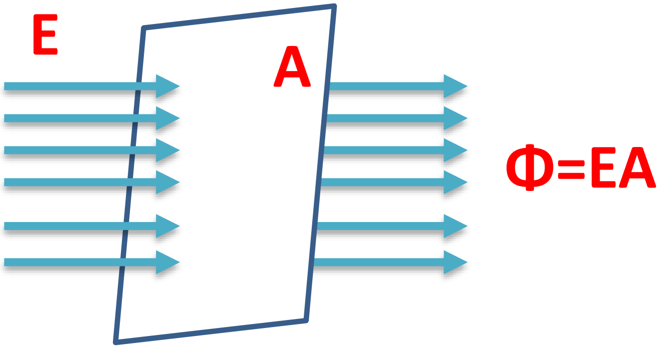
$$\boldsymbol{E}(r)_{\boldsymbol{Volume}} = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(r')d\tau'}{r^3}\vec{r}$$

Gauss Law (1777 – 1855)

- The normal component of the electric field (net outward flux) over any closed surface of any shape drawn in an electric field is equal to times $^{1}/_{\varepsilon_0}$ net charge enclosed by the surface

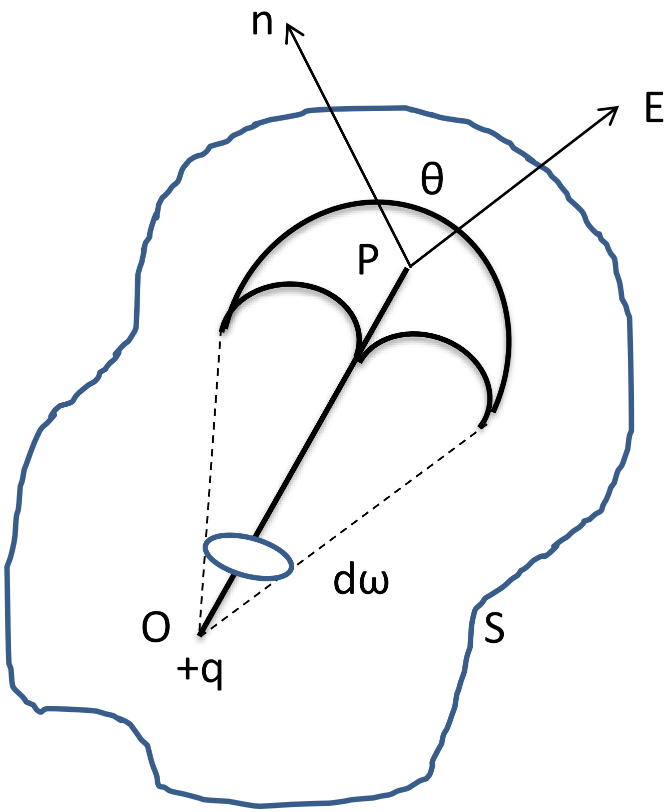
$$\oint \vec{E}.\hat{n}\, da = \frac{q}{\varepsilon_0}$$

- **Electric flux** is proportional to the number of electrical lines passing through infinitesimal area da
- Dot product will consider only the normal component of E.



$$\Phi_E = \int \vec{E}.da$$

Gauss Law : Internal Point



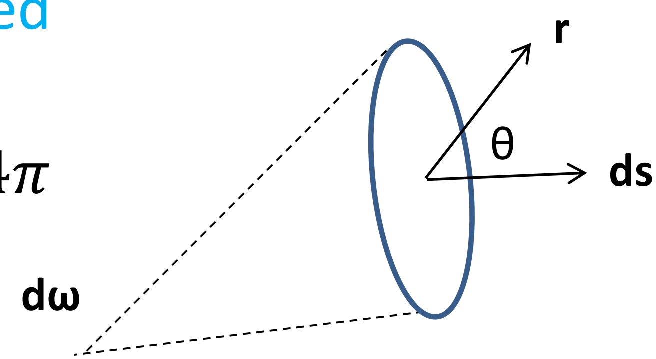
- Electric field at P with charge q at O is,

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^3}\vec{r}$$

$$\oint \vec{E}.\hat{n}\,da = \frac{q}{4\pi\varepsilon_0}\oint \frac{\vec{r}.\hat{n}}{r^3}da$$

Projection of area da perpendicular to $\boldsymbol{r}$ = Solid angle subtended

$$\boldsymbol{d\omega} = \frac{\boldsymbol{ds.r}}{r^2} = 4\pi$$



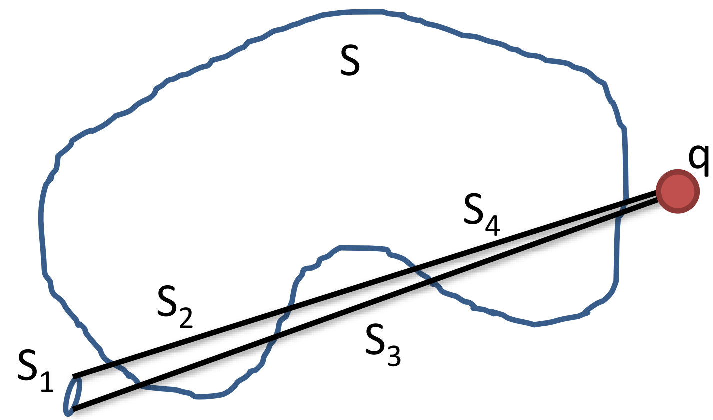
$$\oint \vec{E}.\hat{n}\,da = \frac{q}{4\pi\varepsilon_0}d\omega = \frac{q}{4\pi\varepsilon_0}4\pi = \frac{q}{\varepsilon_0}$$

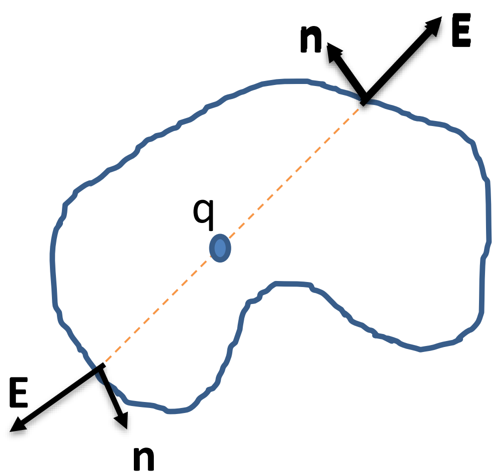
This is Gauss law for point charge enclosed by closed surface of any shape

Gauss Law : External Point

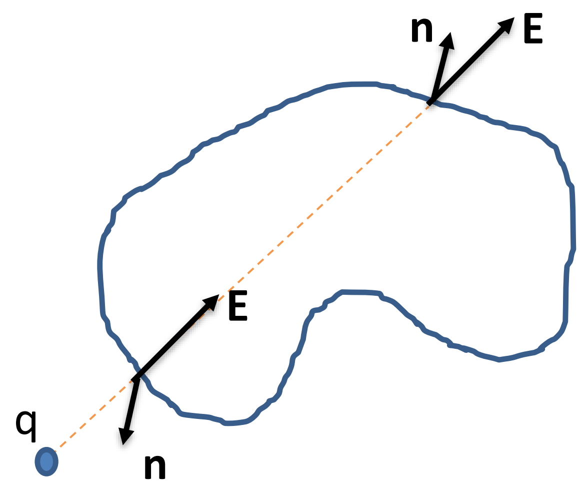
q is outside the surface
S is divided into S_1, S_2, S_3, S_4
S_1, S_3 is outward & S_2, S_4 is inward

$$\oint \vec{E}.\hat{n}\, da = 0$$





$$\boldsymbol{d\omega} = 4\pi$$



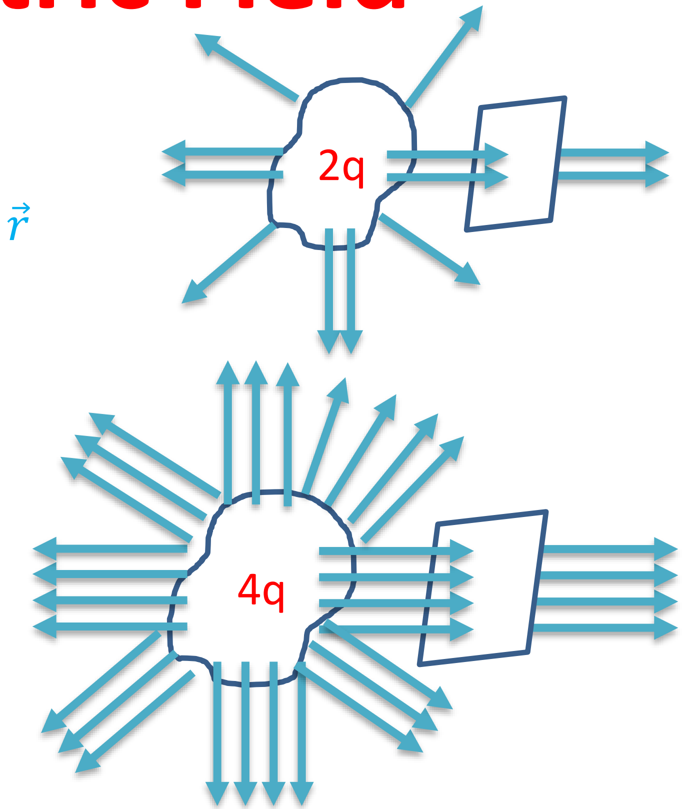
$$\boldsymbol{d\omega} = 0$$

$$\oint \vec{E}.\hat{n}\, da = \begin{cases} \dfrac{q}{\varepsilon_0}, & q \text{ inside } S \\ 0, & q \text{ outside } S \end{cases}$$

Divergence of Electric Field

Electric field over a volume is,

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(r')d\tau'}{r^3}\vec{r} \qquad \nabla.\vec{E} = \frac{1}{4\pi\varepsilon_0}\int \nabla.\frac{\rho(r')d\tau'}{r^3}\vec{r}$$



Gauss's law states,

$$\oint \vec{E}.\hat{n}\, da = \frac{1}{\varepsilon_0}\oint \rho\, dv$$

Using Gauss-divergence theorem

$$\oint \vec{E}.\hat{n}\, da = \oint \nabla.\vec{E}\, dv$$

$$\oint \mathbf{\nabla}.\vec{E}\, dv = \frac{1}{\varepsilon_0}\oint \rho\, dv$$

$$\mathbf{\nabla}.\vec{E} = \frac{\rho}{\varepsilon_0}$$

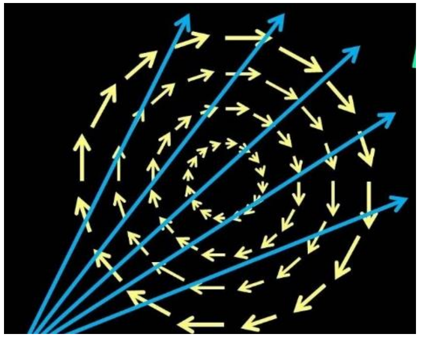
$$\Phi_E = \int \vec{E}.da = \mathbf{\nabla}.\vec{E} = \frac{\rho}{\varepsilon_0}$$

Thus flux through any closed surface is measure of total charge inside.

Differential form of Gauss Law

Curl of Electric Field

$$\int \vec{E}.dl = \int_a^b \frac{q}{r^2} dr = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r_a} - \frac{q}{r_b}\right)$$

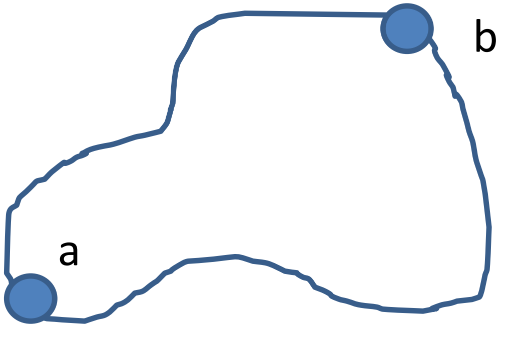


$$r_a = r_b \; ; \int \vec{E}.dl = 0$$

$$\nabla \text{ x } \vec{E} = 0$$

Electric Potential

Since curl of electric field is zero, the line integral of E over any closed path is zero. As line integral is independent of path, it allows to define a function called as Electric potential

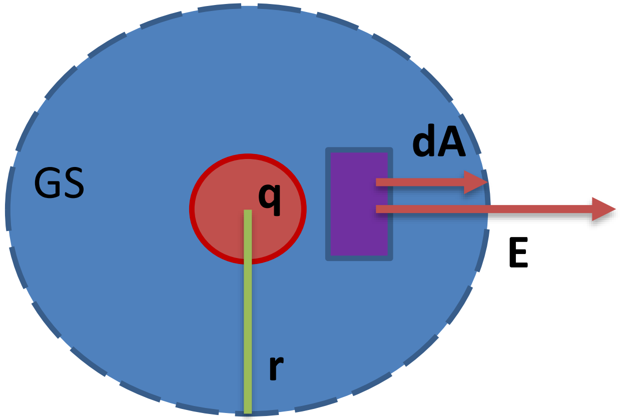


$$V\,(\boldsymbol{r}) = -\int_o^r \vec{E}.dl$$

$$V\,(\boldsymbol{b}) - V\,(\boldsymbol{a}) = \int_a^b \boldsymbol{\nabla} V.dl = -\int_a^b \vec{E}.dl$$

$$\mathbf{E} = -\boldsymbol{\nabla} V$$

Deduce Coulomb's law from Gauss Law



$$\boldsymbol{\Phi_E} = \int \vec{\boldsymbol{E}}.\boldsymbol{da}$$

$$|\vec{E}| \int da = \frac{q}{\varepsilon_0}$$

E and **dA** are parallel and **E** is constant

$$\boldsymbol{E} 4\pi r^2 = \frac{q}{\varepsilon_0}$$

Surface area of sphere

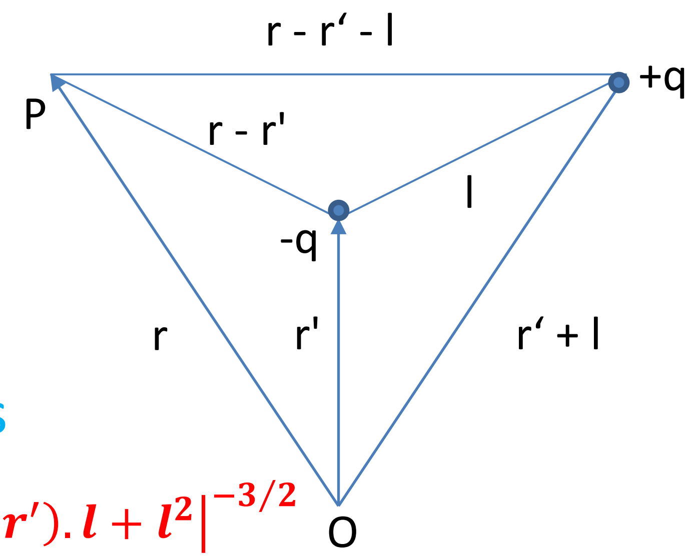
$$\boldsymbol{E} = \frac{q}{4\pi\varepsilon_0 r^2}$$

$$\boldsymbol{E} = \frac{\boldsymbol{kq}}{\boldsymbol{r^2}}$$

This is Coulomb's law

Electric Field and Potential Due to Dipole

Two equal and opposite charges separated by a finite small distance form an dipole



Electric field at P due to dipole is

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q(\boldsymbol{r}-\boldsymbol{r}'-\boldsymbol{l})}{|\boldsymbol{r}-\boldsymbol{r}'-\boldsymbol{l}|^3} - \frac{1}{4\pi\varepsilon_0}\frac{q(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}$$

As l << (r-r’), consider the non-vanishing terms

$$|\boldsymbol{r}-\boldsymbol{r}'-\boldsymbol{l}|^{-3} = |(\boldsymbol{r}-\boldsymbol{r}'-\boldsymbol{l})^2|^{-3/2} = \left|(\boldsymbol{r}-\boldsymbol{r}')^2 - 2(\boldsymbol{r}-\boldsymbol{r}').\boldsymbol{l} + \boldsymbol{l}^2\right|^{-3/2}$$

$$|\boldsymbol{r}-\boldsymbol{r}'-\boldsymbol{l}|^{-3} = (\boldsymbol{r}-\boldsymbol{r}')^{-3}\left[1 - \frac{2(\boldsymbol{r}-\boldsymbol{r}').\boldsymbol{l}}{(\boldsymbol{r}-\boldsymbol{r}')^2} + \frac{\boldsymbol{l}^2}{(\boldsymbol{r}-\boldsymbol{r}')^2}\right]^{-3/2}$$

As l << (r-r’), expand using binomial expression keeping only the linear term of l

$$|\boldsymbol{r}-\boldsymbol{r}'-\boldsymbol{l}|^{-3} = (\boldsymbol{r}-\boldsymbol{r}')^{-3}\left[1 + \frac{3(\boldsymbol{r}-\boldsymbol{r}').\boldsymbol{l}}{(\boldsymbol{r}-\boldsymbol{r}')^2} + \cdots\right]$$

Electric Field and Potential Due to Dipole

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{q}{4\pi\varepsilon_0}\left[(\boldsymbol{r}-\boldsymbol{r}'-\boldsymbol{l})(\boldsymbol{r}-\boldsymbol{r}')^{-3}\left(1+\frac{3(\boldsymbol{r}-\boldsymbol{r}').\boldsymbol{l}}{(\boldsymbol{r}-\boldsymbol{r}')^2}\right)-\frac{(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\right]$$

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{q}{4\pi\varepsilon_0}\left[\left(\frac{3(\boldsymbol{r}-\boldsymbol{r}').\boldsymbol{l}}{(\boldsymbol{r}-\boldsymbol{r}')^5}\right)(\boldsymbol{r}-\boldsymbol{r}')-\frac{\boldsymbol{l}}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\right]$$

This represents the electric field due to finite dipole which is proportional to separation of charges. Other contributions comes to the play from the square, cube and higher power of l.

If Electric dipole moment, $\vec{\boldsymbol{P}} = \lim_{l\to 0, q\to\infty} q\vec{\boldsymbol{l}}$

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\boldsymbol{\pi}\boldsymbol{\varepsilon}_0}\left[\left(\frac{3(\boldsymbol{r}-\boldsymbol{r}').\vec{\boldsymbol{P}}}{(\boldsymbol{r}-\boldsymbol{r}')^5}\right)(\boldsymbol{r}-\boldsymbol{r}')-\frac{\vec{\boldsymbol{P}}}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\right]$$

Electric Field and Potential Due to Dipole

Electric potential at P due to dipole is

$$\Phi(r) = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{|\boldsymbol{r} - \boldsymbol{r}' - \boldsymbol{l}|} - \frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|}\right]$$

Expanding the first term with binomial expression and neglecting the higher powers of l

$$\Phi(r) = \frac{q}{4\pi\varepsilon_0}\left[\frac{(\boldsymbol{r} - \boldsymbol{r}')\boldsymbol{l}}{|\boldsymbol{r} - \boldsymbol{r}'|^3}\right]$$

$$\Phi(r) = \frac{1}{4\pi\varepsilon_0}\left[\frac{(\boldsymbol{r} - \boldsymbol{r}')\vec{\boldsymbol{P}}}{|\boldsymbol{r} - \boldsymbol{r}'|^3}\right]$$

$$\vec{\boldsymbol{E}} = -\,grad\,\boldsymbol{\Phi}$$

If r’ is at origin and θ is angle between **r** and **P**

$$\Phi(r) = \frac{1}{4\pi\varepsilon_0}\left[\frac{\boldsymbol{r}.\boldsymbol{P}}{r^3}\right]$$

$$\Phi(r) = \frac{1}{4\pi\varepsilon_0}\left[\frac{P\cos\theta}{r^2}\right]$$

Poisson and Laplace Equation

For known charge distribution

$$\boldsymbol{E}(r) = \frac{1}{4\pi\varepsilon_0}\int \frac{dq}{|r - r'|^3}\left(\vec{r} - \vec{r'}\right) \qquad \Phi(r) = \frac{1}{4\pi\varepsilon_0}\int \frac{dq}{|r - r'|}$$

This equation fails miserably at the material interface. If the **charge distribution is unknown**, Gauss law fails to evaluate electric field and potential. Hence an alternate method is proposed.

Gauss law in differntial form is, $\nabla.\vec{E} = \dfrac{\rho}{\varepsilon_0}$

For pure electrostatic field, $\vec{E} = -\nabla\Phi$

$$\nabla.\nabla\Phi = \frac{-\rho}{\varepsilon_0} \qquad \nabla^2\Phi = \frac{-\rho}{\varepsilon_0}$$

This is Poisson's equation. ∇^2 is Laplacian scalar differential operator. Thus Poisson equation is a partial differential equation.

Poisson and Laplace Equation

In free charge region, Laplace equation is

$$\nabla^2 \Phi = 0 \ for \ \rho = 0$$

For arbitrary charge distribution

$$\Phi(r) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(r')\, dv'}{|r - r'|}$$

$$\nabla^2 \Phi = \frac{1}{4\pi\varepsilon_0} \nabla^2 \int \frac{\rho(r')\, dv'}{|r - r'|}$$

$$\nabla^2 \left(\frac{1}{r - r'} \right) = \nabla^2 \left(\frac{1}{r} \right)$$

$$\int \nabla^2 \left(\frac{1}{r} \right) = \int \nabla . \nabla \left(\frac{1}{r} \right) dv = \int \nabla \left(\frac{1}{r} \right) \hat{n} da = -4\pi$$

$$\nabla^2 \left(\frac{1}{r - r'} \right) = -4\pi\delta(r - r')$$

$$\nabla^2 \Phi = \frac{1}{4\pi\varepsilon_0} \int \rho(r') \left[-4\pi\delta(r - r') \right] dv'$$

$$\nabla^2 \Phi = \frac{-\rho}{\varepsilon_0}$$

Laplace Equation

- Laplace equation does not itself determine the potential and hence a suitable boundary condition is required
- Conventional approach to estimate electric field and potential by Gauss's law fails in case of conductors, as ρ is not known in advance.
- By considering the total charge of conductor, Poisson equation gives differential form which when taken with appropriate boundary conditions gives Gauss law.

In One Dimension

$$\frac{\partial^2 V}{\partial x^2} = 0 \qquad V(x) = mx + b$$

In Two Dimension

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \qquad V(x,y) = \frac{1}{2\pi R}\oint_{circle} V\,dl$$

In Three Dimension

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \qquad V(x,y,z) = \frac{1}{4\pi R^2}\oint_{sphere} V\,da$$

Applications of Gauss Law

- Gauss law by statement allows to evaluate electric field for any Gaussian shape.
- Although it is true always, it is not useful when,
  - For non-uniform charge distribution (When **E** is not a constant, it cannot be taken out of integral)
  - For Gaussian surface without symmetry
- Symmetry place a crucial role in imposing Gauss law
  - Spherical Symmetry : Concentric Sphere
  - Cylindrical Symmetry : Coaxial cylinder
  - Plane Symmetry : Pill Box

Books for Reference

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