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Unit V
Electromagnetism

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Origin of Electrodynamics

Electro statics

Charges at Rest

Magneto statics

Charges in Steady Motion

Steady state problem deals electric and magnetic fields independent of each other.

Electro magnetism

Charges in Unsteady Motion

Time varying magnetic field gives rise to electric field is electromagnetic induction, which is treated as gateway of electromagnetism.

Coupling the time independent behaviour of electric and magnetic field in time-dependent environment is electrodynamics, which deals with unsteady flow of current or varying magnetic field. Its equation of continuity is given as,

$$\nabla . \boldsymbol{J} + \frac{\partial \rho}{\partial t} = 0$$

Electrodynamics Before Maxwell

Divergence and curl of electric and magnetic fields are

$$\nabla . \boldsymbol{E} = \frac{1}{\varepsilon_0}\rho$$ Electrostatics (Gauss's law)

$$\nabla . \boldsymbol{B} = 0$$ Magnetostatics (no name)

$$\nabla \, X \, \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$ Electromagnetism (Faraday's law)

$$\nabla \, X \, \boldsymbol{B} = \mu_0 \boldsymbol{J}$$ Magnetostatics (Ampere's law)

Before Maxwell, when electrodynamics was proposed from the existence of electromagnetic induction, the laws governing electrostatics and magnetostatics were coupled. Here the curl of electric field which is zero in electrostatics is replaced by faraday's law. But the organized equations showed inconsistency and Maxwell attempted to fix it.

Maxwell Postulate

It is well-known that the divergence of curl is always is zero. To check the validity, take divergence of Faraday's and Ampere's law,

$$\nabla.(\nabla\ X\ \boldsymbol{E}) = \nabla.\left(-\frac{\partial \boldsymbol{B}}{\partial t}\right) = -\frac{\partial}{\partial t}(\nabla.\boldsymbol{B}) = 0$$

$$\nabla.(\nabla\ X\ \boldsymbol{B}) = \mu_0(\nabla.\boldsymbol{J}) \neq 0$$

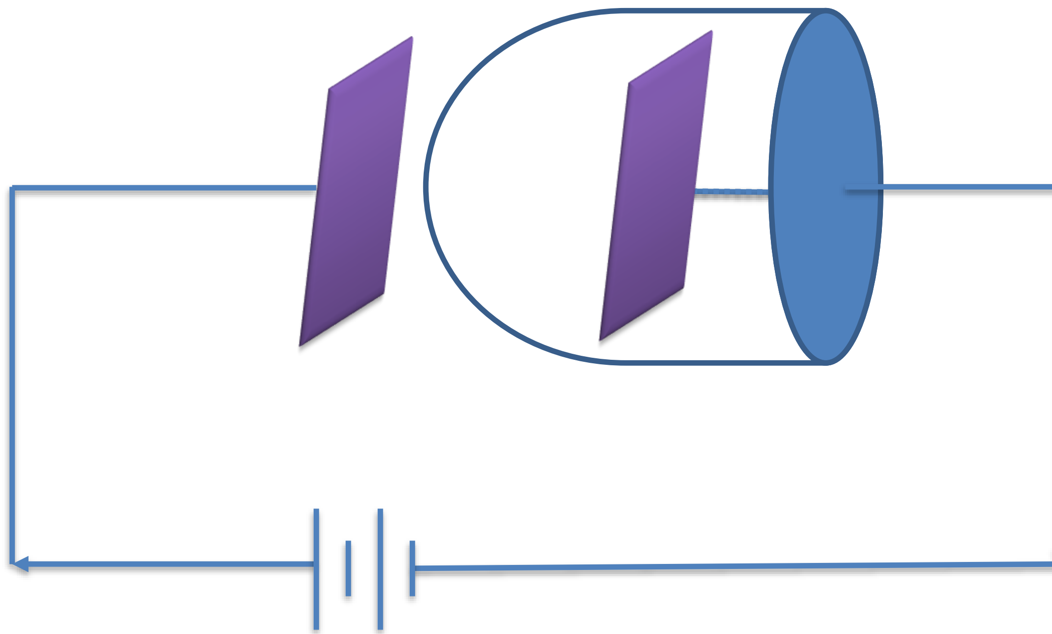
The above contradiction does not come to play in magnetostatics as divergence of current density is zero. Thus for steady current, divergence of curl of magnetic filed is always zero. But in electrodynamics, Ampere's law looks incomplete.

Ampere's law is bound to fail for non-steady current and Maxwell attempted to fix the problem of fitting Ampere's law suitable for electrodynamics through theoretical arguments. The modified law is Maxwell-Ampere's law and the term introduced is popularly called as Maxwell displacement current.

Maxwell Postulate

To understand the failure of Ampere's law for non-steady current, consider the process of charging a capacitor. The integral form of Ampere's law states,

$$\oint \boldsymbol{B}.d\boldsymbol{l} = \mu_0 I_{enc}$$



Here I_{enc} is the total current passing through the loop. Here the surface lies in the plane of the loop and the wire punctures this surface.

Instead of a loop, if balloon shaped surface is placed. No current passes through surface and I_{enc}=0.

Such cases does not prevail in magnetostatics and arises only in non-steady current. This is a paradox and Maxwell proposed a postulate called Maxwell current.

Maxwell Displacement Current

Consider the contradiction in divergence of curl of magnetic filed is always zero, only if divergence of current density is zero. But From equation of continuity,

$$\nabla.(\nabla \; X \; \boldsymbol{B}) = \mu_0(\nabla.\boldsymbol{J}) \neq 0$$

$$\nabla.\boldsymbol{J} = -\frac{\partial \rho}{\partial t}$$

To validate, Ampere's law for non-steady current, Maxwell attempted to introduce a change in current density as,

$$\nabla.(\nabla \; X \; \boldsymbol{B}) = \mu_0 \nabla.(\boldsymbol{J} + \boldsymbol{J_d}) = 0$$

$$\nabla.(\boldsymbol{J} + \boldsymbol{J_d}) = 0 \qquad \nabla.\boldsymbol{J} + \nabla.\boldsymbol{J_d} = 0 \qquad \nabla.\boldsymbol{J} = -\nabla.\boldsymbol{J_d}$$

Using equation of continuity and gauss's law,

$$\nabla.\boldsymbol{J_d} = \frac{\partial \rho}{\partial t} = \frac{\partial}{\partial t}(\varepsilon_0 \nabla.\boldsymbol{E}) = \nabla.\left(\varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}\right) \qquad \boldsymbol{J_d} = \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}$$

Maxwell Displacement Current

Therefore the modified Ampere's law is

$$\nabla\ X\ \boldsymbol{B} = \mu_0 \boldsymbol{J} + \mu_0 \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}$$

In Magnetostatics, E is constant and hence the second term contributes nothing, taking to the simple Ampere's law.

The second term resolves the paradox of the charging capacitor. The integral form of Ampere's law states,

$$\oint \boldsymbol{B}.\,d\boldsymbol{l} = \mu_0 I_{enc} + \mu_0 \varepsilon_0 \int \left(\frac{\partial \boldsymbol{E}}{\partial t}\right).\,d\boldsymbol{a}$$

For flat surface, E=0 and I_{enc} = I. While for balloon surface, I_{enc} =0 and $\int \left(\frac{\partial \boldsymbol{E}}{\partial t}\right).\,d\boldsymbol{a} = {}^{\boldsymbol{I}}/_{\varepsilon_0}$.

Maxwell called the extra term as displacement current and it produces only a magnetic field. Its magnitude is equal to rate of change of displacement current. Displacement current is negligible compared to genuine conduction current in conductors.

Maxwell Equations

Maxwell's equation gives the divergence and curl of electric and magnetic fields.

$$\nabla . \boldsymbol{E} = \frac{1}{\varepsilon_0}\rho$$ (Gauss's law)

$$\nabla . \boldsymbol{B} = 0$$ (no name)

$$\nabla \, X \, \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$ (Faraday's law)

$$\nabla \, X \, \boldsymbol{B} = \mu_0 \boldsymbol{J} + \mu_0 \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}$$ (Maxwell-Ampere's law)

Together with Lorentz force and continuity equation summarize the entire classical electrodynamics.

$$\boldsymbol{F} = q[\boldsymbol{E} + (\boldsymbol{v} \, X \, \boldsymbol{B})]$$

$$\nabla . \boldsymbol{J} + \frac{\partial \rho}{\partial t} = 0$$

Maxwell equation explains, how charges produce fields and Force law tells, how fields affect charges.

Maxwell Current and Equations- Recap

Maxwell's equation gives the divergence and curl of electric and magnetic fields.

$$\nabla . \boldsymbol{E} = \frac{1}{\varepsilon_0}\rho$$ (Gauss's law)

$$\boldsymbol{F} = q[\boldsymbol{E} + (\boldsymbol{v}\ X\ \boldsymbol{B})]$$

$$\nabla . \boldsymbol{B} = 0$$ (no name)

$$\nabla . \boldsymbol{J} + \frac{\partial \rho}{\partial t} = 0$$

$$\nabla\ X\ \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$ (Faraday's law)

$$\nabla\ X\ \boldsymbol{B} = \mu_0 \boldsymbol{J} + \mu_0 \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}$$ (Maxwell-Ampere's law)

Maxwell current term resolves the paradox of the Amperian loop experiment.

$$\oint \boldsymbol{B} . d\boldsymbol{l} = \mu_0 I_{enc} + \mu_0 \varepsilon_0 \int \left(\frac{\partial \boldsymbol{E}}{\partial t}\right) . d\boldsymbol{a}$$

ME-1: Differential Form of Gauss Law in Electrostatics

Consider a surface S bounding a volume V in a dielectric medium. Let ρ and ρ_P be charge densities of free charge and polarization charge at a point in small volume element dV, then Gauss's law is

$$\int \boldsymbol{E}.\,d\boldsymbol{S} = \frac{1}{\varepsilon_0}\int (\rho + \rho_P)\, dV$$

But polarization density is, $\rho_P = -\nabla.\,\boldsymbol{P}$

$$\int (\varepsilon_0 \boldsymbol{E}).\,d\boldsymbol{S} = \int \rho\, dV - \int \nabla.\,\boldsymbol{P}\, dV$$

Using Gauss Divergence theorem

$$\int \nabla.\,(\varepsilon_0 \boldsymbol{E})\, dV = \int \rho\, dV - \int \nabla.\,\boldsymbol{P}\, dV$$

$$\int \nabla.\,(\varepsilon_0 \boldsymbol{E} + \boldsymbol{P})\, dV = \int \rho\, dV$$

But electric displacement is, $\varepsilon_0 \boldsymbol{E} + \boldsymbol{P} = \boldsymbol{D}$

$$\int \nabla.\,\boldsymbol{D}\, dV = \int \rho\, dV$$

$$\nabla.\,\boldsymbol{D} = \rho$$

ME-2: Differential Form of Gauss Law in Magnetostatics

Since isolated magnetic poles have no significance, their magnetic lines of force are closed curves. Number of magnetic line of force entering any arbitrary closed surface is equal to the lines of force leaving out. It means magnetic flux over any closed surface is zero.

$$\int \boldsymbol{B}.\, d\boldsymbol{S} = 0$$

Using Gauss Divergence theorem

$$\int \nabla.\, \boldsymbol{B}\; dV = 0$$

As surface bounding the volume is arbitrary, integrand vanishes

$$\nabla.\, \boldsymbol{B} = 0$$

ME-3: Differential Form of Faraday's Law of Electromagnetic Induction

According to Faraday's law of induction,

$$e = -\frac{d\emptyset}{dt}$$

But magnetic flux is, $\emptyset = \int \boldsymbol{B}.d\boldsymbol{S}$

$$e = -\frac{d}{dt}\int \boldsymbol{B}.d\boldsymbol{S} = -\int \frac{\partial \boldsymbol{B}}{\partial t}.d\boldsymbol{S}$$

If **E** is electric field at a small element d**l** of loop, work done in carrying a charge round the loop, $e = \int \boldsymbol{E}.d\boldsymbol{l}$

$$\int \boldsymbol{E}.d\boldsymbol{l} = -\int \frac{\partial \boldsymbol{B}}{\partial t}.d\boldsymbol{S}$$

Using Stokes theorem

$$\int \nabla\, X\, \boldsymbol{E}.d\boldsymbol{S} = -\int \frac{\partial \boldsymbol{B}}{\partial t}.d\boldsymbol{S} \qquad \int \left(\nabla\, X\, \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t}\right).d\boldsymbol{S} = 0$$

As surface is arbitrary, integrand vanishes

$$\nabla\, X\, \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

ME-4: Maxwell Modified Ampere's Law

Ampere's circuital law states, $\oint \boldsymbol{H}.d\boldsymbol{l} = I$

$Since, I = \int \boldsymbol{J}.d\boldsymbol{S}$ $\oint \boldsymbol{H}.d\boldsymbol{l} = \int \boldsymbol{J}.d\boldsymbol{S}$

Applying Stoke's theorem, $\oint \nabla X\ \boldsymbol{H}.d\boldsymbol{S} = \int \boldsymbol{J}.d\boldsymbol{S}$

$$\oint (\nabla X\ \boldsymbol{H} - \boldsymbol{J}).d\boldsymbol{S} = 0$$

As surface is arbitrary, integrand vanishes

$$\nabla X\ \boldsymbol{H} = \boldsymbol{J}$$

This equation is valid only for steady current (magnetostatics). But for time-varying fields, this equation is insufficient.

To make it consistent with equation of continuity, Maxwell investigated mathematically and introduced an additional current element $\mathbf{J_d}$.

$$\nabla X\ \boldsymbol{H} = \boldsymbol{J} + \boldsymbol{J}_d$$

ME-4: Maxwell Modified Ampere's Law

Taking the divergence and imposing divergence of curl of any vector is zero.

$$\nabla.(\nabla \ X \ \boldsymbol{H}) = \nabla.(\boldsymbol{J} + \boldsymbol{J_d})$$

$$\nabla.(\boldsymbol{J} + \boldsymbol{J_d}) = 0$$

$$\nabla.\boldsymbol{J} + \nabla.\boldsymbol{J_d} = 0$$

$$\nabla.\boldsymbol{J} = -\nabla.\boldsymbol{J_d}$$

Using equation of continuity and Gauss law,

$$\nabla.\boldsymbol{J_d} = \frac{\partial \rho}{\partial t} = \frac{\partial}{\partial t}(\nabla.\boldsymbol{D}) = \nabla.\left(\frac{\partial \boldsymbol{D}}{\partial t}\right)$$

$$\boldsymbol{J_d} = \frac{\partial \boldsymbol{D}}{\partial t}$$

$$\nabla \ X \ \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t}$$

Maxwell called it as displacement current and it arises when electric displacement vector changes with time (only for time-varying fields).

Maxwell's Equation for Free Space

In free space, volume charge density and current density are zero, its Maxwell's equations are.

$$\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} \; and \; \boldsymbol{B} = \mu_0 \boldsymbol{H}$$

where ε_0 and $\mu_0$is absolute permittivity and permeability of free space respectively.

$$\nabla . \boldsymbol{D} = 0$$

$$\nabla . \boldsymbol{B} = 0$$

$$\nabla \; X \; \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

$$\nabla \; X \; \boldsymbol{H} = \frac{\partial \boldsymbol{D}}{\partial t}$$

Maxwell's Equation in Linear Isotropic Medium

If ε and μ is absolute permittivity and permeability of medium respectively,

$$\boldsymbol{D} = \varepsilon\boldsymbol{E} \; and \; \boldsymbol{B} = \mu\boldsymbol{H}$$

Its Maxwell's equations are,

$$\nabla.\boldsymbol{D} = \frac{\rho}{\varepsilon}$$

$$\nabla.\boldsymbol{B} = 0$$

$$\nabla \; X \; \boldsymbol{E} = -\mu\frac{\partial \boldsymbol{H}}{\partial t}$$

$$\nabla \; X \; \boldsymbol{H} = \boldsymbol{J} + \varepsilon\frac{\partial \boldsymbol{E}}{\partial t}$$

Maxwell's Equation for Harmonically Varying Fields

If electromagnetic fields vary harmonically with time,

$$\boldsymbol{D} = \boldsymbol{D}_0 e^{i\omega t} \; and \; \boldsymbol{B} = \boldsymbol{B}_0 e^{i\omega t}$$

$$\frac{\partial \boldsymbol{D}}{\partial t} = \boldsymbol{D}_0 i\omega e^{i\omega t} = i\omega \boldsymbol{D}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{B}_0 i\omega e^{i\omega t} = i\omega \boldsymbol{B}$$

Its Maxwell's equations are,

$$\nabla . \boldsymbol{D} = \rho$$

$$\nabla . \boldsymbol{B} = 0$$

$$\nabla \, X \, \boldsymbol{E} + i\omega \boldsymbol{B} = 0$$

$$\nabla \, X \, \boldsymbol{H} - i\omega \boldsymbol{D} = \boldsymbol{J}$$

Vector and Scalar Potentials

Maxwell equations consist of a set of coupled first-order partial differential equations relating components of fields. For convenience it is necessary to introduce potentials, obtaining a smaller number of second-order equations.

Since $\boldsymbol{\nabla}.\boldsymbol{B} = 0$ still holds, $\boldsymbol{B} = \nabla\, X\, \boldsymbol{A}$

The other homogenous equation, Faraday's law is

$$\nabla\, X \left(\boldsymbol{E} + \frac{\partial \boldsymbol{A}}{\partial t}\right) = 0$$

This vanishing curl can be written as the gradient of some scalar potential Φ

$$\boldsymbol{E} + \frac{\partial \boldsymbol{A}}{\partial t} = -\nabla\emptyset$$

$$\boldsymbol{E} = -\nabla\emptyset - \frac{\partial \boldsymbol{A}}{\partial t}$$

Vector and Scalar Potentials

Then the inhomogeneous equations can be written in terms of potentials as

$$\nabla^2\emptyset + \frac{\partial}{\partial t}(\nabla.\boldsymbol{A}) = \frac{-\rho}{\varepsilon_0}$$

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\boldsymbol{A}}{\partial t^2} - \nabla\left(\nabla.\boldsymbol{A} + \frac{1}{c^2}\frac{\partial\emptyset}{\partial t}\right) = -\mu_0\boldsymbol{J}$$

Now we have set of four Maxwell equation into two coupled equations. Uncoupling can be done through transformation as

$$\boldsymbol{A} \rightarrow \boldsymbol{A}' = \boldsymbol{A} + \nabla\boldsymbol{\Lambda} \qquad\qquad \emptyset \rightarrow \emptyset' = \emptyset - \frac{\partial\boldsymbol{\Lambda}}{\partial t}$$

$$\nabla.\boldsymbol{A} + \frac{1}{c^2}\frac{\partial\emptyset}{\partial t} = 0$$

This will uncouple and leave two inhomogeneous equations as

$$\nabla^2\emptyset - \frac{1}{c^2}\frac{\partial^2\emptyset}{\partial t^2} = \frac{-\rho}{\varepsilon_0}$$

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\boldsymbol{A}}{\partial t^2} = -\mu_0\boldsymbol{J}$$

Books for Reference

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3. **R. P. Feynman, R. B. Leighton and M. Sands**, *The Feynman Lectures on Physics: Vol. II (*Narosa Book Distributors, New Delhi, 1989)
4. **Satya Prakash**, *Electromagnetic Theory and Electrodynamics* (Kedar Nath Ram Nath, Meerut, 2015)