Nonlinear Dynamics

(Lecture Notes)

by Paulsamy MURUGANANADAM

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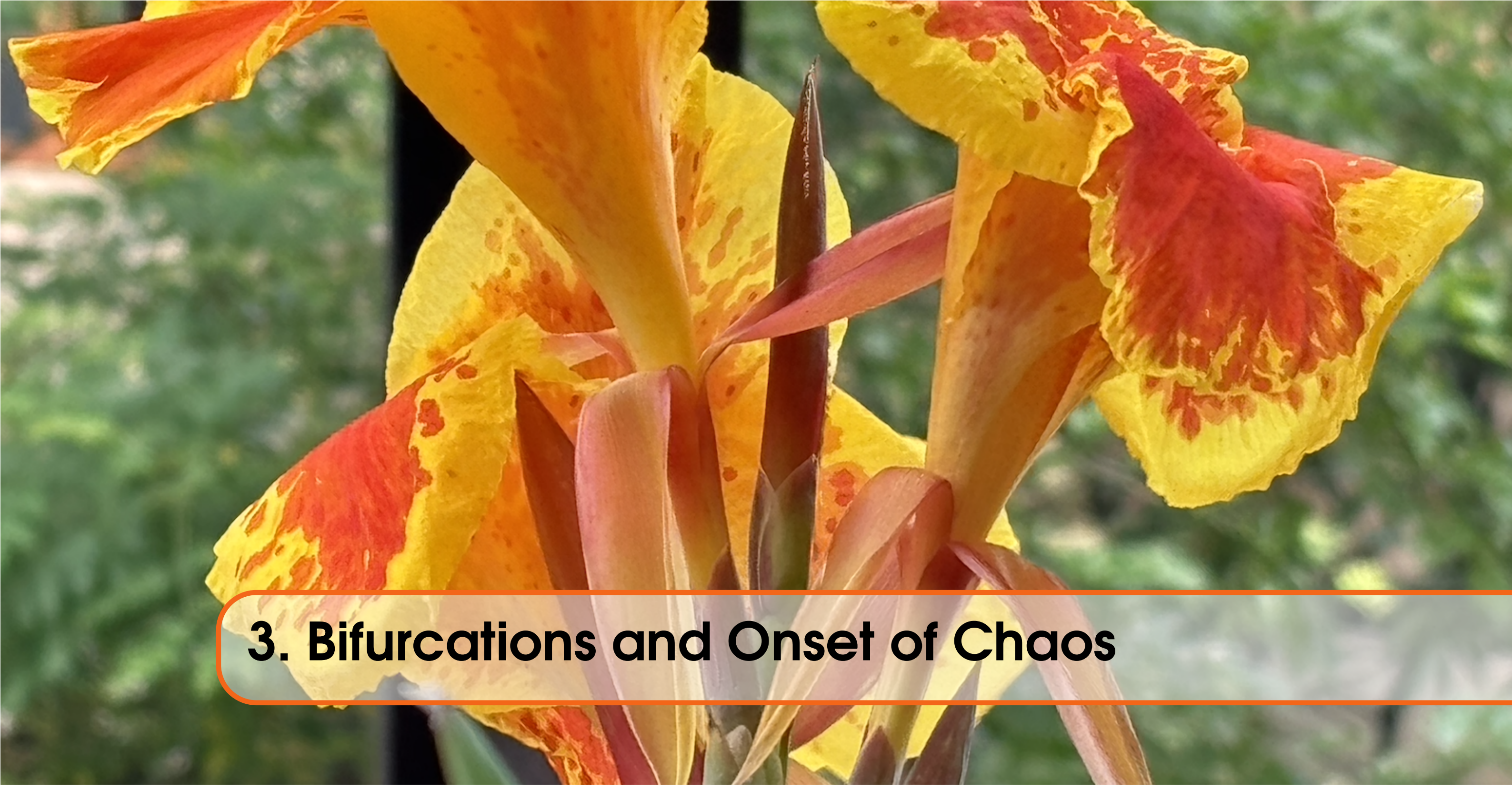

3. Bifurcations and Onset of Chaos

3.1 Bifurcations

Bifurcations in dynamical systems refer to ***sudden qualitative changes in the behavior (dynamics) of a system that occur at critical values of a control parameter as the parameter is varied***. These changes often involve transitions in the number or stability of equilibrium points, periodic orbits, or other invariant sets of the system. Below are the basic types of bifurcations:

1. Saddle-Node Bifurcation
2. Pitchfork Bifurcation
3. Transcritical Bifurcation
4. Hopf Bifurcation

3.1.1 Bifurcation diagram

A bifurcation diagram is a visual representation used in dynamical systems to illustrate how the qualitative behavior of a system changes as a parameter is varied. It provides a map of the system's steady states, periodic orbits, or chaotic behavior against the parameter values, helping to identify bifurcations and transitions in dynamics.

In a bifurcation diagram, the horizontal axis (the x-axis or abscissa) represents the bifurcation parameter, typically denoted by μ, and the vertical axis (the y-axis or ordinate) represents the variable(s) of interest in the system, such as equilibrium points or the amplitude of periodic orbits.

3.1.2 Saddle-Node Bifurcation

A saddle-node bifurcation is one of the simplest and most fundamental types of bifurcations in dynamical systems. It occurs when two equilibrium points – one stable (a node) and one unstable (a saddle) – either collide and annihilate each other or emerge from nothing as a control parameter, such as μ, passes through a critical value.

This bifurcation leads to the creation or destruction of two equilibrium points. The bifurcation occurs when the parameter $\mu = 0$. For $\mu > 0$, two equilibrium points exist; for $\mu < 0$, there are none. Saddle-node bifurcations are commonly observed in various physical, biological, and engineering systems.

Normal form

The standard normal form of a saddle-node bifurcation is given by the equations:

$$\dot{x} = \mu - x^2 \equiv P(x,y) \tag{3.1a}$$
$$\dot{y} = -y \equiv Q(x,y) \tag{3.1b}$$

where μ is the control parameter. The equation for $\dot{x}$ describes the evolution of x with a nonlinear term $-x^2$ and a parameter-dependent constant μ. The equation for $\dot{y}$ describes an independent decay of y, which remains uncoupled from x.

Equilibrium points

The equilibrium points of the above system of equations can be found by setting $\dot{x} = \dot{y} = 0$, i.e., the time-independent solutions of the equations.

- $\mu < 0$ – none
- $\mu > 0$ – $(\pm\sqrt{\mu}, 0)$

Stability analysis

To analyze the stability of the equilibrium points, we compute the Jacobian matrix as

$$M = \begin{pmatrix} \left.\frac{\partial P}{\partial x}\right|_{(x^*,y^*)} & \left.\frac{\partial P}{\partial y}\right|_{(x^*,y^*)} \\ \left.\frac{\partial Q}{\partial x}\right|_{(x^*,y^*)} & \left.\frac{\partial Q}{\partial y}\right|_{(x^*,y^*)} \end{pmatrix} \equiv \begin{pmatrix} -2x^* & 0 \\ 0 & -1 \end{pmatrix} \tag{3.2}$$

The eigenvalues of the Jacobian matrix are:

$$\lambda_1 = -2x^*, \text{ and } \lambda_2 = -1$$

. The stability of equilibrium points for $\mu > 0$ can be described as follows:

1. At $(x^*, y^*) = (+\sqrt{\mu}, 0)$:
   - $\lambda_1 = -2\sqrt{\mu} < 0$ (stable).
   - $\lambda_2 = -1 < 0$ (stable).

   This equilibrium is a *stable node*.
2. At $(x^*, y^*) = (-\sqrt{\mu}, 0)$:
   - $\lambda_1 = -2(-\sqrt{\mu}) = 2\sqrt{\mu} > 0$ (unstable).
   - $\lambda_2 = -1 < 0$ (stable in one direction).

   This equilibrium is a *saddle point*.

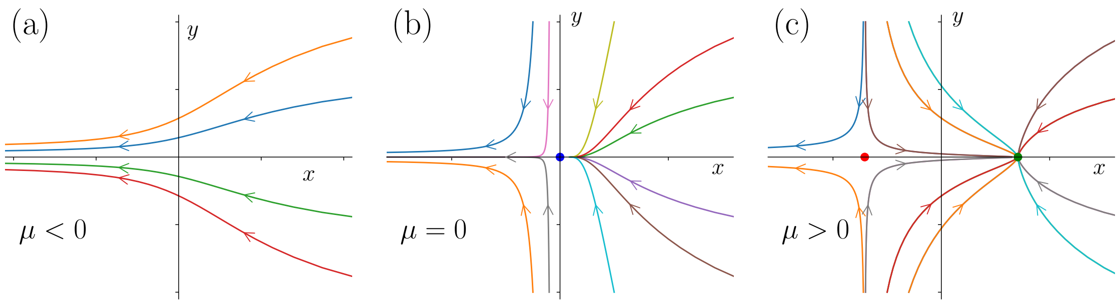

Figure 3.1: Phase trajectories in the phase plane for different values of the bifurcation parameter μ in a saddle-node bifurcation, obtained from the numerical solution of (3.1) with select set of initial conditions: (a) $\mu < 0$, (b) $\mu = 0$, and (c) $\mu > 0$.

Figure 3.1 illustrates the phase trajectories in the phase plane for different values of the bifurcation parameter μ: (a) $\mu < 0$, (b) $\mu = 0$, and (c) $\mu > 0$. As shown in Figure 3.1, before the bifurcation

($\mu < 0$), no equilibrium points exist. At the bifurcation point ($\mu = 0$), a qualitative change occurs, and for $\mu > 0$, two equilibrium points are created: one is a saddle point, and the other is a stable node. This behaviour gives rise to the term *saddle-node* bifurcation.

Bifurcation diagram

In Figure 3.2(a), we present the typical bifurcation diagram for a saddle-node bifurcation in a two-dimensional plane. The x-axis represents the parameter variation, while the y-axis corresponds to the equilibrium points. From Figure 3.2(a), it is evident that two branches (one stable and one

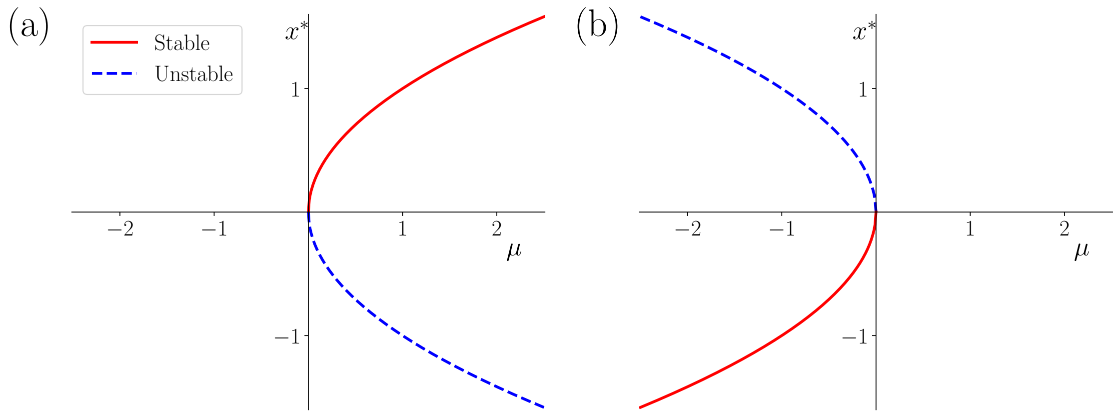

Figure 3.2: Bifurcation diagrams illustrating (a) supercritical saddle-node bifurcation and (b) subcritical saddle-node bifurcation. Stable branches are represented by solid red lines, while unstable branches are shown as dashed blue lines.

unstable) emerge as the parameter crosses the bifurcation point, $\mu = 0$.

This type of bifurcation is referred to as a ***supercritical*** saddle-node bifurcation. As μ increases from a negative value ($\mu < 0$), a stable and an unstable equilibrium point emerge from a region where no equilibrium points exist, characterizing the supercritical nature of the bifurcation.

On the other hand, if we consider the following normal form:

$$\dot{x} = \mu + x^2, \tag{3.3a}$$
$$\dot{y} = -y, \tag{3.3b}$$

we observe a ***subcritical*** saddle-node bifurcation. In this case, the two branches (stable and unstable) merge and disappear as the parameter crosses the bifurcation point. The bifurcation diagram for a subcritical saddle-node bifurcation is illustrated in Figure 3.2(b).

3.1.3 Pitchfork Bifurcation

In this case, a single equilibrium splits into multiple equilibrium points (subcritical or supercritical bifurcation) as the parameter changes. In the supercritical case, a stable equilibrium becomes unstable, and two new stable equilibrium points emerge. The key feature of a pitchfork bifurcation is ***symmetry breaking*** or ***symmetry restoration***.

Normal form

The standard form of a saddle-node bifurcation is given by the equations:

$$\dot{x} = \mu x - x^3, \tag{3.4a}$$
$$\dot{y} = -y, \tag{3.4b}$$

where μ is the control parameter.

Equilibrium points and their stability

- When $\mu < 0$, $(0,0)$ is the only one equilibrium point.
- On the other hand if, $\mu > 0$ there are three equilibrium points: $(0,0)$, $(\pm\sqrt{\mu},0)$

The stability of the above equilibrium points is determined by the eigenvalues of the following Jacobian matrix evaluated at these equilibrium points:

$$M = \begin{pmatrix} \mu - 3x^{*2} & 0 \\ 0 & -1 \end{pmatrix} \tag{3.5}$$

The stability-determining eigenvalues are $\lambda_1 = \mu - 3x^{*2}$ and $\lambda_2 = -1$. It is evident from these eigenvalues that when $\mu < 0$, the only equilibrium point $(0,0)$ is stable. However, when $\mu > 0$, $(0,0)$ becomes unstable, and two new stable equilibrium points $(\pm\sqrt{\mu},0)$ emerge. Figure 3.5(a)

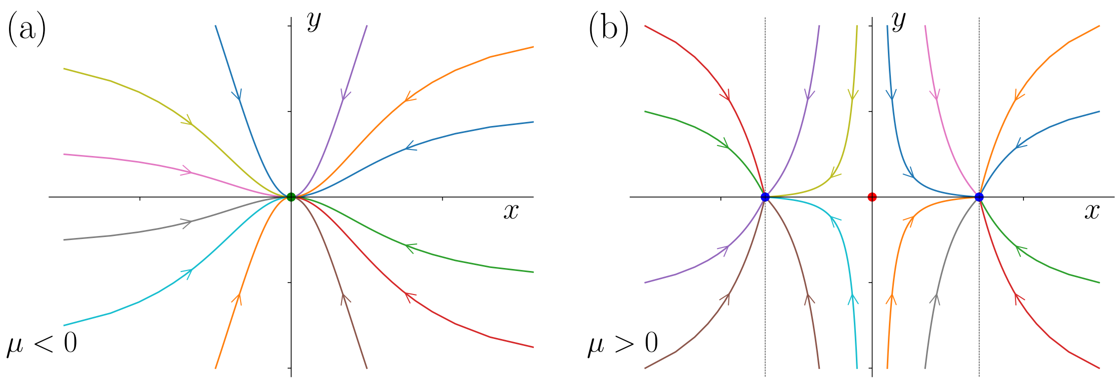

Figure 3.3: Phase trajectories in the phase plane for different values of the bifurcation parameter μ in a pitchfork bifurcation, obtained from the numerical solution of (3.4) with select set of initial conditions: (a) $\mu < 0$, (b) $\mu > 0$.

illustrates the behavior of phase trajectories corresponding to various initial conditions for $\mu < 0$, while Figure 3.5(b) shows them for $\mu > 0$.

Bifurcation diagram

Figure 3.4 illustrates the bifurcation diagram in the $\mu - x^*$ plane. In this diagram, stable equilibrium

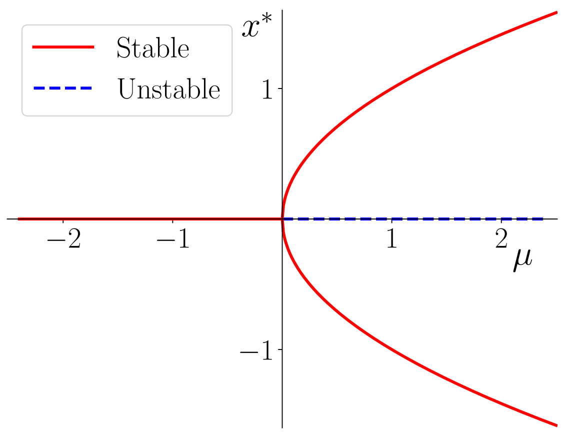

Figure 3.4: Bifurcation diagrams illustrating a pitchfork bifurcation. Stable branches are represented by solid red lines, while unstable branches are shown as dashed blue lines.

branches are represented by solid red lines, while dashed blue lines indicate the unstable equilibrium branch. The structure of the above bifurcation diagram resembles the shape of a pitchfork, hence the name ***pitchfork bifurcation***.

3.1.4 Transcritical bifurcation

Another example of a bifurcation involving only equilibrium points is the transcritical bifurcation. In this type of bifurcation, the stability of the equilibrium points is exchanged as the control parameter passes through a critical bifurcation value. This behavior is characterized by the interaction and exchange of stability between two equilibrium points, making it distinct from other bifurcation types such as saddle-node or pitchfork bifurcations.

Normal form

To illustrate transcritical bifurcation, we consider the following normal form of a two-dimensional dynamical system:

$$\dot{x} = \mu x - x^2, \tag{3.6a}$$
$$\dot{y} = -y, \tag{3.6b}$$

where μ serves as the bifurcation parameter. The equation for $\dot{x}$ describes the evolution of x with a nonlinear term $-x^2$ and a parameter-dependent linear term μx. The equation for $\dot{y}$ describes an independent decay of y, which remains uncoupled from x.

Equilibrium points and their stability

Analyzing the system for equilibrium points, we solve $\dot{x} = 0$ and $\dot{y} = 0$:

$$x^*(\mu - x^*) = 0 \implies x^* = 0 \text{ or } x^* = \mu,$$
$$y^* = 0.$$

Thus, the system has two equilibrium points: $(x^*, y^*) = (0,0)$ and $(x^*, y^*) = (\mu, 0)$. The stability of these equilibrium points are determined from the eigenvalues of the following Jacobian matrix:

$$M = \begin{pmatrix} \mu - 2x^* & 0 \\ 0 & -1 \end{pmatrix} \tag{3.7}$$

The eigenvalues are $\lambda_1 = \mu - 2x^*$ and $\lambda_2 = -1$. The stability of these equilibrium points depends on the value of the parameter μ.

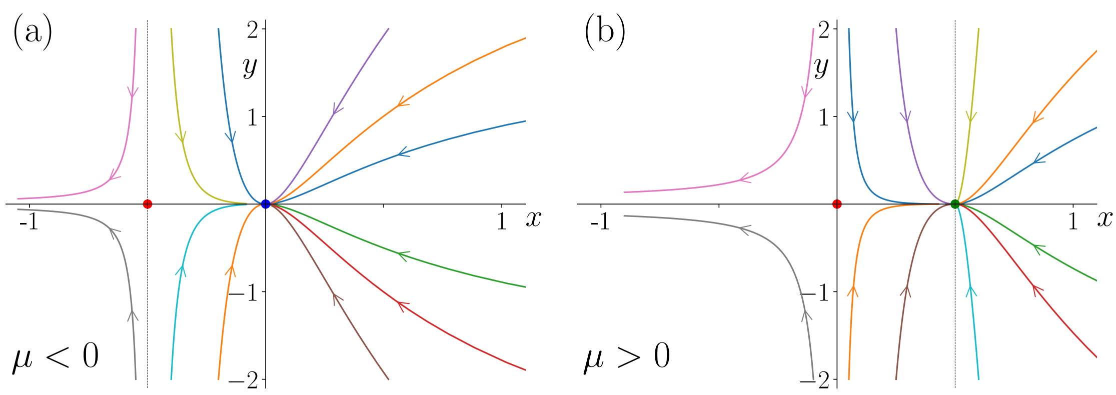

Figure 3.5: Phase trajectories in the phase plane for different values of the bifurcation parameter μ in a transcritical bifurcation, obtained from the numerical solution of (3.6) with select set of initial conditions: (a) $\mu < 0$, (b) $\mu > 0$.

- For $(x^*, y^*) = (0, 0)$, the linear stability analysis reveals that its stability changes as μ crosses zero. When $\mu > 0$, this equilibrium is unstable, while for $\mu < 0$, it is stable.
- For $(x^*, y^*) = (\mu, 0)$, the stability is the opposite: it is stable when $\mu > 0$ and unstable when $\mu < 0$.

This exchange of stability between the two equilibrium points is the hallmark of a transcritical bifurcation. As μ passes through the bifurcation value ($\mu = 0$), the equilibrium points *trade* their roles in terms of stability, creating a dynamic transition in the system's behavior.

Bifurcation diagram

Figure 3.6 illustrates the bifurcation diagram in the $\mu - x^*$ plane. In this diagram, stable equilibrium

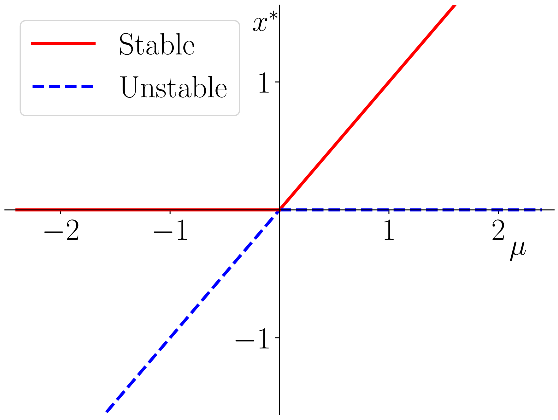

Figure 3.6: Bifurcation diagrams illustrating a transcritical bifurcation. Stable branches are represented by solid red lines, while unstable branches are shown as dashed blue lines.

branches are represented by solid red lines, while dashed blue lines indicate the unstable equilibrium branches. The structure of the above bifurcation diagram clearly reveals the exchange of stability of the equilibrium points at $\mu = 0$.

3.2 Hopf bifurcation

So far, we have examined three fundamental types of bifurcations: saddle-node, pitchfork, and transcritical, which primarily occur in effectively one-dimensional systems. Although we used simple two-dimensional systems involving x and y variables, this approach was intended to provide a clearer understanding of the asymptotic behavior of phase trajectories and the various types of equilibrium points discussed earlier. In these systems, the second dimension y was introduced solely to enhance the visualization of dynamics in phase space, while y remains entirely uncoupled from x.

A different type of bifurcation occurs in genuinely two-dimensional systems, where the system's equilibrium transitions from a stable fixed point to either a stable or unstable limit cycle, depending on the parameter values. Unlike the bifurcations discussed earlier, which focus primarily on the existence and stability of isolated equilibrium points in phase space, the Hopf bifurcation involves a transition from a fixed point (equilibrium) to a periodic orbit (limit cycle), which can be either stable or unstable. A stable limit cycle is a closed curve in phase space that attracts nearby trajectories, while an unstable limit cycle repels nearby trajectories.

In a Hopf bifurcation, as the system's parameters change, a pair of complex conjugate eigenvalues of the Jacobian matrix at the equilibrium point cross the imaginary axis. When the real parts of these eigenvalues change sign (from negative to positive), a limit cycle emerges.

Consider the following system of ordinary differential equations:

$$\dot{x} = \mu x - y - x(x^2 + y^2), \tag{3.8a}$$
$$\dot{y} = x + \mu y - y(x^2 + y^2), \tag{3.8b}$$

where μ is a bifurcation parameter, and $x, y \in \mathbb{R}^2$ are state variables.

- For $\mu < 0$, the system has a stable equilibrium point at the origin.
- When $\mu > 0$, the equilibrium point at the origin loses stability, and a limit cycle (a periodic solution) emerges around the origin.

One of the most well-known examples of a Hopf bifurcation is the ***van der Pol oscillator***, which models the behavior of an electrical circuit with nonlinear resistance. The governing equation is given by

$$\ddot{x} - \mu(1 - x^2)\dot{x} + \omega_0^2 x = 0. \tag{3.9}$$

This second-order ordinary differential equation can be rewritten as two first-order equations of the following form:

$$\dot{x} = y, \tag{3.10a}$$
$$\dot{y} = \mu(1 - x^2)y - \omega_0^2 x. \tag{3.10b}$$

The system has an equilibrium point at the origin, $(x^*, y^*) = (0, 0)$. From stability analysis, it is

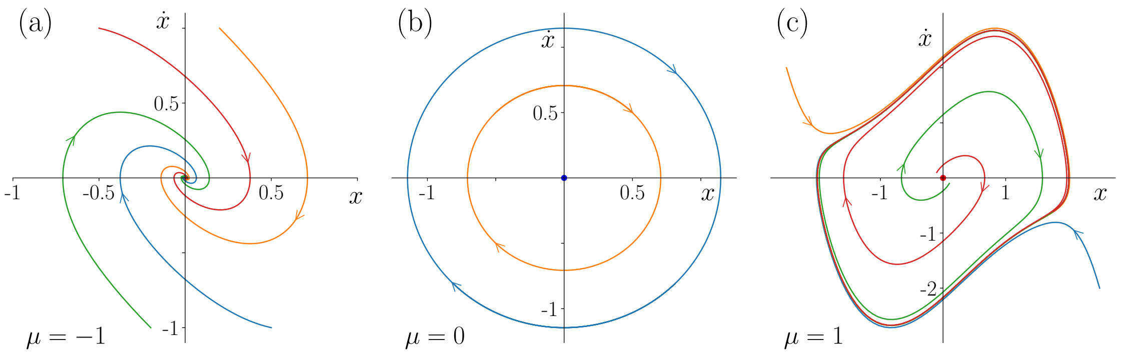

Figure 3.7: Phase trajectories in the phase plane for different values of the bifurcation parameter μ in a supercritical Hopf bifurcation, obtained from the numerical solution of (3.10) with a selected set of initial conditions: (a) $\mu = -1 < 0$, (b) $\mu = 0$, and (c) $\mu = 1 > 0$. The parameter ω_0^2 is fixed at 1.

clear that $(0, 0)$ is stable for $\mu < 0$ and loses its stability when μ crosses zero from below. However, the system does not diverge or blow up. Instead, it gives rise to a stable limit cycle (periodic orbit). Figure 3.7 illustrates the behavior of phase trajectories during the Hopf bifurcation in the van der Pol oscillator (3.10). Figure 3.7(a) shows the phase trajectories, highlighting the stable focus nature of the equilibrium point for $\mu = -1 < 0$. When $\mu = 0$, the system reduces to a linear harmonic oscillator, exhibiting periodic oscillations, as shown in Figure 3.7(b). Figure 3.7(c) illustrates the stable limit cycle of the van der Pol oscillator, which attracts all nearby trajectories. Figure 3.8(a) illustrates the time series of the x variable before ($\mu < 0$), at ($\mu = 0$), and after ($\mu > 0$) the Hopf bifurcation. Figure 3.8(b) shows the limit cycle in the phase space (or phase plane) after the transients are discarded.

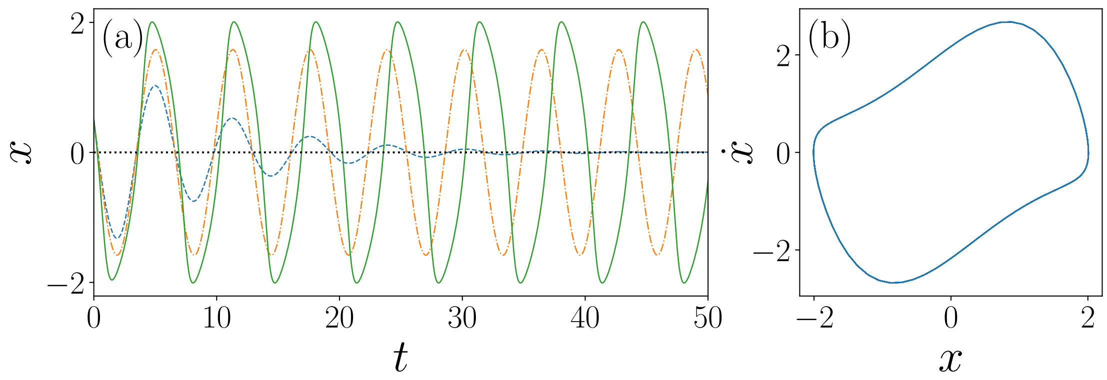

Figure 3.8: (a) Time series plots of van der Pol oscillator for different values of the bifurcation parameter μ in a supercritical Hopf bifurcation, obtained from the numerical solution of (3.10): (a) $\mu = -0.25$ (dashed blue line), $\mu = 0$ (dash-dotted orange/red line) and $\mu = 1$ (solid green line). (b) limit cycle in the phase plane at $\mu = 1$

3.2.1 Hopf theorem

It is evident from the above that the van der Pol oscillator (3.10) exhibits limit cycle oscillations through a Hopf bifurcation when the eigenvalues of the Jacobian matrix cross the imaginary axis. However, not all nonlinear systems with similar behavior in the eigenvalues of the linearized system around the equilibrium undergo a Hopf bifurcation. The eigenvalues crossing the imaginary axis is a necessary condition only. For a nonlinear system to undergo a Hopf bifurcation, the necessary and sufficient condition is provided by the Hopf theorem, stated as follows:

Theorem 3.2.1 Suppose that the autonomous system

$$\dot{x} = P(x,y), \quad \dot{y} = Q(x,y) \tag{3.11}$$

has an equilibrium point at (x^*, y^*), and that the associated Jacobian matrix.

$$M = \begin{pmatrix} \left.\frac{\partial P}{\partial x}\right|_{(x^*,y^*)} & \left.\frac{\partial P}{\partial y}\right|_{(x^*,y^*)} \\ \left.\frac{\partial Q}{\partial x}\right|_{(x^*,y^*)} & \left.\frac{\partial Q}{\partial y}\right|_{(x^*,y^*)} \end{pmatrix} \tag{3.12}$$

has a pair of purely imaginary eigenvalues $\lambda(\mu_0) = \mathrm{i}\omega$ and $\lambda^*(\mu_0) = -\mathrm{i}\omega\cdot$ If

1. $\left.\frac{d}{d\mu}\mathrm{Re}\,\lambda(\mu)\right|_{\mu_0} > 0$ for some μ_0.
2. $P_{\mu x} + Q_{\mu y} \neq 0$, where $P_{\mu x} = \frac{\partial^2 P}{\partial \mu \partial x}$ and $Q_{\mu y} = \frac{\partial^2 Q}{\partial \mu \partial y}$.
3. $a \neq 0$, where

$$\begin{aligned} a = & \frac{1}{16}\left(P_{xxx} + Q_{xxy} + P_{xyy}Q_{yyy}\right) \\ & + \frac{1}{16\omega}\left[P_{xy}\left(P_{xx} + P_{yy}\right) - Q_{xy}\left(Q_{xx} + Q_{yy}\right) - P_{xx}Q_{xx} + P_{yy}Q_{yy}\right] \end{aligned} \tag{3.13}$$

evaluated at (x^*, y^*), then a periodic solution occurs for $\mu < \mu_0$ if $a\left(P_{\mu x} + Q_{\mu y}\right) > 0$ or $\mu > \mu_0$ if $a\left(P_{\mu x} + Q_{\mu y}\right) < 0$. The equilibrium point is stable for $\mu < \mu_0$ and unstable for $\mu > \mu_0$ if

$P_{\mu x} + Q_{\mu y} > 0$. On the other hand, the equilibrium point is stable for $\mu > \mu_0$ and unstable for $\mu < \mu_0$ if $P_{\mu x} + Q_{\mu y} < 0$. In both the cases the periodic solution is stable if the equilibrium points is unstable while it is unstable if the equilibrium point is table on the side of $\mu = \mu_0$ for which the periodic solution exists.

Example

The van der Pol equation (3.9) discussed above has an equilibrium point at $(x^*, y^*) = (0,0)$. The eigenvalues of the Jacobian matrix M are given by:

$$\lambda_{\pm} = \frac{1}{2}\left(\mu \pm \sqrt{\mu^2 - 4}\right).$$

For $-2 < \mu < 2$, the eigenvalues form a complex conjugate pair, while for $|\mu| > 2$, they are real. When $\mu = 0$, the eigenvalues become $\lambda_{\pm} = \pm\mathrm{i}$, indicating pure imaginary values.

1. $\frac{d}{d\mu}\mathrm{Re}\lambda(\mu) > 0$
2. $P_{\mu x} + Q_{\mu y} = 1$
3. $a = -\frac{\mu}{8}$

These satisfy the three conditions for a Hopf bifurcation. Furthermore, the product $a\left(P_{\mu x} + Q_{\mu y}\right) = -\frac{\mu}{8} < 0$, indicating the existence of a stable limit cycle for $0 < \mu < 2$.

The stability of the limit cycle depends on the sign of $P_{\mu x} + Q_{\mu y}$. For the van der Pol oscillator, this quantity is 1 at the equilibrium point $(0,0)$. According to the Hopf bifurcation theorem, for $0 < \mu < 2$, the origin becomes unstable, and the limit cycle solution is stable. This is illustrated in Figure 3.7.

3.2.2 Poincaré; Bendixson Theorem

Consider a two-dimensional system

$$\dot{x} = P(x,y) \tag{3.14}$$
$$\dot{y} = Q(x,y), \tag{3.15}$$

where $(x,y) \in \mathbb{R}^2$, $P : \mathbb{R}^2 \to \mathbb{R}$ and $Q : \mathbb{R}^2 \to \mathbb{R}$.

Theorem 3.2.2 If a solution curve $\mathbb{C} : [x(t), y(t)]$ of the above two-dimensional system remains within a domain of the x-y phase space, where P and Q have continuous first partial derivatives for all $t > \tau$ (for some τ), and does not approach equilibrium points, then a limit cycle exists in the domain. Moreover, $\mathbb{C}$ is either a limit cycle itself or it approaches a limit cycle as $t \to \infty$.

Thus, the possible attractors in two-dimensional systems are *point attractors* and *limit cycle attractors*. No other attractors are generally possible. However, in systems with dimensions greater than two, additional types of motion, such as quasiperiodic or chaotic behavior, can occur.

3.2.3 Bendixson Theorem: Nonexistence of limit cycle motion

The Bendixson theorem for the nonexistence of limit cycles can be stated as follows:

Theorem 3.2.3 For the given system, if $s = \partial P/\partial x + \partial Q/\partial y$ does not change sign and is not identically zero in a region $\mathscr{D}$ of the (x,y)-phase space enclosed by a single closed curve, then no limit cycles exist entirely within $\mathscr{D}$.

Example

For the damped and unforced cubic anharmonic oscillator system:

$$\dot{x} = y \quad (3.16)$$

$$\dot{y} = -dy - x + x^3 \quad (3.17)$$

the quantity $s = -d$ does not change sign in any region of the phase space for a given value of d. According to Bendixson's theorem, this implies that a closed orbit cannot occur in the system described above.

3.2.4 Python code

A sample Python program used to generate Figure 3.8 is provided below:

```
# Python code used to generate Figure 3.8
from numpy import array, pi, arange, matrix
from scipy.integrate import odeint
import matplotlib.pyplot as plt
import matplotlib.gridspec as gridspec

plt.rc('text', usetex=True)
plt.rc('font',family='serif',size='25')

def vanderpol(u, t, b): # defines the vdP system
    xdot = u[1]
    ydot = b[1] * (1. - u[0]*u[0]) * u[1] - b[0] * u[0]
    return (xdot,ydot)

fig = plt.figure(1, figsize=(12, 4))
fig.subplots_adjust(left=0.08, right=0.98, top=0.98, bottom=0.21,
  wspace=0.2)
gs = gridspec.GridSpec(1,2, width_ratios=(0.67, 0.33))
# Figure (a)
ax = fig.add_subplot(gs[0])
#set_axis(ax)
dt = 2.* pi / 100
t = arange(0, 20.*pi + 1e-3, dt)
ic1, b1, line1 = [ 0.5, -1.5], [1.0, -0.25], '--'
ic2, b2, line2 = [ 0.5, -1.5], [1.0,  0.0], '-.'
ic3, b3, line3 = [ 0.5, -1.5], [1.0,  1.0], '-'
for ic, b, line in ((ic1, b1, line1), (ic2, b2, line2), (ic3, b3, line3)):
  sol = odeint(vanderpol, ic, t, args=(b, ) )
  myplot = plt.plot(t, sol[:,0], line, lw=1.)
plt.axhline(y=0, xmin=0, xmax=50, ls=':', color='k')
plt.xlim(0,50)
plt.xlabel('$t$', fontsize='x-large')
plt.ylabel('$x$', fontsize='x-large')
plt.annotate('(a)', xy=(0.0, 0.92), xycoords='axes fraction',
    fontsize='large', xytext=(5, 10), textcoords='offset points',
    color='k', horizontalalignment='left', verticalalignment='top')
# Figure (b)
ax = fig.add_subplot(gs[1])
sol = odeint(vanderpol, ic3, t, args=(b3, ) )
n = 500
```

```python
myplot = plt.plot(sol[n:,0], sol[n:,1], '-', lw=1.)
plt.xlabel('$x$', fontsize='x-large')
plt.ylabel('$\\dot x$', fontsize='x-large', labelpad=-8)
plt.annotate('(b)', xy=(0.0, 0.92), xycoords='axes fraction',
    fontsize='large', xytext=(5, 10), textcoords='offset points',
    color='k', horizontalalignment='left', verticalalignment='top')

# saving figure
plt.savefig('fig-vdp-ts.pdf', bbox_inches='tight')
```

3.3 Review Questions and Problems

Q. 3.1 What is a bifurcation in the context of dynamical systems?

Q. 3.2 What are the primary types of bifurcations in dynamical systems? Provide a brief description of each.

Q. 3.3 Determine the stability of the equilibrium points for $\dot{x} = -rx + x^2$ as r changes and plot the bifurcation diagram. Add a term εx to $\dot{x} = -rx + x^2$ and describe how the bifurcation changes.

Q. 3.4 Analyze the nature of the bifurcation in the system $\dot{x} = -\mu x + x^3$. Draw the bifurcation diagram and indicate the stable and unstable branches. Investigate how a small symmetry-breaking term, i.e., $\dot{x} = -\mu x + x^3 + \varepsilon$, where $\varepsilon \ll 1$, modifies the bifurcation.

Q. 3.5 Define a transcritical bifurcation. How do the stability and number of equilibrium points change as the system undergoes this bifurcation?

Q. 3.6 What is a Hopf bifurcation? How does it relate to the transition from stable fixed points to periodic orbits?

Q. 3.7 For a two-dimensional system, describe how to determine if a Hopf bifurcation occurs at the origin. What is the role of the eigenvalues of the Jacobian matrix?

Q. 3.8 Consider the system $\dot{x} = \mu x - x^3$ and $\dot{y} = \mu y - y^3$, where μ is a bifurcation parameter. Describe the behavior of the system as μ changes. Does a Hopf bifurcation occur?

Q. 3.9 Show that the system

$$\dot{x} = \mu x - \omega y - x(x^2 + y^2), \tag{3.18a}$$

$$\dot{y} = \mu y + \omega x - y(x^2 + y^2), \tag{3.18b}$$

undergoes a Hopf bifurcation at $\mu = 0$. Simulate the above system of equations (3.18) for $\mu < 0$ and $\mu > 0$, and plot the trajectories in the phase plane using Python.

3.4 Reference Books

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