Nonlinear Dynamics

(Lecture Notes)

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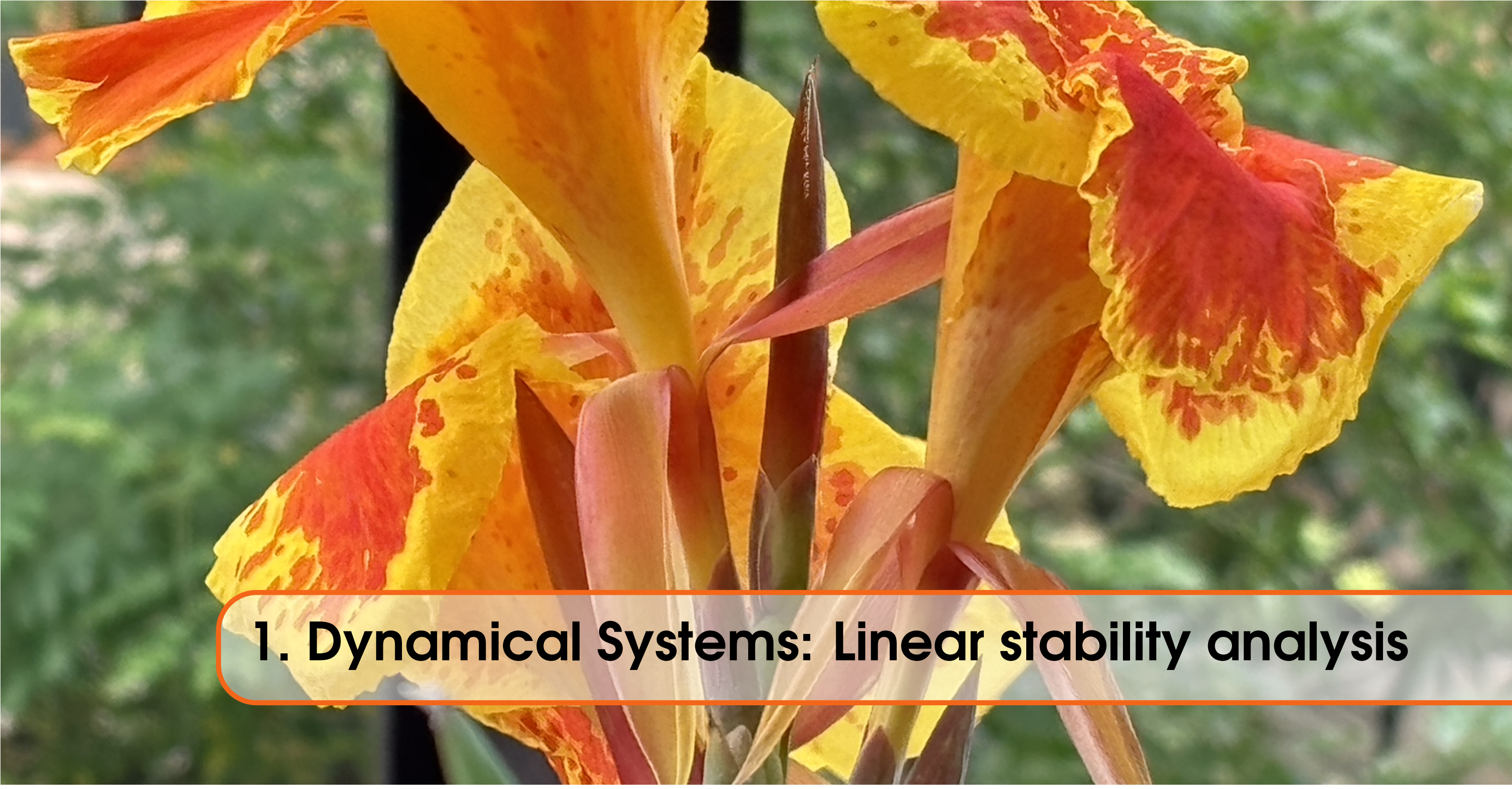

1. Dynamical Systems: Linear stability analysis

1.1 Introduction

The change in the state of physical systems as a function of time is referred to as their ***evolution***, the study of which constitutes the field of ***dynamics***. These changes occur due to the interplay of forces, both simple and complex, acting on the systems.

Dynamical systems are entities (e.g., particles, ensembles of particles, etc.) whose states vary over time. Examples include:

- Physical systems (e.g., linear harmonic oscillator, simple pendulum, Kepler problem, atoms, molecules, etc.)
- Chemical reactions
- Biological systems
- Populations of competing species and ecological systems
- Societal structures and financial markets
- Climate systems

The evolution of different physical systems depends on the nature of the forces acting upon them and their initial states.

Newton's laws form the foundation for describing the evolution of physical systems. Based on these laws, suitable mathematical formulations can be developed in the form of differential equations, either ordinary or partial, or difference equations.

When the forces acting on a system are linear, the system can be described, according to Newton's laws, using linear ordinary differential equations. However, when the forces are nonlinear, they give rise to nonlinear dynamical systems. One can also determine whether the differential equation is linear or nonlinear by examining the total degree of the dependent variables. A differential equation is linear if each term has a total degree of either 0 or 1 in the dependent variable and its derivatives. Even if one term has a degree different from 0 or 1, the equation is nonlinear.

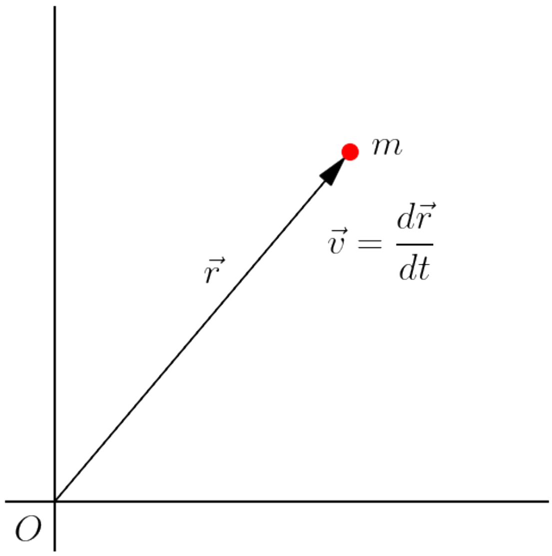

Figure 1.1: Illustration of single particle in Cartesian coordinate system.

1.1.1 Newton's Second Law

An object (or) a particle responds to an external force through a change in its momentum, $\vec{p}$, at a temporal rate $\frac{d\vec{p}}{dt}$, which is exactly equal to the external force, $\vec{F}$. The linear momentum $\vec{p}$ is defined as the product of the object's mass with its velocity, that is,

$$\vec{p} = m\vec{v}. \tag{1.1}$$

The mechanics of the object are *encapsulated* in this law. It states that ***there exist frames of reference in which the motion of the object is governed by the following differential equation:***

$$\vec{F} = \frac{d\vec{p}}{dt} \equiv \dot{\vec{p}}. \tag{1.2}$$

A frame of reference in which the above equation holds is called *an inertial frame* or *a Galilean frame*.

1.1.2 Linear Harmonic Oscillator

Consider a one-dimensional linear harmonic oscillator, where the restoring force is given by $F = -kx$, with $k > 0$ being a constant.

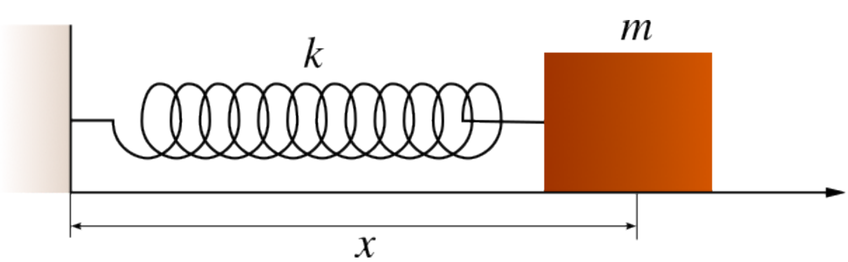

Figure 1.2: Illustration of a spring-mass system (linear harmonic oscillator).

The equation of motion, according to Newton's second law, can be expressed as:

$$\frac{dp}{dt} \equiv m\frac{d^2x}{dt^2} = -kx, \tag{1.3}$$

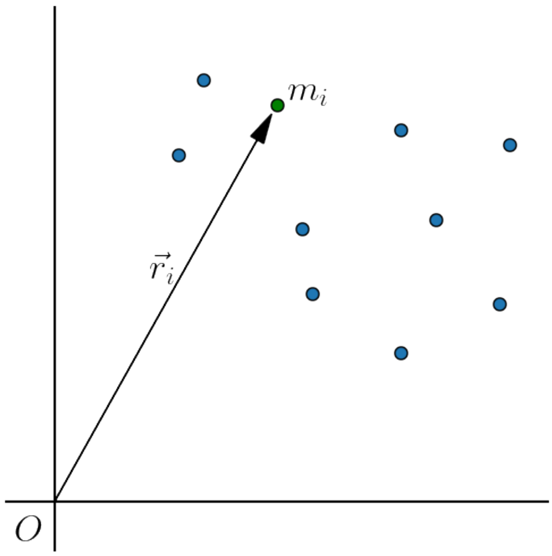

Figure 1.3: Illustration of a system of particles in Cartesian coordinate system.

where m is the mass. The above equation can also be rewritten as:

$$\ddot{x} + \omega_0^2 x = 0, \tag{1.4}$$

where $\omega_0 = \sqrt{k/m}$ is the *natural frequency*. In this expression, the double dot denotes the second derivative with respect to time.

1.1.3 Phase Space

For a system of N particles with masses m_i and position vectors $\vec{r}_i$, $i = 1, 2, \ldots, N$, the dynamics (i.e., the time evolution) are represented by

$$m_i \ddot{\vec{r}}_i = \vec{F}_i \left(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N, \dot{\vec{r}}_1, \dot{\vec{r}}_2, \ldots, \dot{\vec{r}}_N, t \right), \tag{4}$$

where $\vec{F}_i$ is the total force acting on the ith particle of the system. In general, $\vec{F}_i$ depends on the coordinates, velocities, and time. There are $3N$ equations in total.

The space spanned by the coordinates and momenta (or by the state variables) is known as ***phase space***.

In the case of a one-dimensional linear harmonic oscillator the state variables are x and $p = m\dot{x}$. Equation (1.4) is indeed a ***second order linear homogeneous ordinary differential equation*** (ODE).

A general solution to the harmonic oscillator equation (1.4) can be written as

$$x(t) = A \cos \omega_0 t + B \sin \omega_0 t, \tag{1.5}$$

where A and B are arbitrary constants determined by the initial conditions at $t = 0$, namely $x(0)$ and $\dot{x}(0)$.

Figures 1.4(a) and 1.4(b) show the plots of $x(t)$ and $p = m\dot{x}(t)$ for the initial condition $\{x(0), \dot{x}(0)\} = \{0, 1.3\}$, with $m = 1$ and $\omega_0 = 1$. Note that, for the above choice of parameters and initial conditions, the constants A and B are determined as $A = 0$ and $B = 1.3$. The phase space is two-dimensional (***phase plane***) and is represented with x as the abscissa and momentum $p = mv$ as the ordinate, as illustrated in Figure 1.4(c). The solution curve for a given initial condition in the phase plane is called a ***phase trajectory***. The phase trajectory for the initial condition $\{x(0), \dot{x}(0)\} = \{0, 1.3\}$ is a closed curve, shown as the black circle in Figure 1.4(c). The magenta circle corresponds to the initial condition $(x(0), \dot{x}(0)) = (0, 1.7)$.

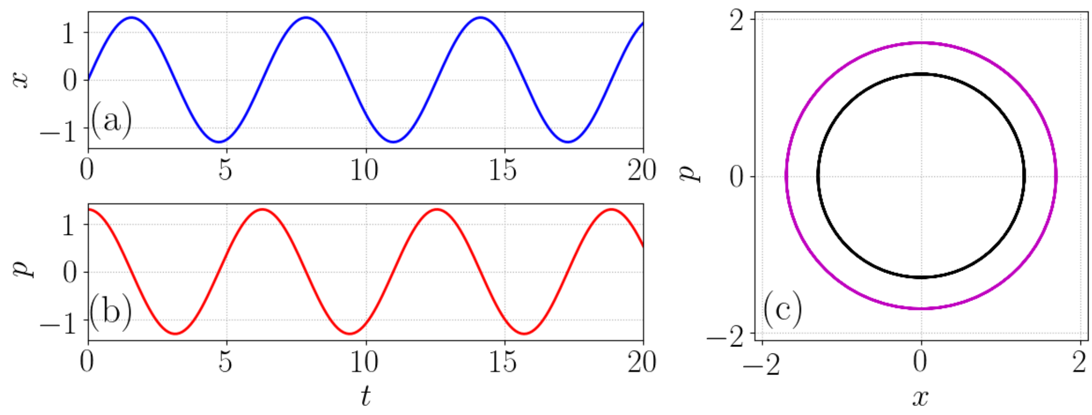

Figure 1.4: Plots (a) coordinate x and (b) momentum p with respect to time (time series), and (c) the phase phase of one-dimensional linear harmonic oscillator (1.4) with $m = 1$, $\omega_0 = \sqrt{k/m} = 1$, and with initial conditions $(x(0), \dot{x}(0)) = (0, 1.3)$ (black) and $(0, 1.7)$ (magenta)

1.1.4 The Simple Pendulum

Consider a pendulum placed in an air medium (see Figure below). The restoring force is proportional

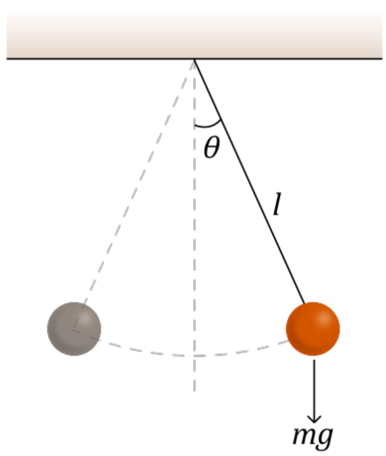

Figure 1.5: Illustration of a Simple Pendulum

to $\sin\theta$, which is nonlinear in θ. The equation of motion is given by

$$\ddot{\theta} + \alpha\dot{\theta} + \frac{g}{l}\sin\theta = 0, \tag{1.6}$$

where α is the damping coefficient. For small displacements, $\sin\theta \approx \theta$, and the pendulum behaves as a linear system. Under this approximation, the equation of motion simplifies to a linear differential equation:

$$\ddot{\theta} + \alpha\dot{\theta} + \frac{g}{l}\theta = 0. \tag{1.7}$$

When the pendulum bob is disturbed from its equilibrium position, the amplitude of oscillation decreases over time due to damping, eventually bringing the system to rest.

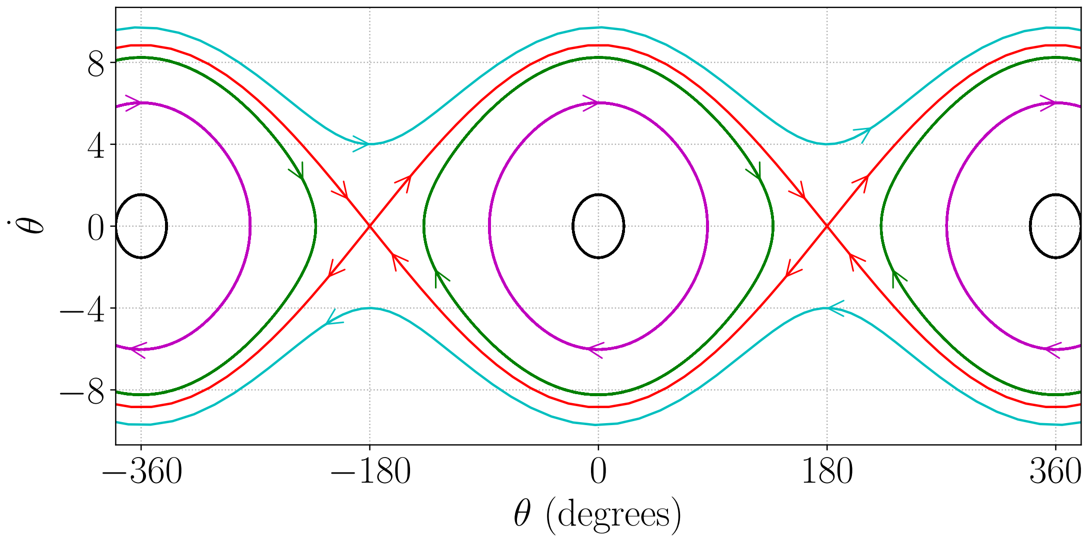

Figure 1.6: Phase portraits (phase trajectories in phase space) of a pendulum with $\alpha = 0$, $g = 9.8$ ms^{-2} and $l = 0.5$ m.

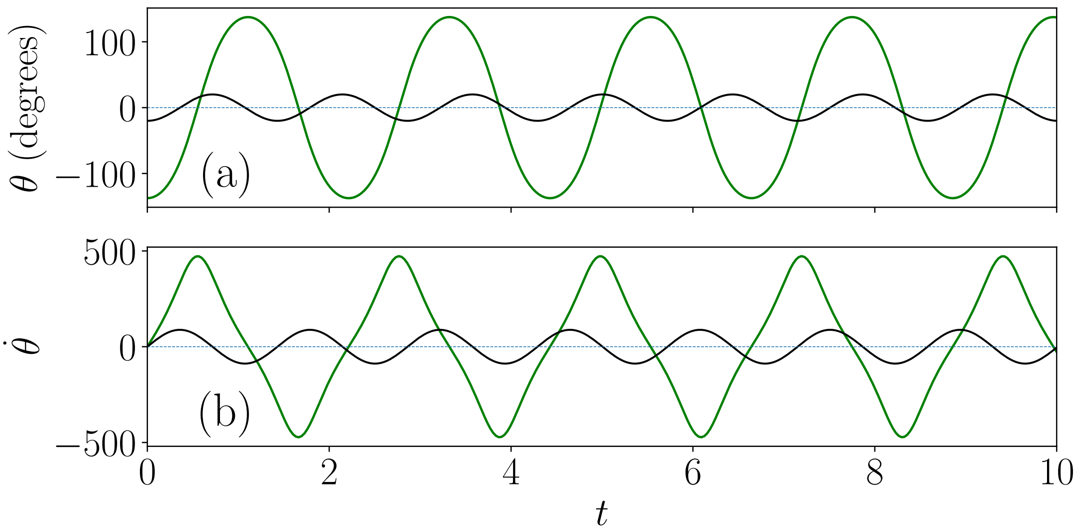

Figure 1.7: Time series plots showing (a) θ and (b) $\dot{\theta}$ for an undamped pendulum ($\alpha = 0$). The parameters are $g = 9.8\,\mathrm{m/s^2}$ and $l = 0.5$ m, with initial conditions $\theta, \dot{\theta} = (-20.0, 0)$ (black) and $(-135, 0)$ (green). Note that the frequencies (or periods) of oscillation differ for the two initial conditions, illustrating the ***amplitude-dependent frequency characteristic of nonlinear systems***. Additionally, for small θ, the waveform is nearly sinusoidal, while at larger amplitudes, it deviates significantly from standard sin or cos profiles.

1.2 Linear Stability Analysis

The dynamics of a particle in one dimension subjected to an external force can be represented by the following set of ordinary differential equations:

$$\dot{x}_1 = f_1(x_1, x_2), \tag{1.8a}$$
$$\dot{x}_2 = f_2(x_1, x_2), \tag{1.8b}$$

where $(x_1, x_2) \in \mathbb{R}^2$, $f_1 : \mathbb{R}^2 \to \mathbb{R}$, and $f_2 : \mathbb{R}^2 \to \mathbb{R}$.

In the case of a simple pendulum (1.7), the above equations can be written as

$$\dot{\theta} = \omega \equiv f_1(\theta, \omega), \tag{1.9a}$$

$$\dot{\omega} = -\alpha\omega - \frac{g}{l}\sin\theta \equiv f_2(\theta, \omega), \tag{1.9b}$$

where $x_1 = \theta$ and $x_2 = \omega$.

1.2.1 Equilibrium Point

One can often identify a time-independent solution (steady state) to equations (1.9a) and (1.9b), such as (x_1^*, x_2^*), where $f_1(x_1^*, x_2^*) = 0$ and $f_2(x_1^*, x_2^*) = 0$. Such a solution is referred to as an ***equilibrium point*** or a ***fixed point***. It is sometimes also called a ***singular point***.

To determine the stability of the equilibrium point, we introduce an infinitesimal perturbation near the equilibrium point. For instance,

$$x_1 = x_1^* + \varepsilon x_1', \tag{1.10a}$$

$$x_2 = x_2^* + \varepsilon x_2', \tag{1.10b}$$

where $\varepsilon \ll 1$.

To analyze the resulting dynamics, we perform a Taylor expansion about the equilibrium point:

$$\begin{aligned} f_1(x_1, x_2) &= f_1(x_1^* + \varepsilon x_1', x_2^* + \varepsilon x_2') \\ &= f_1(x_1^*, x_2^*) + \varepsilon \left.\frac{\partial f_1}{\partial x_1}\right|_{x_1^*, x_2^*} x_1' + \varepsilon \left.\frac{\partial f_1}{\partial x_2}\right|_{x_1^*, x_2^*} x_2' + \mathscr{O}(\varepsilon^2). \end{aligned} \tag{1.11}$$

Here, $\left.\frac{\partial f_1}{\partial x_1}\right|_{x_1^*, x_2^*}$ and $\left.\frac{\partial f_1}{\partial x_2}\right|_{x_1^*, x_2^*}$ denote the partial derivatives evaluated at the equilibrium point (x_1^*, x_2^*). Similarly, we can expand $f_2(x_1, x_2)$ as

$$\begin{aligned} f_2(x_1, x_2) &= f_2(x_1^* + \varepsilon x_1', x_2^* + \varepsilon x_2') \\ &= f_2(x_1^*, x_2^*) + \varepsilon \left.\frac{\partial f_2}{\partial x_1}\right|_{x_1^*, x_2^*} x_1' + \varepsilon \left.\frac{\partial f_2}{\partial x_2}\right|_{x_1^*, x_2^*} x_2' + \mathscr{O}(\varepsilon^2), \end{aligned} \tag{1.12}$$

Because $f_1(x_1^*, x_2^*) = f_2(x_1^*, x_2^*) = 0$, and the higher-order terms $\mathscr{O}(\varepsilon^2)$ on the right-hand sides can be neglected in most cases, equations (1.9a) and (1.9b) can be approximated as a system of two coupled first-order linear differential equations:

$$\dot{x}_1' = a x_1' + b x_2', \tag{1.13a}$$

$$\dot{x}_2' = c x_1' + d x_2', \tag{1.13b}$$

where

$$a = \left.\frac{\partial f_1}{\partial x_1}\right|_{(x_1^*, x_2^*)}, \quad b = \left.\frac{\partial f_1}{\partial x_2}\right|_{(x_1^*, x_2^*)}, \quad c = \left.\frac{\partial f_2}{\partial x_1}\right|_{(x_1^*, x_2^*)}, \quad \text{and} \quad d = \left.\frac{\partial f_2}{\partial x_2}\right|_{(x_1^*, x_2^*)} \tag{1.14}$$

The above equations (1.13) can be equivalently expressed in matrix form as:

$$\begin{pmatrix} \dot{x}_1' \\ \dot{x}_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} \equiv \mathbf{M} \begin{pmatrix} x_1' \\ x_2' \end{pmatrix}, \tag{1.15}$$

where $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is called the ***Jacobian matrix***.

Differentiating (1.13a) with respect to t and eliminating x_2' from it, we obtain a second-order differential equation in x_1':

$$\ddot{x}_1' - (a+d)\dot{x}_1' + (ad-bc)x_1' = 0. \tag{1.16}$$

A formal solution to the above can be expressed as

$$x_1'(t) = A\exp(\lambda_1 t) + B\exp(\lambda_2 t), \tag{1.17}$$

where A and B are integration constants, and

$$\lambda_{1,2} = \frac{1}{2}\left[(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}\right], \tag{1.18}$$

are the eigenvalues of the linear system (1.13a) and (1.13b) or that of the Jacobian matrix $\mathbf{M}$. Note that the eigenvalues λ_1 and λ_2 may, in general, be complex.

Substituting (1.17) into (1.13a), we obtain

$$x_2' = C\exp(\lambda_1 t) + D\exp(\lambda_2 t), \quad \text{where} \quad C = \frac{A(\lambda_1 - a)}{b}, \quad \text{and} \quad D = \frac{B(\lambda_2 - a)}{b}. \tag{1.19}$$

- **Stable**: Both eigenvalues have negative real parts.
- **Unstable**: At least one eigenvalue has a positive real part.
- **Neutral**: Both eigenvalues have zero real parts.

1.2.2 Classification of Equilibrium (Singular) Points

Based on the above criteria for determining the stability or instability of equilibrium points, the following broad classification can be made, depending on the nature of the eigenvalues.

Case 1: $\lambda_1 \le \lambda_2 < 0$ – stable node/star

When both eigenvalues are real and negative (less than zero), it follows that $x_1'(t) \to 0$ and $x_2'(t) \to 0$ as $t \to \infty$. Consequently, any trajectory starting in the neighborhood of the equilibrium point (x_1^*, x_2^*) in the (x_1, x_2) phase plane approaches this point exponentially in the long time (asymptotic) limit .

To analyze how the trajectories approach the equilibrium point in the (x_1, x_2) phase plane, we examine the slope $\dfrac{dx_2}{dx_1}$ of the trajectories.

Shifting the origin to the equilibrium point (x_1^*, x_2^*) for convenience (which involves replacing $x_1 \to x_1 - x_1^*$ and $x_2 \to x_2 - x_2^*$ in (1.10), and then rewriting the equations for the new variables (x_1, x_2)), we obtain the following from (1.19) and (1.17):

$$\frac{dx_2}{dx_1} = \frac{C\lambda_1 e^{(\lambda_1 - \lambda_2)t} + D\lambda_2}{A\lambda_1 e^{(\lambda_1 - \lambda_2)t} + B\lambda_2} \tag{1.20}$$

Here we can distinguish two cases: (i) $\lambda_1 = \lambda_2 = \lambda$ and (ii) $\lambda_1 \neq \lambda_2$.

(i) When $\lambda_1 = \lambda_2$ we have

$$\frac{dx_2}{dx_1} = \frac{(C+D)\lambda}{(A+B)\lambda} = \text{constant} = m. \tag{1.21}$$

Integrating the above yields

$$x_2 = mx_1 + c_0 \tag{1.22}$$

where c_0 is a constant. Using the fact that the origin is a fixed point–since we have shifted the fixed point to the origin through the transformations $x_1 \to x_1 - x_1^*$ and $x_2 \to x_2 - x_2^*$–we may choose $c_0 = 0$. Consequently, equation (1.22) represents a straight line passing through the origin (the fixed point). Therefore, in the $x_1 - x_2$ phase space, trajectories approach the equilibrium point along straight-line paths, as illustrated in Figure 1.8(a). An equilibrium point of this type is called a ***stable star*** (due to its starlike structure).

(ii) For $\lambda_1 < \lambda_2$, the slope of the trajectories varies with time and decreases exponentially to the value D/B, since $\lambda_1 < \lambda_2$. It can be shown that the trajectories in the (x_1, x_2)-plane approach the equilibrium point along parabolic paths. Figure 1.8(b) illustrates the phase trajectories in the neighborhood of the equilibrium point after a suitable rotation of the coordinate axes. This type of equilibrium point is called a ***stable node***.

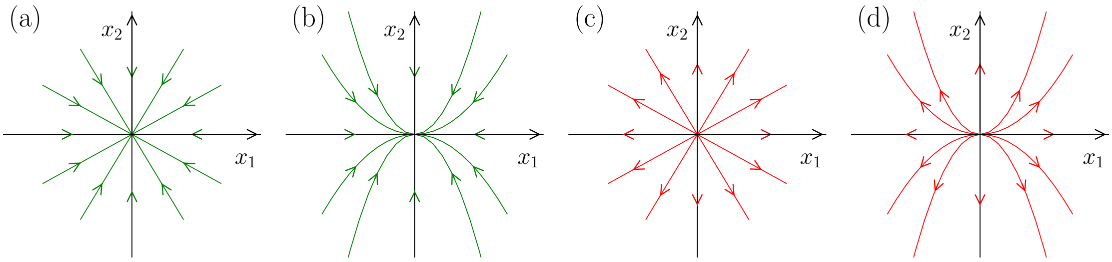

Figure 1.8: Classification of equilibrium points: Phase trajectories near the equilibrium points: (a) stable star, (b) stable node, (c) unstable star, and (d) unstable node. In all the plots, the equilibrium point is located at the origin.

Case 2: $\lambda_1 \geq \lambda_2 > 0$ – unstable node/star

Next, when λ_1 and λ_2 are real and positive, $x_1(t) \to 0$ as $t \to \infty$. Therefore, the trajectories starting from a neighbourhood of the equilibrium point diverge from it, and the equilibrium point is unstable. For $\lambda_1 = \lambda_2 = \lambda$, the slope dy/dx is given again by (1.21), and the trajectories now diverge along straight lines. The equilibrium point is referred to as an ***unstable star***. When $\lambda_1 \neq \lambda_2$, it is classified as an ***unstable node***. The corresponding phase trajectories are illustrated in Figures 1.8(c) and 1.8(d).

Case 3: λ_1, λ_2 complex conjugates - stable/unstable focus

Let λ_1 and λ_2 be complex conjugates given by

$$\lambda_1 = \alpha + i\beta, \quad \lambda_2 = \alpha - i\beta, \tag{1.23}$$

where α and β are real constants with $\beta > 0$. Now the solution for this case becomes

$$x_1(t) = |A|e^{\alpha t}\cos(\beta t + \phi), \tag{1.24a}$$
$$x_2(t) = |D|e^{\alpha t}\sin(\beta t + \phi'), \tag{1.24b}$$

where $A = A_R + iA_I$, $D = D_R + iD_I$, $\phi = \arctan\left(\frac{A_I}{A_R}\right)$, and $\phi' = \arctan\left(\frac{D_I}{D_R}\right)$. In the above $A = B^*$ is an integration constant while $D = C^*$ given by Eq. (1.19).

For $\alpha < 0$, both $x_1 \to 0$ and $x_2 \to 0$ as $t \to \infty$. Hence, the equilibrium point is stable. Figure 1.9(a) illustrates the trajectories near the equilibrium point. These trajectories spiral around the equilibrium point several times before converging to it asymptotically. In this scenario, the equilibrium point is referred to as a ***stable spiral point*** or ***stable focus***. Conversely, when $\alpha > 0$, the trajectories diverge from the equilibrium point along spiral paths [see Figure 1.9(b)]. In this case, the equilibrium point is classified as an ***unstable focus***.

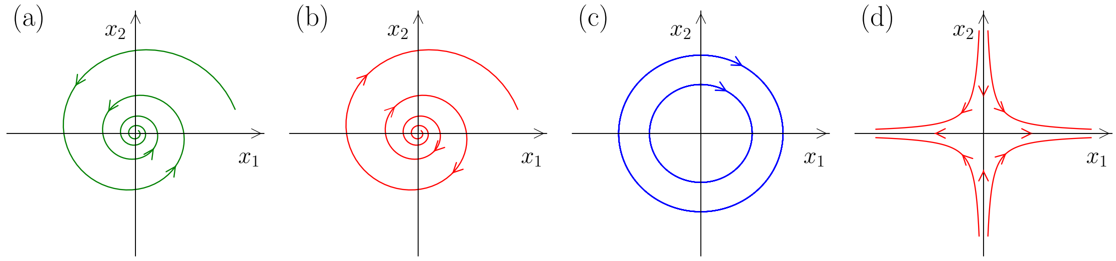

Figure 1.9: Classification of equilibrium points: Phase trajectories near the equilibrium points: (a) stable focus, (b) unstable focus, (c) center, and (d) saddle (hyperbolic). In all the plots, the equilibrium point is located at the origin.

Case 4: λ_1, λ_2 pure imaginary (complex conjugates) - center/elliptic

In this case the solution becomes

$$x_1(t) = |A|\cos(\beta t + \phi), \tag{1.25a}$$
$$x_2(t) = |D|\sin(\beta t + \phi'), \tag{1.25b}$$

The perturbation neither decays to zero nor diverges to infinity; instead, it varies periodically with time. In this case, the trajectories form closed orbits around the equilibrium point, as shown in Figure 1.9(c). The trajectories do not approach the equilibrium point as $t \to \infty$. This type of equilibrium point is called a ***center type*** or ***elliptic equilibrium point*** and is neutrally stable, i.e., neither stable nor unstable.

Case 5: $\lambda_1 < 0 < \lambda_2$ saddle or hyperbolic

For $\lambda_1 < 0$ and $\lambda_2 > 0$ (or vice versa), the first terms in equations (1.17) and (1.19) approach zero as $t \to \infty$, while the second terms diverge to infinity as $t \to \infty$. When both terms are considered, we have $|x_1|, |x_2| \to \infty$ as $t \to \infty$, and the solution curves are hyperbolic.

However, for certain initial conditions, specifically when $B = D = 0$, we have $|x_1|, |x_2| \to 0$ as $t \to \infty$, and hence the trajectories approach the equilibrium point. Figure 1.9(d) illustrates the trajectories in the vicinity of the equilibrium point.

In Figure 1.9(d), we observe that trajectories reach the equilibrium point along two specific directions only, while in all other directions, the trajectories diverge from it. Therefore, we can conclude that, in general, the trajectories diverge from the equilibrium point. This type of equilibrium point is called a ***saddle point***, which is inherently unstable. It is also called a ***hyperbolic equilibrium point***.

1.2.3 The Pendulum Example

Let us again consider the pendulum case (1.6). The equilibrium points are obtained by solving

$$\dot{\theta} = \omega = 0, \tag{1.26}$$
$$\dot{\omega} = -\alpha\omega - \frac{g}{l}\sin\theta = 0, \tag{1.27}$$

which gives the equilibrium points as

$$(\theta^*, \omega^*) = (n\pi, 0), \;\; n = 0, \pm 1, \pm 2, \ldots \tag{1.28}$$

The stability of (θ^*, ω^*) is determined by linearizing (1.7) in the neighborhood of the equilibrium points. For this purpose, we assume the solutions in the form $\theta(t) = \theta^* + \varepsilon\theta'(t)$ and $\omega(t) =$

$\omega^* + \varepsilon\omega'(t)$. The linear system is then given by

$$\dot{\theta}' = a\theta' + b\omega', \tag{1.29a}$$

$$\dot{\omega}' = c\theta' + d\omega', \tag{1.29b}$$

where $a = 0$, $b = 1$, $c = -\frac{g}{l}\cos\theta^*$, and $d = -\alpha$. Or in matrix form

$$\begin{pmatrix} \dot{\theta}' \\ \dot{\omega}' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{g}{l}\cos\theta^* & -\alpha \end{pmatrix} \begin{pmatrix} \theta' \\ \omega' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{g}{l}(-1)^n & -\alpha \end{pmatrix} \begin{pmatrix} \theta' \\ \omega' \end{pmatrix} \equiv M \begin{pmatrix} \theta' \\ \omega' \end{pmatrix}, \tag{1.30}$$

where $\theta^* = n\pi$, $n = 0, \pm 1, \pm 2, \ldots$ is used.

The eigenvalues of M that determine the stability are obtained by solving

$$\lambda^2 + \alpha\lambda + \frac{g}{l}(-1)^n = 0, \tag{22}$$

and are given by

$$\lambda_{1,2} = \frac{1}{2}\left[-\alpha \pm \sqrt{\alpha^2 - \frac{4g}{l}(-1)^n}\right]. \tag{23}$$

- If we consider the equilibrium point $(\theta^*, \omega^*) = (0, 0)$, and assume $\alpha = 0$ (no damping) and $g/l = 1$, the eigenvalues are $\lambda_{1,2} = \pm\mathrm{i}$, where $\mathrm{i} = \sqrt{-1}$. Thus, the equilibrium point $(0, 0)$ is classified as a *center* (or *neutral point*).

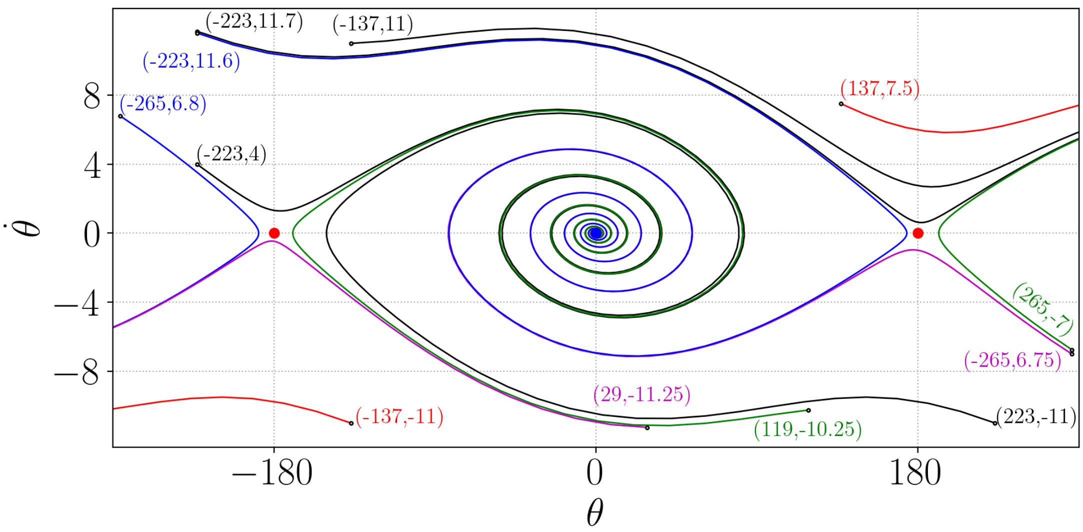

Figure 1.10: Phase portraits (phase trajectories in phase space) of a pendulum illustrating the stability of the equilibrium points, with $\alpha = 1.0\,\mathrm{s}^{-1}$, $g = 9.8\,\mathrm{m/s}^2$, and $l = 0.5\,\mathrm{m}$. The values in parentheses are the starting values (initial conditions) of $(\theta, \dot{\theta})$ for the corresponding trajectory.

- With $\alpha = 0$ and $g/l = 1$, the other equilibrium point $(\theta^*, \omega^*) = (0, \pi)$ for $n = 1$ has eigenvalues $\lambda_{1,2} = \pm 1$. Since one of the eigenvalues has a positive real part, the equilibrium point is *unstable*. However, because the other eigenvalue is negative, the equilibrium point $(0, \pi)$ is also classified as a *saddle point*.

Let us now attempt to draw the phase trajectories of the pendulum by considering damping $(\alpha \neq 0)$.

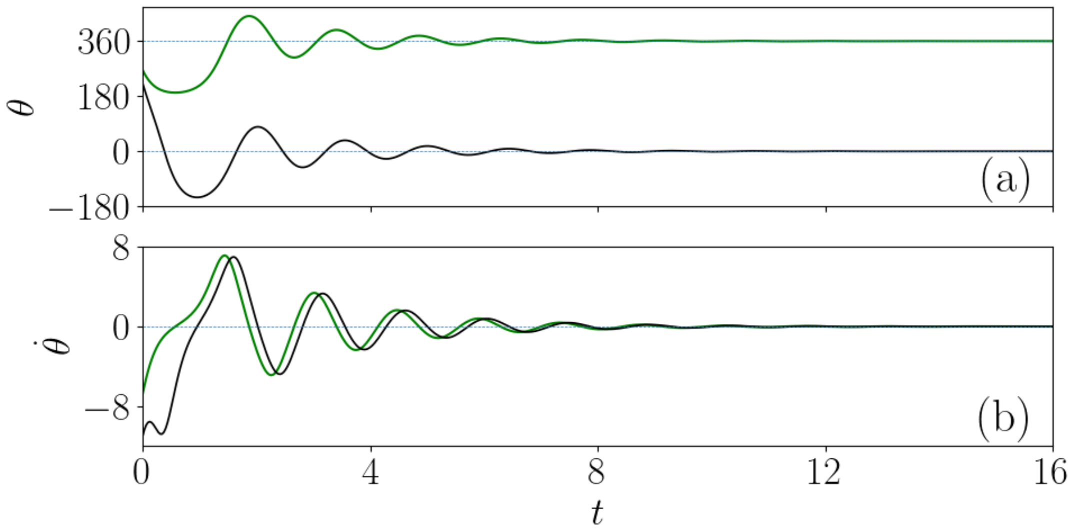

Figure 1.11: Time series plots illustrating damped oscillations of (a) θ and (b) $\dot{\theta}$ in a pendulum with $\alpha = 1.0\ \mathrm{s}^{-1}$, $g = 9.8\ \mathrm{m/s}^2$, and $l = 0.5$ m for two different initial conditions: $(265, -6.75)$ (green line) and $(223, -11)$ (black line). These conditions asymptotically approach the stable equilibrium points $(\theta^*, \dot{\theta}^*) = (0,0)$ and $(2\pi, 0) = (360°, 0)$, respectively.

1.2.4 An Anharmonic Oscillator

Consider a particle of mass m is subjected to a nonlinear spring force $F = -kx + \tilde{\beta}x^3$ (nonlinear stiffness). According to Newton's law, the equation of motion can be written as

$$m\ddot{x} = -kx + \tilde{\beta}x^3$$

The above second order ODE can be written as a set of two first order ODEs

$$\dot{x} = y,$$
$$\dot{y} = -\omega_0^2 x + \beta x^3, \quad \text{where} \quad \omega_0^2 = \frac{k}{m}, \quad \text{and} \quad \beta = \frac{\tilde{\beta}}{m}.$$

The equilibrium points are obtained by setting $\dot{x} = 0$ and $\dot{y} = 0$, and are

$$(x^*, y^*) = (0,0), \quad \left(\sqrt{\frac{\omega_0^2}{\beta}}, 0\right), \quad \text{and} \quad \left(-\sqrt{\frac{\omega_0^2}{\beta}}, 0\right).$$

To determine the stability, we need to linearize the above first order ODEs in the neighbourhood of the equilibrium points. This is done by assuming $x = x^* + \varepsilon x'$ and $y = y^* + \varepsilon y'$ $(\varepsilon \ll 1)$. Taylor expanding and dropping higher order terms in ε results, we get

$$\dot{x}' = y',$$
$$\dot{y}' = -\omega_0^2 x' + 3\beta x^{*2} x'.$$

Or in matrix form

$$\begin{pmatrix} \dot{x}' \\ \dot{y}' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega_0^2 + 3\beta x^{*2} & 0 \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} \equiv M \begin{pmatrix} x' \\ y' \end{pmatrix}.$$

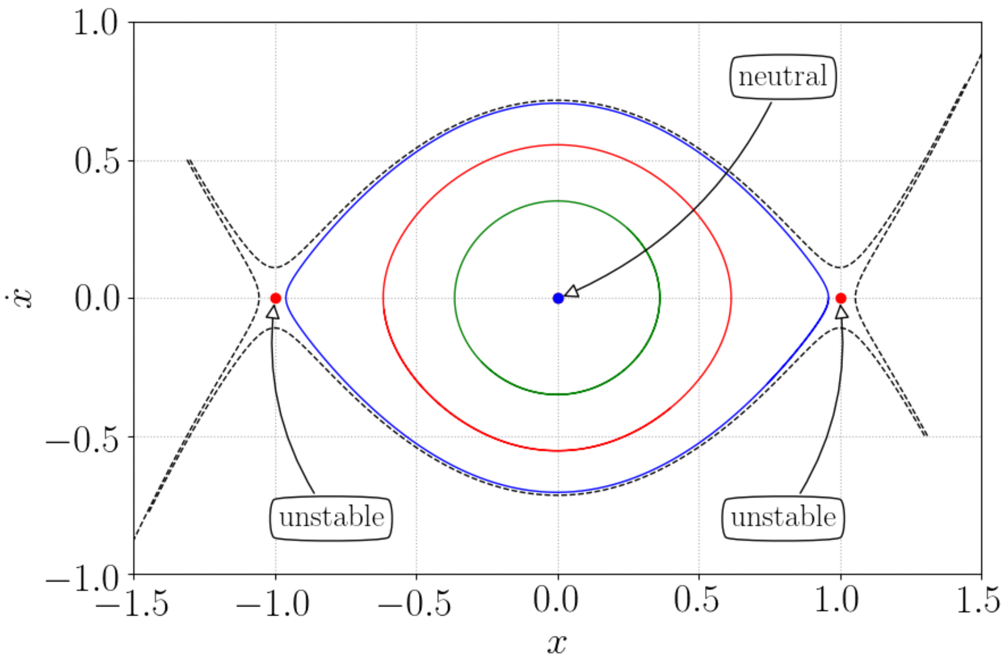

Figure 1.12: Phase portraits (phase trajectories in phase space) of a nonlinear spring illustrating the stability of the equilibrium points for $\omega_0^2 = 1$ and $\beta = 1$ ($k = 1$, $\tilde{\beta} = 1$, and $m = 1$ in the original problem). The equilibrium points are $(0,0)$ (a center or neutral), and $(1,0)$, $(-1,0)$ (both unstable - saddle points). The black dashed lines represent unstable trajectories.

The stability determining eigenvalues are obtained by solving

$$|M - \lambda I| = 0 \implies \lambda^2 + \omega_0^2 - 3\beta x^{*2} = 0.$$

This gives

$$\lambda_{1,2} = \pm\sqrt{3\beta x^{*2} - \omega_0^2}.$$

Thus, the equilibrium point $(0,0)$ has a pair of purely imaginary eigenvalues, $\lambda_{1,2} = \pm i\omega_0$, when $\omega_0^2 > 0$. Therefore, it is classified as *neutral* (represented by solid lines in Figure 1.12 above). In contrast, the other two equilibrium points, $\left(\sqrt{\frac{\omega_0^2}{\beta}}, 0\right)$ and $\left(-\sqrt{\frac{\omega_0^2}{\beta}}, 0\right)$, have eigenvalues $\lambda_{1,2} = \pm\sqrt{2}\omega_0$ and are *unstable saddle points*. Figure 1.12 illustrates the phase trajectories of the nonlinear spring model, with the unstable trajectories indicated by dashed lines.

1.2.5 Damped Anharmonic Oscillator (Double Well Potential Case)

The equation of motion of the damped anharmonic oscillator (in dimensionless form) is given by

$$\ddot{x} + \alpha\dot{x} - \omega_0^2 x + \beta x^3 = 0,$$

where α, β and ω_0 are positive constants. which can be written as a set of first order ODEs of the form

$$\dot{x} = y,$$
$$\dot{y} = \omega_0^2 x - \alpha y - \beta x^3,$$

The equilibrium points are

$$(x^*, y^*) = (0, 0), \quad \left(+\sqrt{\frac{\omega_0^2}{\beta}}, 0\right), \text{ and, } \left(-\sqrt{\frac{\omega_0^2}{\beta}}, 0\right),$$

The stability of these equilibrium points are determined by the eigenvalues of the following matrix

$$M = \begin{pmatrix} 0 & 1 \\ \omega_0^2 - 3\beta x^{*2} & -\alpha \end{pmatrix},$$

which is given by

$$\lambda^2 + \alpha\lambda - \left(\omega_0^2 - 3\beta x^{*2}\right) = 0.$$

The eigenvalues are

$$\lambda_{1,2} = \frac{1}{2}\left[-\alpha \pm \sqrt{\alpha^2 + 4\left(\omega_0^2 - 3\beta x^{*2}\right)}\right].$$

- The equilibrium point $(0,0)$ has eigenvalues $\lambda_{1,2} = \frac{1}{2}\left(-\alpha \pm \sqrt{\alpha^2 + 4\omega_0^2}\right)$.
  - Since $\sqrt{\alpha^2 + 4\omega_0^2}$ is always greater than α, we have $\lambda_1 > 0$ and $\lambda_2 < 0$, and therefore the equilibrium point $(0,0)$ is *unstable* (*saddle or hyperbolic fixed point*).
- The other two equilibrium points $\left(\pm\sqrt{\frac{\omega_0^2}{\beta}}, 0\right)$ have eigenvalues $\lambda_{1,2} = \frac{1}{2}\left[-\alpha \pm \sqrt{\alpha^2 - 8\omega_0^2}\right]$.
  - For $\alpha^2 > 8\omega_0^2$ both the eigenvalues are negative and the equilibrium points are *stable* (*stable node*).
  - For $\alpha^2 < 8\omega_0^2$, $\lambda_{1,2} = \frac{1}{2}\left[-\alpha \pm i\sqrt{|\alpha^2 - 8\omega_0^2|}\right]$. The eigenvalues are complex conjugates with negative real parts, and they are *stable* (*stable focus*).
  - When $\alpha^2 = 8\omega_0^2$, the eigenvalues are negative real and identical (*stable star*).

The phase trajectories in phase space are shown in Fig. 1.13. The Python code used to generate this figure is provided below.

1.2.6 Python Code

```python
# Python code used to generate Figure 1.13
from numpy import *
from scipy.integrate import odeint
from matplotlib import *
from matplotlib.pyplot import *
import matplotlib.gridspec as gspec

rc('text', usetex=True) # to use LaTeX for math symbols in plots
rc('font', family='serif', size='25') # to set font size

def cubic(u, t, b): # defines the system of odes
    xdot = u[1]
    ydot = (b[0] - b[2] * u[0] * u[0]) * u[0] - b[1] * u[1]
    return (xdot, ydot)

```

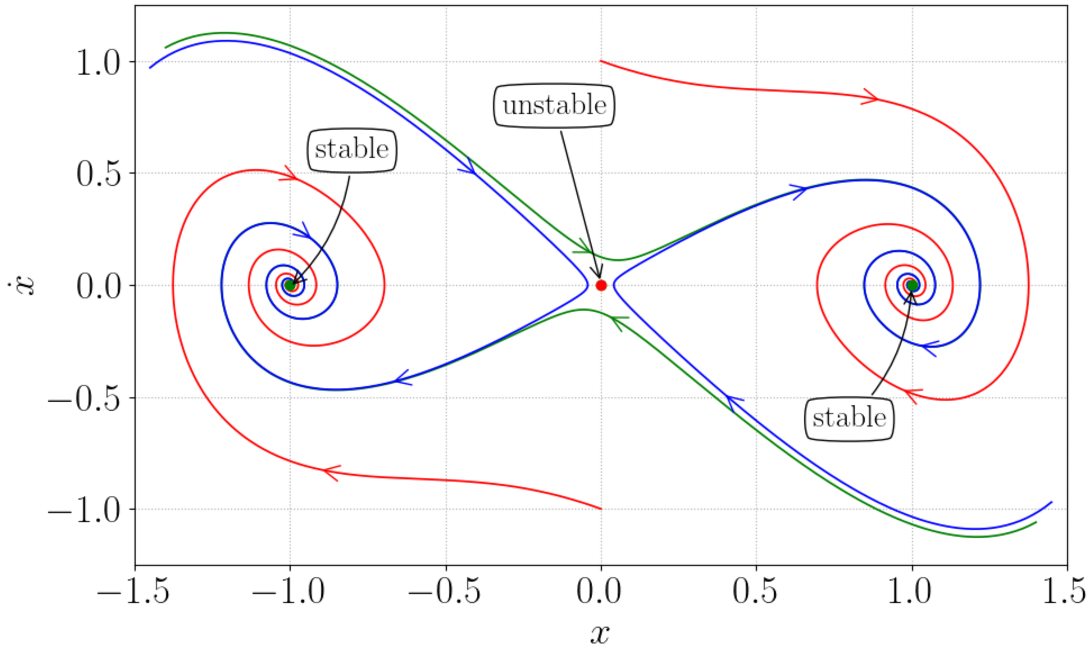

Figure 1.13: Phase trajectories in the phase space of an anharmonic oscillator illustrating the stability of its equilibrium points. The parameters are $\omega_0^2 = 1$, $\alpha = 0.5$, and $\beta = 1$. The equilibrium point $(0,0)$, denoted by a red circle, is unstable (a saddle point), while the other two equilibrium points $(\pm 1, 0)$, marked by green circles, are stable (stable focus).

```
b = [1., 0.5, 1.]

dt = 1.0 / 100
t = arange(0, 25.01, dt)
t1 = arange(0, 8.51, dt)
t2 = arange(0, 15.01, dt)
t3 = arange(0, 4.01, dt)

nm = 0

xx = array([-1, 0, 1])
yy = array([0, 0, 0])

u10 = [1.4, -1.06]
u1 = odeint(cubic, u10, t, (b,))

u20 = [-1.4, 1.06]
u2 = odeint(cubic, u20, t, (b,))

u30 = [1.45, -0.97]
u3 = odeint(cubic, u30, t, (b,))

u40 = [-1.45, 0.97]
u4 = odeint(cubic, u40, t, (b,))

u50 = [0., 1]
u5 = odeint(cubic, u50, t, (b,))
```

```
u60 = [0., -1]
u6 = odeint(cubic, u60, t, (b,))

label =['(a)', '(b)', '(c)']
pos = [(-5, 10), (-5, 10), (15, -25)]

fig = figure(1, figsize=(10, 6))
fig.subplots_adjust(left=0.13, right=0.96, top=0.96, bottom=0.13)
gs = gspec.GridSpec(1,1)

sp1 = Subplot(fig, gs[0])
fig1 = fig.add_subplot(sp1)
plot(u1[nm:,0], u1[nm:,1], 'g-', lw=1.25)
plot(u2[nm:,0], u2[nm:,1], 'g-', lw=1.25)
plot(u3[nm:,0], u3[nm:,1], 'b-', lw=1.25)
plot(u4[nm:,0], u4[nm:,1], 'b-', lw=1.25)
plot(u5[nm:,0], u5[nm:,1], 'r-', lw=1.25)
plot(u6[nm:,0], u6[nm:,1], 'r-', lw=1.25)

n1, n2 = nm+240, nm+241
annotate(text='', xy=(u1[n1,0], u1[n1,1]), xytext=(u1[n2,0], u1[n2,1]),
  arrowprops=dict(arrowstyle='<-', color='g'))
annotate(text='', xy=(u2[n1,0], u2[n1,1]), xytext=(u2[n2,0], u2[n2,1]),
  arrowprops=dict(arrowstyle='<-', color='g'))

n1, n2 = nm+120, nm+121
annotate(text='', xy=(u3[n1,0], u3[n1,1]), xytext=(u3[n2,0], u3[n2,1]),
  arrowprops=dict(arrowstyle='<-', color='b'))
annotate(text='', xy=(u4[n1,0], u4[n1,1]), xytext=(u4[n2,0], u4[n2,1]),
  arrowprops=dict(arrowstyle='<-', color='b'))

n1, n2 = nm+800, nm+801
annotate(text='', xy=(u3[n1,0], u3[n1,1]), xytext=(u3[n2,0], u3[n2,1]),
  arrowprops=dict(arrowstyle='<-', color='b'))
annotate(text='', xy=(u4[n1,0], u4[n1,1]), xytext=(u4[n2,0], u4[n2,1]),
  arrowprops=dict(arrowstyle='<-', color='b'))

n1, n2 = nm+1060, nm+1061
annotate(text='', xy=(u3[n1,0], u3[n1,1]), xytext=(u3[n2,0], u3[n2,1]),
  arrowprops=dict(arrowstyle='<-', color='b'))
n1, n2 = nm+1100, nm+1101
annotate(text='', xy=(u4[n1,0], u4[n1,1]), xytext=(u4[n2,0], u4[n2,1]),
  arrowprops=dict(arrowstyle='<-', color='b'))

n1, n2 = nm+100, nm+101
annotate(text='', xy=(u5[n1,0], u5[n1,1]), xytext=(u5[n2,0], u5[n2,1]),
  arrowprops=dict(arrowstyle='<-', color='r'))
annotate(text='', xy=(u6[n1,0], u6[n1,1]), xytext=(u6[n2,0], u6[n2,1]),
  arrowprops=dict(arrowstyle='<-', color='r'))

n1, n2 = nm+300, nm+301
annotate(text='', xy=(u5[n1,0], u5[n1,1]), xytext=(u5[n2,0], u5[n2,1]),
  arrowprops=dict(arrowstyle='<-', color='r'))
annotate(text='', xy=(u6[n1,0], u6[n1,1]), xytext=(u6[n2,0], u6[n2,1]),
```

```
  arrowprops=dict(arrowstyle='<-', color='r'))

plot(xx, yy, 'go')
plot(xx[1], yy[1], 'ro')

xlim(-1.5, 1.5)
ylim(-1.25, 1.25)
grid(True, which='both', linestyle=":")
xlabel("$x$")
ylabel("$\\dot x$")

sp1.annotate("unstable", xy=(0.0, 0.01), xycoords='data',
  xytext=(-0.15, 0.8), textcoords='data', size=20, va="center", ha="center",
  bbox=dict(boxstyle="round4", fc="w"), arrowprops=dict(arrowstyle="->"))

sp1.annotate("stable", xy=(-1.0, -0.01), xycoords='data',
  xytext=(-0.8, 0.6), textcoords='data', size=20, va="center", ha="center",
  bbox=dict(boxstyle="round4", fc="w"), arrowprops=dict(arrowstyle="->",
  connectionstyle="arc3,rad=-0.2", fc="w"))

sp1.annotate("stable", xy=(1.0, -0.01), xycoords='data',
  xytext=(0.8, -0.6), textcoords='data', size=20, va="center", ha="center",
  bbox=dict(boxstyle="round4", fc="w"), arrowprops=dict(arrowstyle="->",
  connectionstyle="arc3,rad=0.2", fc="w"))

show()
```

1.3 Problems

Exercise 1.1 The equation of motion for a damped anharmonic oscillator is given by:

$$\ddot{x} + \alpha\dot{x} - \omega_0^2 x + \beta x^3 = 0,$$

where $\ddot{x}$ represents the acceleration, $\dot{x}$ is the velocity, α is the damping coefficient, ω_0 is the natural angular frequency, and β characterizes the anharmonicity of the oscillator. Determine the equilibrium points of the system and analyze their stability.

Exercise 1.2 The Lotka-Volterra Predator-Prey Model is described by the following set of first-order ordinary differential equations:

$$\dot{x} = x(\alpha - \beta y),$$
$$\dot{y} = y(-\gamma + \delta x),$$

where x represents the prey population, y represents the predator population, and $\alpha, \beta, \gamma, \delta$ are positive constants. Find the equilibrium points and perform linear stability analysis around each equilibrium.

Exercise 1.3 The equation of motion for the Duffing Oscillator is:

$$\ddot{x} + \delta\dot{x} + \alpha x + \beta x^3 = 0,$$

where δ is the damping coefficient, and α, β characterize the stiffness and nonlinearity. Rewrite the second-order ODE as a system of first-order equations, identify equilibrium points, and analyze their stability.

Exercise 1.4 The nonlinear system describing chemical reaction dynamics is given by:

$$\dot{x} = k_1 - k_2 x + k_3 x^2 y,$$
$$\dot{y} = -k_3 x^2 y + k_4 y,$$

where x and y are the concentrations of two species, and k_1, k_2, k_3, k_4 are positive rate constants. Identify the equilibrium points and classify their stability.

Exercise 1.5 The dynamics of a Van der Pol Oscillator are described by:

$$\ddot{x} - \mu(1 - x^2)\dot{x} + x = 0,$$

where $\mu > 0$ is a parameter. Convert the second-order equation into a first-order system, determine the equilibrium points, and analyze their stability for different values of μ.

Exercise 1.6 The dynamics of the SIR Epidemic Model are governed by:

$$\dot{S} = -\beta S I,$$
$$\dot{I} = \beta S I - \gamma I,$$
$$\dot{R} = \gamma I,$$

where S, I, R represent the susceptible, infected, and recovered populations, respectively, and $\beta, \gamma > 0$ are rate parameters. Reduce the system to two equations (using $S + I + R = N$), find the equilibrium points, and perform linear stability analysis.

Exercise 1.7 The equation of motion for a pendulum with damping is:

$$\ddot{\theta} + \alpha\dot{\theta} + \omega_0^2 \sin\theta = 0,$$

where θ is the angular displacement, α is the damping coefficient, and ω_0 is the natural angular frequency. Approximate $\sin\theta \approx \theta$ for small perturbations, identify the equilibrium points, and analyze stability for small θ.

Exercise 1.8 The dynamics of a nonlinear circuit (Chua's circuit) are described by the following set of equations in dimensionless variables:

$$\dot{x} = \alpha(y - x - f(x)),$$
$$\dot{y} = x - y + z,$$
$$\dot{z} = -\beta y,$$

where $f(x) = m_1 x + 0.5(m_0 - m_1)(|x+1| - |x-1|)$, and α, β, m_0, m_1 are parameters. Find the equilibrium points and analyze their stability for specific parameter values.

Exercise 1.9 A two-species ecological model with logistic growth is given by:

$$\dot{x} = r_1 x \left(1 - \frac{x}{K_1} - a_{12}\frac{y}{K_1}\right),$$
$$\dot{y} = r_2 y \left(1 - \frac{y}{K_2} - a_{21}\frac{x}{K_2}\right),$$

where x and y represent the populations of two species, r_1, r_2 are intrinsic growth rates, K_1, K_2 are carrying capacities, and a_{12}, a_{21} are competition coefficients. Determine the equilibrium points and investigate their stability.

1.4 Reference Books

[1] H. Goldstein, J. L. Safko, and C. P. Poole. *Classical Mechanics, 3e*. Pearson Education, 2011.

[2] M. Lakshmanan and S. Rajasekar. *Nonlinear Dynamics*. Springer Berlin Heidelberg, 2003.

[3] S. Strogatz. *Nonlinear Dynamics and Chaos. With applications to physics, biology, chemistry, and engineering*. CRC Press, 2019.